BUSINESS MATH

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& Technology

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Red River College of Applied Arts, Science, & Technology

Business Math

Jean-Paul Olivier

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Licensing

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Business math is the study of mathematics required by the field of business. Business professionals will work with taxes, gross earnings, product prices, and currency exchange; they will be offered loans, lines of credit, mortgages, leases, savings bonds, and other financial tools. This textbook covers all of these topics and how these financial tools can maximize their earnings and minimize their costs. It discusses how to execute smart monetary decisions both personally and for their business.

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Textbook Features

How To Approach The Mathematical Problem (The PUPP Model)

Our students struggle with completing math exercises from beginning to end. They are unfamiliar with mathematical thought process and the techniques involved. A systematic approach is needed to help students learn how to think. This textbook's core design centers around a structured thinking process termed The PUPP Model (Plan, Understand, Perform, and Present). This process is found throughout the textbook in every guided example to help students develop a step-by-step problem-solving approach.

1. **Plan**. One of the most common difficulties is when students rush into calculations and have no clue what they are solving for! This step focuses and reminds students about what it is they are doing. The goal is to encourage students to recognize what question has been asked and what variable(s) need to be calculated.
2. **Understand**. It is critical to know what you are doing and how you are going to get there. Before performing calculations, this step urges students to think through the entire process required in order to logically arrive at the solution. There are two critical elements:
    i. **What You Already Know**. Students must read through the question and assign the correct values to appropriate variables. If necessary, diagrams such as timelines are drawn to help visualize the information being provided and what to do with that information.
    ii. **How You Will Get There**. Everybody needs a roadmap. This step focuses students on describing the steps, procedures, and formulas required to arrive at the solution.
3. **Perform**. This stage is all about the mathematics. Students execute their roadmap and perform the needed calculations and procedures to arrive at the root(s) of the problem.
4. **Present**. The final part of the model encourages students to express their solution(s) in the context of the question. It's not good enough just to calculate a number. They have to understand what it means and how it fits into the situation. This also makes it easier to see if the solution makes sense. If not, there is a higher chance of spotting an error. Additionally, visual charts and graphs assist students in understanding how the various numbers fit into the cohesive puzzle.

What Are The Steps Needed To Arrive At The Final Answer? (How It Works)

Have you sometimes known what the destination is but you just couldn't figure out how to get there? Students commonly experience this problem in financial mathematics, particularly when questions become grander in scope. Integrated into the PUPP Model is another core feature of this textbook called "How It Works". In this section, students can find:

1. Detailed, step-by-step sequential procedures to address common business mathematical problems.
2. Handy reference flowchart guides that summarize the key steps required to arrive at the solutions.
3. Where possible, quick and simple examples to further illustrate the procedure. In every section of the textbook, the "How It Works" is introduced. These steps are then utilized in all guided examples to assist students in taking a problem from beginning to end.

The Repetitiveness Of Annuities

I have heard from instructors, professors, textbook reviewers, and students alike the same concerns about the repetitiveness in teaching the four types of annuities. It is no secret that there are only minor differences between the annuity types. Many have expressed that their courses are highly time-constrained. Other textbooks address most or all annuity types separately. Each annuity type is then solved for the five common unknown variables. This can result in up to twenty different lectures.

An overwhelming number of educators have indicated that they find it better to teach the four annuity types up front. Once students understand how to recognize the annuity type and the key differences, students can solve any annuity for any variable!

In this textbook, the four annuity types are thoroughly introduced and discussed, followed by sections discussing the solving for each formula variable. Key knowledge is offered in a single chapter. This textbook also utilizes only four annuity formulas - two for ordinary annuities and another two for annuities due. Incorporated into the design of these formulas are the mathematics needed to address both simple and general formats. To reduce the chance of error, students are not required to use combinations of annuity formulas nor substitute solutions of one formula into another.

When solving for other annuity variables, students use their algebra skills to substitute and rearrange the four formulas. There are no long lists of formulas that are nothing more than algebraic rearrangements of existing formulas. There are two formulas for ordinary annuities and two formulas for annuities due. Clean and simple.

Relevant Algebraic Symbols

There are two key student concerns here: irrelevant algebraic symbols and algebraic symbols with multiple usages. Both of these concerns are addressed through symbol choices and formulas utilized in this textbook.

Nothing is more frustrating for students than algebraic symbols that have nothing to do with the variable. For example, it is common to let "j" represent the nominal interest rate. To students this is like apples and oranges. What makes more sense is to use relevant algebraic symbols that help students to understand the formulas being presented. Doesn't IY for interest rate per year make much more sense than “j”? This text utilizes representative singular (such as P for profit) or plural (such as PMT for annuity payments) symbols to further student understanding.

Additionally, having identical algebraic symbols with different interpretations is very confusing for students. For example, the symbol for future value (FV) is used to represent the future value of a single payment, an ordinary annuity, or an annuity due. However, many questions require students to execute combinations of single payments and annuities. This requires multiple appearances of the same FV symbol each requiring different computational formulas. Except where industry standards exist, this textbook differentiates algebraic symbols so that each symbol has only one meaning. Thus FV is for the future value of a single payment, FVORD is the future value of an ordinary annuity payment stream, and FVDUE is the future value of an annuity due payment stream. This means less confusion for our students.

How And Why A Formula Works

We all agree that understanding a formula and not just memorizing it goes a long way in business mathematics. In this respect, this textbook offers annotated and detailed formulas allowing students to simultaneously visualize the formula as a whole and by its various components. Instead of cross-referencing variable definitions and concept explanations, each part of the formula is visually addressed and the elements explained as a cohesive whole. Handbook Design It is difficult to get students to look at and use their textbook. An easy reference handbook design forms a cornerstone to aiding our students in two ways.

1. The book offers a friendly, strong visual appeal. It is not just page after page of endless and uninviting formulas. Instead, students find modern graphics, pictures, and an appearance designed to encourage their interaction.

2. Unlike other textbooks, every section and chapter follows an identical structure. The goal is to make material easy to find. Throughout the book there are constant guides and locators.

- In addition to the table of contents, every chapter introduction and summary provides key topic summaries.
- If a student needs to know the steps in solving a problem, students always find the details in the "How It Works" section. If they just need a summary or reminder, they can always find a nutshell summary at the end of each chapter.
- If students need calculator function instructions, they always find it in the Technology section of each chapter summary. Call the book a reference manual if you will. It is about students finding what they need when they need it made easy. Relevance As educators, we constantly hear, “…and when will I ever use that in my life?” This textbook demonstrates to students both personal and professional applications in discussions, guided examples, case studies, and even their homework questions. 1. On a personal level, students are shown realistic scenarios, companies, and products that they commonly are exposed to. It is always easier to learn something when students can relate to it.

On a professional level, this textbook demonstrates business situations in different business fields so that students can see applicability regardless of their chosen career path. Whether students are planning on becoming accountants or marketers, there are applications to be found in this textbook. This book offers to your student: Math organized by business function using products and companies that are familiar to students. Both business and personal application sections ranging from HELOCs and Promissory Notes to RRSP planning and purchasing vehicles. Integration Of Technology & Resources Texas Instruments BAII+ Calculator One of the most widely used calculators in Canada, the Texas Instruments BAII+, is illustrated and fully integrated into this textbook. For those instructors and professors utilizing this calculator, step-by-step calculator button solutions are provided at the bottom of relevant PUPP model examples. Improving this integration is the usage of similar formula and calculator symbols that allows students to make the easy transfer between algebraic variables and calculator buttons. If calculators are not used in your course or you use an alternate calculator, the layout and design of the textbook strategically places these calculator solutions in an inconspicuous manner in the guided solutions. Thus, they do not interfere with your students' coverage of the material and are easily skipped. Structured Exercises Every section, as well as chapter end, has approximately 18 practice questions for the students. Students comment that having more than that appears daunting and discourages doing their homework. Imagine reading a section and finding eighty homework questions at the end! Solutions to all exercises are found at the back of the book. The exercises are broken into three sections:

1. Mechanics. This section focuses on fundamental mathematical skills and practicing of the formulas. These questions are straightforward and may appear in a table format that encourages students to become more familiar with the formulas and how to solve them. After completing this section, your students should have the basic mathematical skills however will not be able to demonstrate adequate competency in the subject matter without continuing to the next section.
2. Applications. Once students has mastered the mathematics, it is time to put their skills to work and solve various business and personal applications. The questions in this section are in word format and require the students to execute their problem-solving skills. After completing this section, students will be able to demonstrate adequate competency in the subject matter.
3. Challenge, Critical Thinking, and Other Applications. This section raises the difficulty bar and asks challenging questions. Solutions may require multi-stage approaches or integration of different concepts. Strong problem-solving skills are needed. Questions may also take students in new directions by having them interrelate different concepts. Students completing this section demonstrate a high level of competency in the subject matter.

Identification Of Common Errors & Finding Shortcuts

Built-in to the design of this book are the standard elements that make students aware of the common errors (Things To Watch Out For) as well as some tips and tricks (Paths To Success) along the way. What makes these different in this book is that they are an integrated component of the textbook and not separated into text boxes that are generally ignored by our students.

How Do All The Pieces Fit Together?

Helping your students integrate their concepts is accomplished in two main ways: 1. End Of Chapter Exercises. These questions allow students to explore the various concepts outside of their chapter section. Where possible, various concepts are combined in questions that allow students to integrate their acquired knowledge. 2. Short Case Studies. A fictional company called Lightning Wholesale is used throughout this textbook. Almost every chapter features a short case study showing how a single business applies all the various mathematical concepts into its daily operations. Students can learn how various chapter elements combine together into business scenarios. Basic Version Business Math: A Step-by-Step Handbook 2018 Revision A Creative Commons License (CC BY-NC-SA) J. OLIVIER vii

Am I Understanding The Material? (Give It Some Thought)

The students have read the material, but did they take the time to understand the material? Throughout the textbook, after concepts have been introduced, is a question and answer section called "Give It Some Thought". This is not a mathematical computation section. It features conceptually based questions whereby no calculations are required. Instead, this section offers students periodic checkpoints to determine if they are understanding the relationships of various variables and the concepts being presented.

A Note on Rounding Rules

- The rules utilized in this textbook follow a very basic premise: no rounding until the final answer is achieved unless there is a logical reason to round an interim solution.
- All interim solutions display up to six decimal places, limited only by the graphical display capabilities of the BAII+ calculator. Thus, what the student sees on the screen of the calculator is exactly what is presented in the examples.
- Unless there is a standard for rounding the final answer (such as currency), the solution displays up to six decimals in decimal format and four decimals in percentage format.

- An introduction to the global rounding rules is presented in the appendix to Chapter 1.
- The basic principles of rounding are discussed in Chapter 2.
- Any nuances, textbook standards, or industry standards are presented as needed throughout the book as various topics are introduced. These key rules are summarized in Appendix C.

About The Author

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- *University of Manitoba*, Bachelor of Commerce (Honours)
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Jean-Paul has been teaching Business and Financial Mathematics, as well as Business Statistics and Quantitative Methods for the past 21 years. He is a dedicated instructor interested in helping his students succeed through multi-media teaching involving PowerPoints, videos, whiteboards, in-class discussions, readings, online software, and homework practice. He regularly facilitates these quantitative courses and leads a team of instructors. You may have met Jean-Paul at various mathematical and statistical symposiums held throughout Canada (and the world). He has contributed to various mathematical publications for the major publishers with respect to reviews, PowerPoint development, online algorithmic authoring, technical checks, and co-authoring of textbooks. This text represents Jean-Paul's first venture into solo authoring.

Jean-Paul resides in Winnipeg, Manitoba, Canada, along with his wife and two daughters. When he is not teaching, he loves vacationing in the warm climes of the south with his family in California and Florida

CHAPTER OVERVIEW

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1.0: Introduction

Math! You may really dread the “M” word. Not only that, but this course is business mathematics, which some people say is even harder. Is that true, or is it just a rumor?

Take a look at your world from a different perspective—notice that math is everywhere! Before you went to bed last night, you performed some math and solved a complex algebraic equation (and you probably didn’t even realize it!). That equation helped you to figure out what time to get up this morning so that you could arrive at your destination on time. You factored in variables such as the required arrival time, commuting time (including the unknown variables of traffic and weather delays), morning routine time, and even snooze button time. When you solved this complex algebraic equation, you then set your alarm clock.

From the moment you woke up this morning, numbers have appeared everywhere. Perhaps you figured out how many calories are in your breakfast cereal. Maybe your car’s gas tank was low, so you calculated whether you had enough gas for your commute today. You checked your bank account online to see if you had enough money to pay the bills. At Tim Hortons you figured out what size of coffee you could buy to go with your donut using only the $1.80 in change in your pocket. I hope you get my point.

Students ask, “But why do I want to learn about business math?” The answer is simply this: money. Our world revolves around money. We all work to earn money. We need money to purchase our necessities, such as homes, transportation, food, and utilities. We also need money to enjoy the pleasures of life, including vacations, entertainment, and hobbies. We need money to retire. And businesses need money to survive and prosper.

Quite simply, business math teaches you monetary concepts and how to make smart decisions with your money.

Contributors and Attributions

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1.1: What Is Business Math?

Business math is the study of mathematics required by the field of business. By the fact that you are reading this textbook, you must be interested in a business field such as accounting, marketing, human resources, or economics. Regardless of your path, you cannot avoid dealing with money and numbers.

Both personally and in your career you certainly use elementary arithmetic such as addition, subtraction, multiplication, and division. However, there is a whole field of mathematics that deals specifically with money. You will work with taxes, gross earnings, product prices, and currency exchange; you will be offered loans, lines of credit, mortgages, leases, savings bonds, and other financial tools. Do you know what these are and how these financial tools can maximize your earnings and minimize your costs? Do you have what it takes to execute smart monetary decisions both personally and for your business? Do you know how interest works and how it gets calculated? If you can answer “yes” to these questions, then you are already off to a great start. If not, by the end of this textbook you will have a better understanding of all of these topics and more.

How Do I Learn about Business Math?

Let’s be realistic. In some areas of life and business, you can achieve a reasonable degree of understanding just by reading. However, reading about business mathematics without doing it would be disastrous. To succeed, you must follow a structured approach:

1. Always read the content prior to your professor covering the topic in class.
2. Attend class, ask questions, and explore the topic to advance your understanding.
3. Do the homework and assignments—you absolutely must practice, practice, and practice!
4. Seek help immediately when you need it. Learning mathematics is like constructing a building.

Each floor of the building requires the floor below it to be completed first. In mathematics, each section of a textbook requires the concepts and techniques from the sections that preceded it. If you have trouble with a concept, you must fix it NOW before it causes a large ripple effect on your ability to succeed in subsequent topics.

So the bottom line is that you absolutely cannot replace this approach—you must follow it.

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1.2: How Can I Use The Features Of This Book To My Advantage?

I wrote this textbook with the help of student input specifically for students like you. It is designed to address common student needs in mathematics. Let’s examine each of those needs and learn how you benefit from the features in this textbook.

I Don't Know How to Solve a Math Problem

Do you find it difficult to start a math problem? What do you do first? What is the next step? The reason this is challenging to some students is that they have never been shown problem-solving techniques. This textbook extensively uses an author-created design called the PUPP model in all guided examples. Successfully working through a problem involves four steps: Plan, Understand, Perform, and Present (PUPP). The figure on the next page shows the PUPP model at work. This model provides a structured problem-solving approach that will aid you not only in mathematics but in any problem-solving situation you may encounter.

✓ : Straightforward Interest Rate Calculation - FV and PV Known

When Sandra borrowed $7,100 from Sanchez, she agreed to reimburse him $8,615.19 three years from now including interest compounded quarterly. What interest rate is being charged?

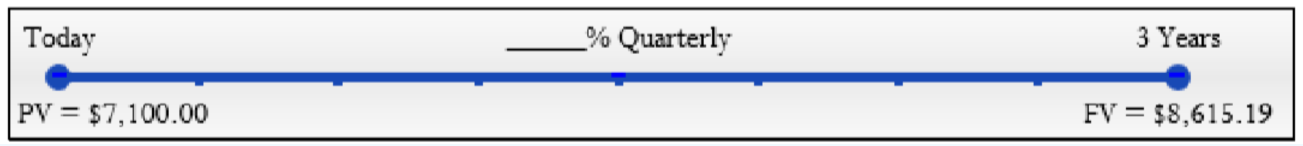
Solution

Find the nominal quarterly compounded rate of interest (IY).

What You Already Know

Step 1:

The present value, future value, term, and compounding are known, as illustrated in the timeline.



CY = quarterly = 4 Term = 3 years

How You Will Get There

Step 2:

Calculate N using Formula 9.2.

Step 3:

Substitute into Formula 9.3 and rearrange for i.

Step 4:

Substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

$N=4\times 3=12$

Step 3:

$$\begin{aligned}\$8,615.10&=\$7,100(1+i)^{12}\\ 1.213394&=(1+i)^{12}\\ 1.213394^{\frac{1}{12}}&=1+i\\ 1.016249&=1+i\\ 0.016249&=i\end{aligned}$$

Step 4:

$$0.016249=\frac{IY}{4}$$

$$IY = 0.064996 = 0.065 \text{ or } 6.5\%$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
12	Answer: 6.499709	-7100	0	8615.10	4	4

Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.

To understand each of these steps, let’s plan a vacation.

1. **Plan**. If you are heading out on vacation, the first thing you want to know is where you are going! Otherwise, how would you get there? In mathematics, you need to figure out what variable you are solving for before you can proceed. This step focuses your problem solving on the end goal. Many students mistakenly rush through this step and start punching numbers on a calculator or plugging numbers into formulas without understanding what it is they are doing. Spend plenty of time on this step, because the remaining three steps do not help at all if you solve for the wrong variable!
2. **Understand**. In planning your vacation, once you figure out where you want to go you need to gather information about your destination. There are things you already know, but there are also things that you do not know, so you need a plan to figure these out. This step of the PUPP model helps your problem solving in two ways:
   - First, you need to assess what information has already been provided to you. You need to assign values to variables. You may need to draw diagrams to understand how all the numbers fit together.
   - Second, you create a plan by which you will solve your problem. Your roadmap identifies how you go from Point A (some variable is unknown) to Point B (knowing what the variable is).
3. **Perform**. Once your vacation is planned, all that is left is to go on vacation! In other words, execute your plan. This step involves all of the mechanics and computations of your roadmap that you developed in the previous step.
4. **Present**. After you return from vacation, you tell everyone about what you just did. In mathematics, this means you must explain solutions in the context of the question asked. If you just calculated x = $700, what exactly does that represent and how should it be interpreted? Calculating a solution without understanding it is not good enough. A further benefit of this last step is that you can detect errors more easily when you have to explain the solution. For example, read this presentation statement: “If an investor puts $1,000 into a savings account that earns 10% interest, I have calculated that after one year the investor has $700 in her savings account.” Did that make sense? If an account is earning interest, the balance should have become bigger, not smaller! Clearly, the solution of x = $700 must be an error!

Once you repeatedly practise this model and ingrain it in your thinking process, you have a structured approach to solving not just math problems but any life problem. When you are first learning the PUPP model you will need to write out every step of the model. As you keep practising, you'll find that the model becomes a way of thinking. You can never skip one of the steps, but you won’t have to write them all out anymore because you’ll execute some steps with rapid-fire thought!

I Can't Find What I Want When I Need It

When practising your business math through homework and assignments, you will need to go back to look up various formulas or techniques or to review concepts. The challenge of most textbooks is in finding what you need when you need it buried in the vast pages of the textbook.

Browse the table of contents of this book and notice the functional layout. This textbook incorporates a handbook design that aids in finding information. When you need to look something up, you will notice the following:

- Every part of every section in every chapter is designed and laid out using an identical structure. While not every part is required in every section, the order is always as follows:
  - Concept introduction
  - Concept fundamentals and characteristics
  - Formula development and explanation
  - Steps and procedures required to solve problems
  - Important concept elements and technology use

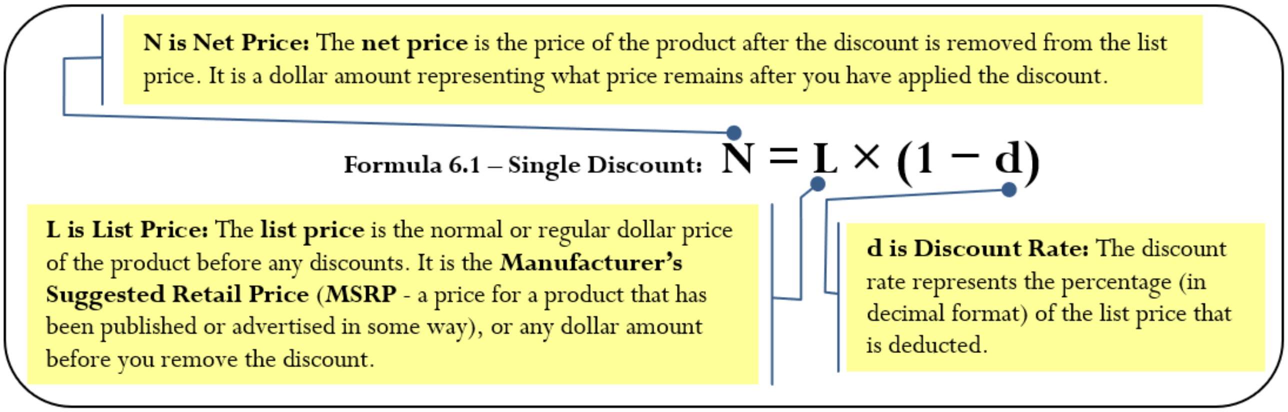
- Common pitfalls and shared tips
    - Conceptual understanding questions
    - Guided examples
    - Homework exercises
- The start of every chapter includes a miniature table of contents allowing you to locate what you need in the chapter without having to flip back to the book's main table of contents.
- End-of-chapter summaries deliver exactly that—each chapter in a nutshell. They contain key concept reviews, key terms, algebraic symbol definitions, formula summaries, and technology discussions on calculator usage. Links to other chapters are also provided as needed. If you need to look something up, this is the place to start!

Are There Any Steps to Help Me Remember?

Students have always asked if they need to follow a logical sequence of steps to solve a problem. The answer, of course, is YES! A section called "How It Works" helps you with this by providing step-by-step procedures, techniques, and formulas you need to solve the problem.

Can This Book Help Me Understand the Formulas Better?

You can't avoid formulas—this is math, after all. What helps, though, is to understand what the formulas do and how each component comes together to produce the solution. If you can understand, you do not need to memorize! This book lays formulas out visually, clearly labelling all symbols. In addition, each component is fully explained (see below). You are not required to cross-reference paragraph-style explanations to formulas, as the visual layout makes it all clear. The goal is for you to understand what exactly the formula does and how, so that you understand enough to recognize when solutions make sense or might be wrong.



N is Net Price: The **net price** is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

Formula 6.1 – Single Discount: $N = L \times (1 - d)$

L is List Price: The **list price** is the normal or regular dollar price of the product before any discounts. It is the **Manufacturer's Suggested Retail Price (MSRP** - a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

d is Discount Rate: The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Additionally, you will not find any of those archaic algebraic symbols in this book. Representative symbols make the algebra easier to remember and understand. Those ancient mathematicians just loved to speak Greek (literally!), but this text speaks modern-day English and understands that we have technology here in the twenty-first century! We'll use symbols like N for net price, L for list price, P for profit, and E for expenses, letters that actually remind you of the variable they stand for!

Does Anyone Actually Do Any of This in the Real World?

As you read through the guided examples and do your homework, pause to consider how the question applies to you. Does it demonstrate a business application you can use to be a smarter business professional? How can you extend the question to similar situations? This textbook shows you examples from both business and personal situations that you either have already encountered or probably will encounter in the future. (Remember: Math is all around you!)

- **Business** Why do business programs require a business math course in the first term or two? How is it relevant to your future? This textbook provides numerous examples of applications in marketing, finance, economics, accounting, and more. Once you complete this textbook, you should see clearly how business mathematics is relevant and important to your chosen future career.
- **Personal** You will see companies and products from your everyday life. Real-world situations show you how to put a few extra dollars in your pocket by making smart mathematical choices. What if you could save $100 per month simply by making smarter choices? Over 40 years at a very conservative rate of interest, that is approximately $100,000 in your pocket. You will also find life lessons. Are you ready to start planning your RRSP? Would you like to know the basics on how to do this? If you

are thinking of buying a home, do you know how the numbers work on your mortgage? To learn how to buy a car at the lowest cost, read on!

Are There Any Secrets?

There are never any secrets. However, I will point out some tricks of the trade and commonly made mistakes throughout this book.

1. **Paths To Success** This feature is designed to make you feel like part of the “in” crowd by letting you take advantage of shortcuts or tricks of the trade, such as some easy way to remember a particular technique or formula. It helps you see how some of the math could be made a little simpler or performed in a different way.
2. **Things to Watch Out For** Awareness of common pitfalls and sources of error in mathematics reduces your chances of making a mistake. A reminder also helps when important concepts are about to be used, reused, or combined in a novel way. It gives you a “heads up” when needed.

Is There Any Way to Know That I'm "Getting It"?

Mastering business math requires two key skills:

1. Executing the required techniques and calculations
2. Understanding and successfully integrating various mathematical concepts. (For example, when a variable changes, can you estimate how this affects the final result?)

The second skill is much harder to acquire than the first. A feature called **Give It Some Thought**, poses scenarios and questions for which no calculations are required. You need to visualize and conceptualize various mathematical theorems. Ultimately, you perform a self-test to see if you are "getting it."

In Today's World, Who Does This By Hand Anymore?

That is a perfectly valid question. You are right: Most people do not do this work by hand and instead use mathematical tools such as calculators, spreadsheet software, and apps. However, as the saying goes, you must learn to walk before you can run. The learning of mathematics requires pencil and paper first. Once you master these, you can then employ technology to speed up the process. This basic version textbook incorporates one technology to aid you in your application of business mathematics:

1. **The Calculator** Integrated throughout this book are instructions for one of the financial calculators most widely used in Canada—the Texas Instruments BAII Plus. Calculator functions with step-by-step button sequences are presented with guided examples illustrating how you solve the math problem not only by hand but also using the financial calculator.

Is There Any Way I Can Assess My Ability?

At the end of almost every section of this book you will find approximately 18 practice questions covering the concepts specifically introduced in that section, with final solutions listed to all questions at the end of this textbook. These questions are divided into three categories to help develop your mathematical skills and assess your abilities:

1. **Mechanics** This group of questions focuses on fundamental mathematical skills and formula usage. They are your first taste of using the section formulas and working with the mathematical concepts at an introductory level. Successful completion of the Mechanics section means that you:
    a. Have at least a basic understanding of the variables at play,
    b. Show rudimentary application of concepts, and
    c. Can calculate solutions using the formulas.

    Do not stop here, though! You do not yet have enough skills to meet the basic requirements of most business math courses. Once you have mastered the Mechanics section, you must progress to the next section, where you learn to apply your knowledge.

2. **Applications** These questions are more typical of business math course expectations (check with your instructor). The key difference from the Mechanics section is that you must now execute your problem-solving skills. These questions require you to determine the unknown variable and figure out how the various pieces of the puzzle come together in the solution. Successful completion of these questions means that you
    a. Understand concepts at a satisfactory level,
    b. Can problem solve typical business math applications, and

   c. Can successfully integrate concepts at an acceptable performance level.
3. **Challenge, Critical Thinking, and Other Applications** These questions put your knowledge and understanding to the test. The difficulty bar is set high, as these questions commonly require multistep solutions with advanced problem-solving techniques. As well, success might require you to integrate formulas, concepts, or procedures in novel ways. You will be challenged! Successful completion of these questions means that you
   a. Understand concepts at an above-average level,
   b. Can problem solve in difficult situations, and
   c. Can successfully integrate concepts at a higher level of thinking.

How Do I Put It All Together?

You may have noticed in your previous studies that many math textbooks teach one topic at a time. However, they do not necessarily help you see how all the little topics come together and interrelate. This leaves you wondering if there is a master plan.

Throughout this textbook, a single fictional company called Lightning Wholesale illustrates such interconnections. Case studies appear at the end of Chapter 3 through Chapter 15. Even if your instructor does not assign these case studies or work through them in class, completing them benefits you in three ways:

1. You learn how to integrate chapter concepts into a cohesive whole.
2. You perform large multitask problems that reflect real-world problem solving. This will make you a much more valuable employee when you graduate.
3. You see how business mathematics works throughout an organization.

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1.3: Where Do We Go From Here?

At this point, I hope that you can see that all the features of this textbook are built for you! They are here to help you learn, understand, and perform math. This book gives you every opportunity to succeed!

Now that you know what business mathematics is and why it is important to you, let's get going! Complete the exercises below. Then look at your course syllabus and read your first section. Remember to take advantage of all the features of this textbook along the way.

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1.4: Rounding Rules (Appendix 1)

How and when to round always is a source of confusion. This Appendix provides you with some rules for rounding your answers. We start with some general guidelines that apply globally throughout the entire textbook, then comment at the rules that apply to some specific sections of text.

Global Rounding Rules

1. All interim solutions never get rounded unless there is a logical reason or business process forcing it to be rounded.
2. When writing non-terminating decimals in this textbook, up to six decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
3. All final numbers are rounded to four decimals in decimal format and two decimals in percent format unless instructions indicate otherwise.
4. Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
5. Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line up a sequence of numbers.

Topic Specific Rounding Rules

Some specific topics have their own rounding standards. Until you learn about these topics, it does not make any sense to put those standards here. However, the standards are introduced with the relevant topic. They are also summarized into Appendix 16.3: Rounding Rules.

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1.E: Succeeding in Business Mathematics (Exercises)

1.3: Where do we go from here?

Mechanics

In questions 1–8, identify where you would look in this textbook if you needed to find the indicated information.

1. The definition of a mathematical concept or term.
2. An explanation of a particular calculator function.
3. Guided solutions and examples to illustrate a mathematical concept.
4. An illustration of how to use the calculator to solve a particular scenario.
5. Step-by-step procedures for solving a mathematical problem.
6. More practice questions.
7. Detailed explanations of how to use the Excel spreadsheet templates.
8. A full discussion of a mathematical concept.
9. What does the acronym PUPP stand for? Why is it important?
10. You just completed the "Mechanics" questions for a particular section. Do you need to do any more questions, or is that enough for you to pass the course?
11. Every chapter includes Review Exercises at the end. Why do you think it is important to complete these exercises?

Applications

12. Think about your current job. List five activities that you perform that involve mathematics.
13. Talk to family members. Note their individual occupations and ask them about what work activities they perform every day that involve mathematics.

For questions 14–17, gather a few of your fellow students to discuss business mathematics.

14. Identify five specific activities or actions that you need to perform to succeed in your business math course.
15. Consider how you will study for a math test. Develop three specific study strategies.
16. Many students find it beneficial to work in study groups. List three ways that a study group can benefit you in your business math course.
17. For those students who may experience high levels of stress or anxiety about mathematics, identify three coping strategies.

Go online and enter the phrase "business math" or "business math help" into a search engine. Look at the table of contents for this book and note any websites that may help you study, practice, or review each chapter.

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CHAPTER OVERVIEW

2: Back To The Basics (Shoulder Check)

2.0: Introduction
2.1: Order of Operations (Proceed In An Orderly Manner)
2.2: Fractions, Decimals, and Rounding (Just One Slice Of Pie, Please)
2.3: Percentages (How Does It All Relate?)
2.4: Algebraic Expressions (The Pieces Of The Puzzle)
2.5: Linear Equations - Manipulating and Solving (Solving the Puzzle)
2.6: Natural Logarithms (How Can I Get That Variable Out of the Exponent?)
2.S: Summary
2.E: Back To The Basics (Exercises)

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2.0: Introduction

Where can you go in life and not be exposed to numbers and mathematics? Whether you are figuring out the price of a product (including shipping) on eBay, or balancing your bank accounts, you use your elementary mathematical skills from both your primary and secondary education.

Think about the math you perform every day:

- At the grocery store, you compare products to calculate the best value. One brand of potato chips retails for $3.99 for 300 g, while the equally satisfying brand beside it is priced at $3.49 for 250 g. Which offers the better value?
- As the host for a large gathering, you are preparing a homemade lasagna and need to triple the original recipe, which calls for 1 ⅔ cups of tomato sauce. In the expanded recipe, how many cups of tomato sauce do you need?
- If you are a sports fan, you know plenty of statistics about your favourite players and teams. Many come in percentage form, such as three-point throws for NBA stars or save percentages for NHL goaltenders. What exactly do those percentages mean?
- Many employers pay out bonuses. Perhaps in your company managers get twice as large a bonus as employees. Your company has five managers and 25 employees. If it announces a $35,000 total bonus, what is your share as an employee?

Mathematics and numbers surround you in the business world, where you must read many numerical reports, interpret how the numbers fit together, and create your own reports showing such metrics as sales and profit projections.

Away from work, you must manage your income and pay your bills. This is a mathematical problem you likely solve on a daily basis, ensuring that the money flowing out of your bank account does not exceed the money flowing in. To purchase groceries, vacations, or entertainment, you need numbers.

This chapter gives you a refresher on your basic mathematical skills, which are essential for success in later chapters. Some instructors will review this chapter with you, while others will leave it up to you to complete this chapter independently. Either way, this chapter is important and should be used to test your basic abilities. For example, it would be unfortunate if you got a leasing calculation wrong because you made an error by breaking an order of operation rule even though you understood basic leasing concepts.

Approach this chapter with confidence, and if you encounter any difficulties ensure that you master the concepts before moving on to future chapters.

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2.1: Order of Operations (Proceed In An Orderly Manner)

You have just won $50,000 in a contest. Congratulations! But before you can claim it, you are required to answer a mathematical skill-testing question, and no calculators are permitted. After you hand over your winning ticket to the redemption agent, she hands you your time-limited skill-testing question: $2 \times 5 + 30 \div 5$. As the clock counts down, you think of the various possibilities. Is the answer 8, 14, 16, or something else altogether? Would it not be terrible to lose $50,000 because you cannot solve the question! If you figured out the solution is 16, you are on the right path. If you thought it was something else, this is a great time to review order of operations.

The Symbols

While some mathematical operations such as addition use a singular symbol (+), there are other operations, like multiplication, for which multiple representations are acceptable. With the advent of computers, even more new symbols have crept into mathematical symbology. The table below lists the various mathematical operations and the corresponding mathematical symbols you can use for them.

Mathematical Operation	Symbol or Appearance	Comments
Brackets	() or [] or {}	In order, these are known as round, square, and curly brackets.
Exponents	2^3 or 2^3	- In 2^3, 3 is the exponent; an exponent is always written as a superscript. - On a computer, the exponent is recognized with the ^ symbol; 2^3 is represented by 2^3.
Multiplication	$\times$ or * or $\cdot$ or 2(2) or (2)(2)	In order, these are known as times, star, or bullet. The last two involve implied multiplication, which is when two terms are written next to each other joined only by brackets.
Division	/ or $\div$ or $\frac{4}{2}$	In order, these are known as the slash, divisor, and divisor line. Note in the last example that the division is represented by the horizontal line.
Addition	+	There are no other symbols.
Subtraction	-	There are no other symbols.

You may wonder if the different types of brackets mean different things. Although mathematical fields like calculus use specialized interpretations for the different brackets, business math uses all the brackets to help the reader visually pair up the brackets. Consider the following two examples:

$$\text{Example 1}: 3 \times (4/(6-(2+2))+2)$$

$$\text{Example 2}: 3 \times [4/\{6-(2+2)\}+2]$$

Notice that in the second example you can pair up the brackets much more easily, but changing the shape of the brackets did not change the mathematical expression. This is important to understand when using a calculator, which usually has only round brackets. Since the shape of the bracket has no mathematical impact, solving example 1 or example 2 would involve the repeated usage of the round brackets.

BEDMAS

In the section opener, your skill-testing question was $2 \times 5 + 30 \div 5$. Do you just solve this expression from left to right, or should you start somewhere else? To prevent any confusion over how to resolve these mathematical operations, there is an agreed-upon sequence of mathematical steps commonly referred to as BEDMAS. **BEDMAS** is an acronym for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

How It Works

Step 1: Brackets must be resolved first. As brackets can be nested inside of each other, you must resolve the innermost set of brackets first before proceeding outwards to the next set of brackets. When resolving a set of brackets, you must perform the mathematical operations within the brackets by following the remaining steps in this model (EDMAS). If there is more than one set of brackets but the sets are not nested, work from left to right and top to bottom.

Step 2: If the expression has any exponents, you must resolve these next. Remember that an exponent indicates how many times you need to multiply the base against itself. For example, $2^3 = 2 \times 2 \times 2$. More review of exponents is found in Section 2.4.

Step 3: The order of appearance for multiplication and division does not matter. However, you must resolve these operations in order from left to right and top to bottom as they appear in the expression.

Step 4: The last operations to be completed are addition and subtraction. The order of appearance doesn't matter; however, you must complete the operations working left to right through the expression.

Important Notes

Before proceeding with the Texas Instruments BAII Plus calculator, you must change some of the factory defaults, as explained in the table below. To change the defaults, open the Format window on your calculator. If for any reason your calculator is reset (either by removing the battery or pressing the reset button), you must perform this sequence again.

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.

Buttons Pushed	Calculator Display	What It Means
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0	Back to regular calculator usage.

Also note that on the BAII Plus calculator you have two ways to key in an exponent:

1. If the exponent is squaring the base (e.g., 3^2), press $3x^2$. It calculates the solution of 9.
2. If the exponent is anything other than a 2, you must use the y^x button. For 2^3, you press $2y^x3$ =. It calculates the solution of 8.

Things To Watch Out For

Negative Signs. Remember that mathematics uses both positive numbers (such as $+3$) and negative numbers (such as -3). Positive numbers do not need to have the + sign placed in front of them since it is implied. Thus $+3$ is written as just 3. Negative numbers, though, must have the negative sign placed in front of them. Be careful not to confuse the terminology of a negative number with a subtraction or minus sign. For example, $4+(-3)$ is read as "four plus negative three" and not "four plus minus three." To key a negative number on a calculator, enter the number first followed by the ± button, which switches the sign of the number.

The Horizontal Divisor Line. One of the areas in which people make the most mistakes involves the “hidden brackets.” This problem almost always occurs when the horizontal line is used to represent division. Consider the following mathematical expression:

$$(4+6) \div (2+3)$$

If you rewrite this expression using the horizontal line to represent the divisor, it looks like this:

$$\frac{4+6}{2+3}$$

Notice that the brackets disappear from the expression when you write it with the horizontal divisor line because they are implied by the manner in which the expression appears. Your best approach when working with a horizontal divisor line is to reinsert the brackets around the terms on both the top and bottom. Thus, the expression looks like this:

$$\frac{(4+6)}{(2+3)}$$

Employing this technique will ensure that you arrive at the correct solution, especially when using calculators.

Paths To Success

Hidden and Implied Symbols. If there are hidden or implied symbols in the expressions, your first step is to reinsert those hidden symbols in their correct locations. In the example below, note how the hidden multiplication and brackets are reinserted into the expression:

$$4\left[\frac{3+2^2\times 3}{(2+8)\div 2}\right] \text{ transforms into } 4\times\left[\frac{\{3+2^2\times 3\}}{\{(2+8)\div 2\}}\right]$$

Once you have reinserted the symbols, you are ready to follow the BEDMAS model.

Calculators are not programmed to be capable of recognizing implied symbols. If you key in “$3(4+2)$” on your calculator, failing to input the multiplication sign between the “3” and the “$(4+2)$,“ you get a solution of 6. Your calculator ignores the “3” since it doesn’t know what mathematical operation to perform on it. To have your calculator solve the expression correctly, you must punch the equation through as “$3 \times (4+2)$ =”. This produces the correct answer of 18.

Simplifying Negatives. If your question involves positive and negative numbers, it is sometimes confusing to know what symbol to put when simplifying or solving. Remember these two rules:

Rule #1: A pair of the same symbols is always positive. Thus "$4+(+3)$" and "$4-(-3)$" both become "$4+3$."

Rule #2: A pair of the opposite symbols is always negative. Thus "$4+(-3)$" and "$4-(+3)$" both become "$4–3$."

A simple way to remember these rules is to count the total sticks involved, where a "+" sign has two sticks and a "−" sign has one stick. If you have an odd number of total sticks, the outcome is a negative sign. If you have an even number of total sticks, the outcome is a positive sign. Note the following examples:

$$4+(-3) =\longrightarrow 3 \text{ total sticks is odd and therefore simplifies to negative } \rightarrow 4-3=1$$

$$(-2)\times(-2) =\longrightarrow 2 \text{ total sticks is even and therefore simplifies to positive } \rightarrow (-2)\times(-2)=+4$$

✓ Example 2.1.1: Solving Expressions Using BEDMAS

Evaluate each of the following expressions.

a. $2 \times 5 + 30 \div 5$
b. $(6+3)^2 + 18 \div 2$
c. $4 \times \left[\dfrac{\{3+2^2 \times 3\}}{\{(2+8) \div 2\}}\right]$

Solution

You need to evaluate each of the expressions. This means you must solve each expression.

What You Already Know

You are provided with the mathematical expressions in a formula format. These expressions are ready for you to solve.

How You Will Get There

Employ the knowledge of BEDMAS to solve each operation in its correct order.

Perform

a.

$2 \times 5 + 30 \div 5$	**Step 1**: No brackets **Step 2**: No exponents **Step 3**: Working left to right, resolve the multiplication first.
$\mathbf{10} + 30 \div 5$	**Step 3**: Now resolve the division.
$10 + \mathbf{6}$	**Step 4**: Perform the remaining addition
$\mathbf{16}$	Final solution

b.

$(6+3)^2 + 18 \div 2$	**Step 1**: Start with the brackets and perform EDMAS; you have only an addition to perform
$\mathbf{9}^2 + 18 \div 2$	**Step 2**: Resolve the exponent.
$\mathbf{81} + 18 \div 2$	**Step 3**: Perform the division.
$81 + \mathbf{9}$	**Step 4**: Perform the addition.

90	Final solution

c.

$4\times\left[\dfrac{\{3+2^2\times 3\}}{\{(2+8)\div 2\}}\right]$	**Step 1**: Start with the innermost set of brackets \$(2+8)\$ and perform EDMAS. In this bracket, you have only an addition to resolve.
$4\times\left[\dfrac{\{3+2^2\times 3\}}{\{\mathbf{10}\div 2\}}\right]$	The innermost brackets are complete, so you now drop them. **Step 1**: You still have two sets of inner {} brackets. Start with the top one and perform EDMAS
$4\times\left[\dfrac{\{3+\mathbf{4}\times 3\}}{\{10\div 2\}}\right]$	Exponent resolved.
$4\times\left[\dfrac{\{3+\mathbf{12}\}}{\{10\div 2\}}\right]$	Multiplication performed.
$4\times\left[\dfrac{\mathbf{15}}{\{10\div 2\}}\right]$	Addition performed. You no longer need the brackets, so you drop them. Step 1: Now do the bottom brackets, performing EDMAS. You have only one division to resolve.
$4\times\left[\dfrac{15}{\mathbf{5}}\right]$	Division completed. You no longer need the brackets, so you drop them. **Step 1**: One last set of brackets to go! Only a division is left.
$4\times\mathbf{3}$	Division performed. You no longer need the brackets, so you drop them. **Step 2**: No exponents. **Step 3**: Perform the multiplication.
12	**Step 4**: Not needed Final solution.

Calculator Instructions

a. $2\times 5+30\div 5=$
b. $(6+3)y^x 2+18\div 2=$
c. $4\times((3+2)y^x 2\times 3)\div((2+8)\div 2))=$

The final solutions for the expressions are:

a. 16
b. 90
c. 12

Excel Instructions In any cell, key in a formula:

a. $=2*5+30/5$
b. $=(6+3)\hat{}2+18/3$
c. $=4*((3+2\hat{}2*3)/((2+8)/2))$

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2.2: Fractions, Decimals, and Rounding (Just One Slice Of Pie, Please)

Your local newspaper quotes a political candidate as saying, “The top half of the students are well-educated, the bottom half receive extra help, but the middle half we are leaving out.”[1] You stare at the sentence for a moment and then laugh. To halve something means to split it into two. However, there are three halves here! You conclude that the speaker was not thinking carefully.

In coming to this conclusion, you are applying your knowledge of fractions. In this section, you will review fraction types, convert fractions into decimals, perform operations on fractions, and also address rounding issues in business mathematics.

Types of Fractions

To understand the characteristics, rules, and procedures for working with fractions, you must become familiar with fraction terminology. First of all, what is a fraction? A fraction is a part of a whole. It is written in one of three formats:

$$1/2 \text{ or } ½ \text{ or } \frac{1}{2}$$

Each of these formats means exactly the same thing. The number on the top, side, or to the left of the line is known as the numerator. The number on the bottom, side, or to the right of the line is known as the denominator. The slash or line in the middle is the divisor line. In the above example, the numerator is 1 and the denominator is 2. There are five different types of fractions, as explained in the table below.

Fraction	Terminology	Characteristics	Result of Division*
$\frac{2}{5}$	Proper	The numerator is smaller than the denominator.	Answer is between 0 and 1
$\frac{5}{2}$	Improper	The numerator is larger than the denominator.	Answer is greater than 1
$3\frac{2}{5}$	Compound	A fraction that combines an integer with either a proper or improper fraction. When the division is performed, the proper or improper fraction is added to the integer.	Answer is greater than the integer
$3\frac{2/5}{7}$	Complex	A fraction that has fractions within fractions, combining elements of compound, proper, or improper fractions together. It is important to follow BEDMAS in resolving these fractions.	Answer varies depending on the fractions involved
$\frac{1}{2}$ and $\frac{2}{4}$	Equivalent	Two or more fractions of any type that have the same numerical value upon completion of the division. Note that both of these examples work out to 0.5.	Answers are equal

*Assuming all numbers are positive.

How It Works

First, focus on the correct identification of proper, improper, compound, equivalent, and complex fractions. In the next section, you will work through how to accurately convert these fractions into their decimal equivalents.

Equivalent fractions require you to either solve for an unknown term or express the fraction in larger or smaller terms.

Solving For An Unknown Term

These situations involve two fractions where only one of the numerators or denominators is missing. Follow this four-step procedure to solve for the unknown:

Step 1: Set up the two fractions.

Step 2: Note that your equation contains two numerators and two denominators. Pick the pair for which you know both values.

Step 3: Determine the multiplication or division relationship between the two numbers.

Step 4: Apply the same relationship to the pair of numerators or denominators containing the unknown.

Assume you are having a party and one of your friends says he would like to eat one-third of the pizza. You notice the pizza has been cut into nine slices. How many slices would you give to your friend?

Step 1: Your friend wants one out of three pieces. This is one-third. You want to know how many pieces out of nine to give him. Assign a meaningful variable to represent your unknown, so have s represent the number of slices to give; you need to give him s out of 9 pieces, or $s/9$.

$$\frac{1}{3}=\frac{s}{9}$$

Step 2: Work with the denominators 3 and 9 since you know both of them.

Step 3: Take the larger number and divide it by the smaller number. We have $9 \div 3 = 3$. Therefore, the denominator on the right is three times larger than the denominator on the left.

Step 4: Take the 1 and multiply it by 3 to get the s. Therefore, $s = 1 \times 3 = 3$.

$$\frac{1\times 3}{3\times 3}=\frac{3}{9}$$

You should give your friend three slices of pizza.

Expressing The Fraction In Larger Or Smaller Terms

When you need to make a fraction easier to understand or you need to express it in a certain format, it helps to try to express it in larger or smaller terms. To express a fraction in larger terms, multiply both the numerator and denominator by the same number. To express a fraction in smaller terms, divide both the numerator and denominator by the same number.

- Larger terms: $\frac{2}{12}$ expressed with terms twice as large would be $\frac{2\times 2}{12\times 2}=\frac{4}{24}$
- Smaller terms: $\frac{2}{12}$ expressed with terms half as large would be $\frac{2\div 2}{12\div 2}=\frac{1}{6}$

When expressing fractions in higher or lower terms, you do not want to introduce decimals into the fraction unless there would be a specific reason for doing so. For example, if you divided 4 into both the numerator and denominator of, you would have, which is not a typical format. To find numbers that divide evenly into the numerator or denominator (called factoring), follow these steps:

- Pick the smallest number in the fraction.
- Use your multiplication tables and start with $1\times$ before proceeding to $2\times$, $3\times$, and so on. When you find a number that works, check to see if it also divides evenly into the other number.

For example, if the fraction is $\frac{12}{18}$, you would factor the numerator of 12. Note that $1 \times 12 = 12$; however, 12 does not divide evenly into the denominator. Next you try 2×6 and discover that 6 does divide evenly into the denominator. Therefore, you reduce the fraction to smaller terms by dividing by 6, or $\frac{12\div 6}{18\div 6}=\frac{2}{3}$.

Things To Watch Out For

With complex fractions, it is critical to obey the rules of BEDMAS. As suggested in Section 2.1, always reinsert the hidden symbols before solving. Note in the following example that an addition sign and two sets of brackets were hidden: You should rewrite $3\frac{2/5}{7}$ as $3+\left[\frac{(2/5)}{7}\right]$ before you attempt to solve with BEDMAS.

Paths To Success

What do you do when there is a negative sign in front of a fraction, such as $-\frac{1}{2}$? Do you put the negative with the numerator or the denominator? The common solution is to multiply the numerator by negative 1, resulting in $\frac{(-1)\times 1}{2}=\frac{-1}{2}$. In the special case of a compound fraction, multiply the entire fraction by -1. Thus, $-1\frac{1}{2}=(-1)\times\left(1+\frac{1}{2}\right)=-1-\frac{1}{2}$.

Example 2.2.1: Identifying Types of Fractions

Identify the type of fraction represented by each of the following:

a. $\frac{2}{3}$
b. $6\frac{7}{8}$
c. $12\frac{4/3}{6\frac{4}{5}}$
d. $\frac{15}{11}$
e. $\frac{5}{6}$
f. $\frac{3}{4}\&\frac{9}{12}$

Solution

For each of these six fractions, identify the type of fraction.

What You Already Know

There are five types of fractions, including proper, improper, compound, complex, or equivalent

How You Will Get There

Examine each fraction for its characteristics and match these characteristics with the definition of the fraction.

Perform

a. The numerator is smaller than the denominator. This matches the characteristics of a proper fraction.
b. This fraction combines an integer with a proper fraction (since the numerator is smaller than the denominator). This matches the characteristics of a compound fraction.
c. There are lots of fractions involving fractions nested inside other fractions. The fraction as a whole is a compound fraction, containing an integer with a proper fraction (since the numerator is smaller than the denominator). Within the proper fraction, the numerator is an improper fraction $\left(\frac{4}{3}\right)$ and the denominator is a compound fraction containing an integer and a proper fraction $\left(6\frac{4}{5}\right)$. This all matches the definition of a complex fraction: nested fractions combining elements of compound, proper, and improper fractions together.
d. The numerator is larger than the denominator. This matches the characteristics of an improper fraction.
e. The numerator is smaller than the denominator. This matches the characteristics of a proper fraction.
f. There are two proper fractions here that are equal to each other. If you were to complete the division, both fractions calculate to 0.75. These are equivalent fractions.

Of the six fractions examined, there are two proper fractions (a and e), one improper fraction (d), one compound fraction (b), one complex fraction (c), and one equivalent fraction (f).

Example 2.2.2: Working with Equivalent Fractions

a. Solve for the unknown term x: $\frac{7}{12}=\frac{49}{x}$
b. Express this fraction in lower terms: $\frac{5}{50}$

Solution

a. Find the value of the unknown term, x.
b. Take the proper fraction and express it in a lower term.

What You Already Know

The needed fractions in a ready-to-solve format are provided.

How You Will Get There

a. Apply the four-step technique to solving equivalent fractions. The first step has already been done for you, in that the equation is already set up.
b. Find a common divisor that divides evenly into both the numerator and denominator. As only 1 and 5 go into the number 5, it makes sense that you should choose 5 to divide into both the numerator and denominator. Note that 5 factors evenly into the denominator, 50, meaning that no remainder or decimals are left over.

Perform

a.

Step 2: You have both of the numerators, so work with that pair.

Step 3: Take the larger number and divide by the smaller number, or $49 \div 7 = 7$. Therefore, multiply the fraction on the left by 7 to get the fraction on the right.

Step 4: Applying the same relationship, $12 \times 7 = 84$.

b. $\frac{5 \div 5}{50 \div 5} = \frac{1}{10}$

Result

a. The unknown denominator on the right is 84, and therefore $\frac{7}{12} = \frac{49}{84}$.
b. In lower terms, $\frac{5}{50}$ is expressed as $\frac{1}{10}$.

Converting to Decimals

Although fractions are common, many people have trouble interpreting them. For example, in comparing $\frac{27}{37}$ to $\frac{57}{73}$, which is the larger number? The solution is not immediately apparent. As well, imagine a retail world where your local Walmart was having a th off sale! It's not that easy to realize that this equates to 15% off. In other words, fractions are converted into decimals by performing the division to make them easier to understand and compare.

How It Works

The rules for converting fractions into decimals are based on the fraction types.

Proper and Improper Fractions

Resolve the division. For example, $\frac{3}{4}$ is the same as $3 \div 4 = 0.75$. As well, $5/4 = 5 \div 4 = 1.25$.

Compound Fractions

The decimal number and the fraction are joined by a hidden addition symbol. Therefore, to convert to a decimal you need to reinsert the addition symbol and apply BEDMAS:

$$3\frac{4}{5} = 3 + 4 \div 5 = 3 + 0.8 = 3.8$$

Complex Fractions

The critical skill here is to reinsert all of the hidden symbols and then apply the rules of BEDMAS:

$$2\frac{11/4}{11/4} = 2 + \left[\frac{(11 \div 4)}{(1 + 1 \div 4)}\right] = 2 + \left[\frac{(11 \div 4)}{(1 + 0.25)}\right] = 2 + \left[\frac{2.75}{1.25}\right] = 2 + 2.2 = 4.2$$

✓ Example 2.2.3: Converting Fractions to Decimals

Convert the following fractions into decimals:

a. $\frac{2}{5}$
b. $6\frac{7}{8}$
c. $12\frac{9/2}{1\frac{2}{10}}$

Solution

Take these fractions and convert them into decimal numbers.

What You Already Know

The three fractions are provided and ready to convert.

How You Will Get There

a. This is a proper fraction requiring you to complete the division.
b. This is a compound fraction requiring you to reinsert the hidden addition symbol and then apply BEDMAS.
c. This is a complex fraction requiring you to reinsert all hidden symbols and apply BEDMAS.

Perform

a. $\frac{2}{5} = 2 \div 5 = 0.4$
b. $6\frac{7}{8} = 6 + 7 \div 8 = 6 + 0.875 = 6.875$
c. $12\frac{9/2}{10} = 12 + \left[\frac{9 \div 2}{1 + 2 \div 10}\right] = 12 + \left[\frac{9 \div 2}{1 + 0.2}\right] = 12 + \left[\frac{4.5}{1.2}\right] = 12 + 3.75 = 15.75$

In decimal format, the fractions have converted to 0.4, 6.875, and 15.75, respectively.

Rounding Principle

Your company needs to take out a loan to cover some short-term debt. The bank has a posted rate of 6.875%. Your bank officer tells you that, for simplicity, she will just round off your interest rate to 6.9%. Is that all right with you? It shouldn't be!

What this example illustrates is the importance of rounding. This is a slightly tricky concept that confuses most students to some degree. In business math, sometimes you should round your calculations off and sometimes you need to retain all of the digits to maintain accuracy.

How It Works

To round a number off, you always look at the number to the right of the digit being rounded. If that number is 5 or higher, you add one to your digit; this is called rounding up. If that number is 4 or less, you leave your digit alone; this is called rounding down.

For example, if you are rounding 8.345 to two decimals, you need to examine the number in the third decimal place (the one to the right). It is a 5, so you add one to the second digit and the number becomes 8.35.

For a second example, let's round 3.6543 to the third decimal place. Therefore, you look at the fourth decimal position, which is a 3. As the rule says, you would leave the digit alone and the number becomes 3.654.

Nonterminating Decimals

What happens when you perform a calculation and the decimal doesn't terminate?

1. You need to assess if there is a pattern in the decimals:

- **The Nonterminating Decimal without a Pattern**: For example, $\frac{6}{17} = 0.352941176...$ with no apparent ending decimal and no pattern to the decimals.
- **The Nonterminating Decimal with a Pattern**: For example, $\frac{2}{11} = 0.18181818...$ endlessly. You can see that the numbers 1 and 8 repeat. A shorthand way of expressing this is to place a horizontal line above the digits that repeat. Thus, you can rewrite

0.18181818 ... as $0.\overline{18}$.

2. You need to know if the number represents an interim or final solution to a problem:

- **Interim Solution**: You must carry forward all of the decimals in your calculations, as the number should not be rounded until you arrive at a final answer. If you are completing the question by hand, write out as many decimals as possible; to save space and time, you can use the shorthand horizontal bar for repeating decimals. If you are completing the question by calculator, store the entire number in a memory cell.
- **Final Solution**: To round this number off, an industry protocol or other clear instruction must apply. If these do not exist, then you would make an arbitrary rounding choice, subject to the condition that you must maintain enough precision to allow for reasonable interpretation of the information.

Important Notes

To assist in your calculations, particularly those that involve multiple steps to resolve, your calculator has 10 memory cells. Your display is limited to 10 digits, but when you store a number into a memory cell the calculator retains all of the decimals associated with the number, not just those displaying on the screen. Your calculator can, in fact, carry up to 13 digit positions. It is strongly recommended that you take advantage of this feature where needed throughout this textbook.

Let's say that you just finished keying in $\frac{6}{17}$ on your calculator, and the resultant number is an interim solution that you need for another step. With 0.352941176 on your display, press STO followed by any numerical digit on the keypad of your calculator. STO stands for store. To store the number into memory cell 1, for example, press STO 1. The number with 13 digits is now in permanent memory. If you clear your calculator (press CE/C) and press RCL # (where # is the memory cell number), you will bring the stored number back. RCL stands for recall. Press RCL 1. The stored number 0.352941176 reappears on the screen.

✓ Example $2.2.4$: Rounding Numbers

Convert the following to decimals. Round each to four decimals or use the repeating decimal notation.

a. $\frac{6}{13}$
b. $\frac{4}{9}$
c. $\frac{4}{11}$
d. $\frac{3}{22}$
e. $5\frac{1/7}{10/27}$

Solution

Convert each of the fractions into decimal format, then round any final answer to four decimals or use repeating decimal notation.

What You Already Know

The fractions and clear instructions on how to round them have been provided.

How You Will Get There

To convert the fractions to decimals, you need to complete the division by obeying the rules of BEDMAS. Watch for hidden symbols and adhere to the rules of rounding.

Perform

a. $\frac{6}{13}=0.461538$. The fifth decimal is a 3, so round down. The answer is 0.4615.
b. $\frac{4}{9}=0.444444$. Note the repeating decimal of 4. Using the horizontal bar, write $0.\overline{4}$.
c. $\frac{4}{11}=0.363636$. Note the repeating decimals of 3 and 6. Using the horizontal bar, write $0.\overline{36}$.
d. $\frac{3}{22}=0.136363$. Note the repeating decimals of 3 and 6 after the 1. Using the horizontal bar, write $0.1\overline{36}$.
e. $5\frac{1/7}{10/27}=5+\frac{(1\div 7)}{(10\div 27)}=5+\frac{0.142857}{0.370}=5+0.385714=5.385714$. Since the fifth digit is a 1, round down. The answer is 5.3857.

> According to the rounding instructions, the solutions are 0.4615, $0.\overline{4}$, $0.\overline{36}$, $0.1\overline{36}$, and 5.3857, respectively.

Rounding Rules

One of the most common sources of difficulties in math is that different people sometimes use different standards for rounding. This seriously interferes with the consistency of final solutions and makes it hard to assess accuracy. So that everyone arrives at the same solution to the exercises/examples in this textbook, these rounding rules apply throughout the book:

1. Never round an interim solution unless there is a logical reason or business process that forces the number to be rounded. Here are some examples of logical reasons or business processes indicating you should round:
   - You withdraw money or transfer it between different bank accounts. In doing so, you can only record two decimals and therefore any money moving between the financial tools must be rounded to two decimals.
   - You need to write the numbers in a financial statement or charge a price for a product. As our currency is in dollars and cents, only two decimals can appear.
2. When you write nonterminating decimals, show only the first six (or up to six) decimals. Use the horizontal line format for repeating decimals. If the number is not a final solution, then assume that all decimals or as many as possible are being carried forward.
3. Round all final numbers to six decimals in decimal format and four decimals in percentage format unless instructions indicate otherwise.
4. Round final solutions according to common business practices, practical limitations, or specific instructions. For example, round any final answer involving dollar currency to two decimals. These types of common business practices and any exceptions are discussed as they arise at various points in this textbook.
5. Generally avoid writing zeros, which are not required at the end of decimals, unless they are required to meet a rounding standard or to visually line up a sequence of numbers. For example, write 6.340 as 6.34 since there is no difference in interpretation through dropping the zero.

Paths To Success

Does your final solution vary from the actual solution by a small amount? Did the question involve multiple steps or calculations to get the final answer? Were lots of decimals or fractions involved? If you answer yes to these questions, the most common source of error lies in rounding. Here are some quick error checks for answers that are "close":

1. Did you remember to obey the rounding rules laid out above? Most importantly, are you carrying decimals for interim solutions and rounding only at final solutions?
2. Did you resolve each fraction or step accurately? Check for incorrect calculations or easy-to-make errors, like transposed numbers.
3. Did you break any rules of BEDMAS?

References

1. Neal, Marcia. Candidate for the 3rd Congressional District Colorado State Board of Education, as quoted in Perez, Gayle. 2008. *"Retired School Teacher Seeks State Board Seat." Pueblo Chieftain.*

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2.3: Percentages (How Does It All Relate?)

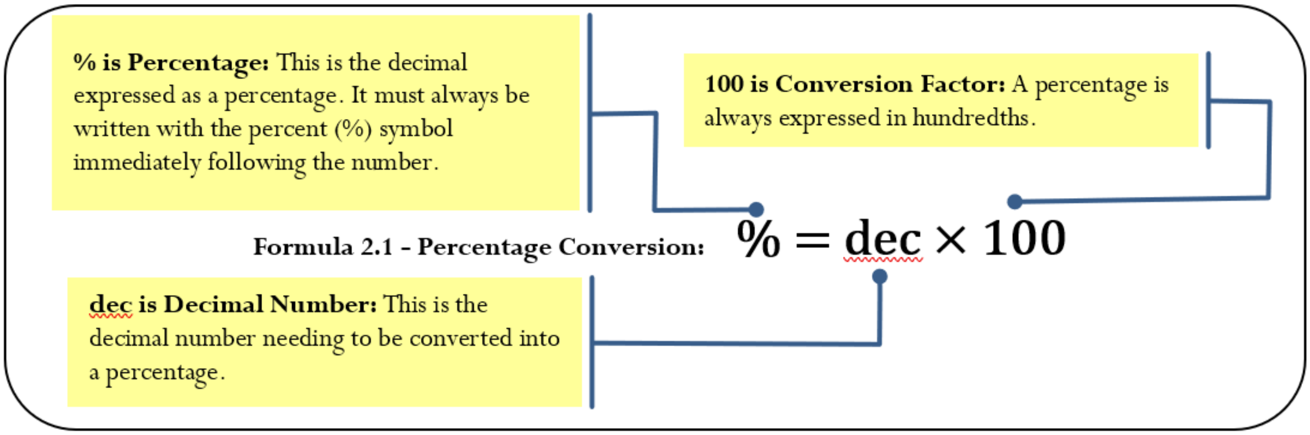
Your class just wrote its first math quiz. You got 13 out of 19 questions correct, or $\frac{13}{19}$. In speaking with your friends Sandhu and Illija, who are in other classes, you find out that they also wrote math quizzes; however, theirs were different. Sandhu scored 16 out of 23, or $\frac{16}{23}$, while Illija got 11 out of 16, or $\frac{11}{16}$. Who achieved the highest grade? Who had the lowest? The answers are not readily apparent, because fractions are difficult to compare.

Now express your grades in percentages rather than fractions. You scored 68%, Sandhu scored 70%, and Illija scored 69%. Notice you can easily answer the questions now. The advantage of percentages is that they facilitate comparison and comprehension.

Converting Decimals to Percentages

A percentage is a part of a whole expressed in hundredths. In other words, it is a value out of 100. For example, 93% means 93 out of 100, or $\frac{93}{100}$.

The Formula



% is Percentage: This is the decimal expressed as a percentage. It must always be written with the percent (%) symbol immediately following the number.

100 is Conversion Factor: A percentage is always expressed in hundredths.

Formula 2.1 - Percentage Conversion: $\% = \text{dec} \times 100$

dec is Decimal Number: This is the decimal number needing to be converted into a percentage.

Formula 2.1

How It Works

Assume you want to convert the decimal number 0.0875 into a percentage. This number represents the dec variable in the formula. Substitute into Formula 2.1:

$$\% = 0.0875 \times 100 = 8.75\%$$

Important Notes

You can also solve this formula for the decimal number. To convert any percentage back into its decimal form, you need to perform a mathematical opposite. Since a percentage is a result of multiplying by 100, the mathematical opposite is achieved by dividing by 100. Therefore, to convert 81% back into decimal form, you take $81\% \div 100 = 0.81$.

Things To Watch Out For

Your Texas Instruments BAII Plus calculator has a % key that can be used to convert any percentage number into its decimal format. For example, if you press 81 and then %, your calculator displays 0.81.

While this function works well when dealing with a single percentage, it causes problems when your math problem involves multiple percentages. For example, try keying 4% + 3% = into the calculator using the % key. This should be the same as $0.04 + 0.03 = 0.07$. Notice, however, that your calculator has 0.0412 on the display.

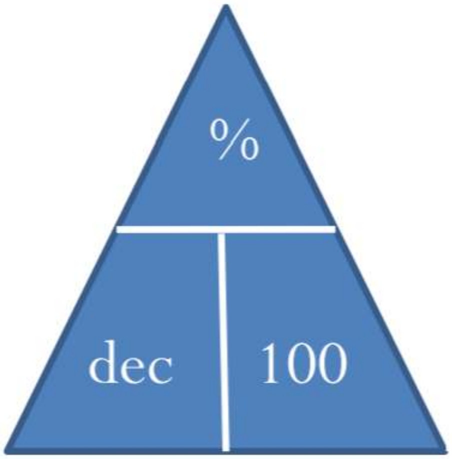
Why is this? As a business calculator, you BAII Plus is programmed to take portions of a whole. When you key 3% into the calculator, it takes 3% of the first number keyed in, which was 4%. As a formula, the calculator sees $4\% + (3\% \times 4\%)$. This works out to $0.04 + 0.0012 = 0.0412$.

To prevent this from happening, your best course of action is not to use the % key on your calculator. It is best to key all percentages as decimal numbers whenever possible, thus eliminating any chance that the % key takes a portion of your whole. Throughout this textbook, all percentages are converted to decimals before calculations take place.

Paths To Success

When working with percentages, you can use some tricks for remembering the formula and moving the decimal point.

Remembering the Formula. When an equation involves only multiplication of all terms on one side with an isolated solution on the other side, use a mnemonic called the triangle technique. In this technique, draw a triangle with a horizontal line through its middle. Above the line goes the solution, and below the line, separated by vertical lines, goes each of the terms involved in the multiplication. The figure to the right shows how Formula 2.1 would be drawn using the triangle technique.



To use this triangle, identify the unknown variable, which you then calculate from the other variables in the triangle:

1. Anything on the same line gets multiplied together. If solving for %, then the other variables are on the same line and multiplied as $dec \times 100$.
2. Any pair of items with one above the other is treated like a fraction and divided. If solving for dec, then the other variables are above/below each other and are divided as $\frac{\%}{100}$.

Moving the Decimal. Another easy way to work with percentages is to remember that multiplying or dividing by 100 moves the decimal over two places.

1. If you are multiplying by 100, the decimal position moves two positions to the right.

$$0.73 \times 100 = 0.\underbrace{\overrightarrow{7}}_{\text{2 positions}}\underbrace{\overrightarrow{3}}_{\text{to the right}} = 73\%$$

2. If you are dividing by 100, the decimal position moves two positions to the left.

$$73\% \div 100 = \underbrace{\overleftarrow{7}}_{\text{2 positions}}\underbrace{\overleftarrow{3}}_{\text{to the left}}\cdot = 0.73$$

✓ Example $2.3.1$: Working with Percentages

Convert (a) and (b) into percentages. Convert (c) back into decimal format.

a. $\frac{3}{8}$
b. 1.3187
c. 12.399%

Solution

For questions (a) and (b), you need to convert these into percentage format. For question (c), you need to convert it back to decimal format.

What You Already Know

a. This is a fraction to be converted into a decimal, or dec.
b. This is dec.
c. This is %.

How You Will Get There

a. Convert the fraction into a decimal to have dec. Then apply Formula 2.1: $\% = dec \times 100$ to get the percentage.
b. As this term is already in decimal format, apply Formula 2.1: $\% = dec \times 100$ to get the percentage.
c. This term is already in percentage format. Using the triangle technique, calculate the decimal number through $dec = \frac{\%}{100}$.

Perform

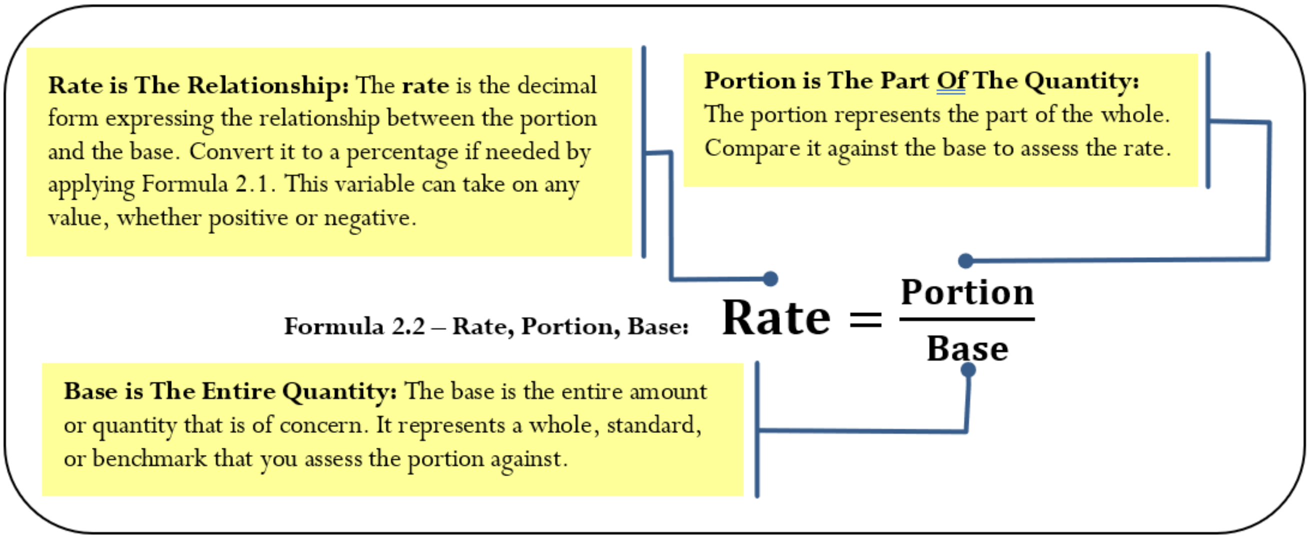
a. $\frac{3}{8} = 3 \div 8 = 0.375 \ \% = 0.375 \times 100 \ \% = 37.5\%$
b. $\% = 1.3187 \times 100 \ \% = 131.87\%$
c. $dec = \frac{12.399\%}{100} = 0.12399$

In percentage format, the first two numbers are 37.5% and 131.87%. In decimal format, the last number is 0.12399.

Rate, Portion, Base

In your personal life and career, you will often need to either calculate or compare various quantities involving fractions. For example, if your income is $3,000 per month and you can't spend more than 30% on housing, what is your maximum housing dollar amount? Or perhaps your manager tells you that this year's sales of $1,487,003 are 102% of last year's sales. What were your sales last year?

The Formula



Rate is The Relationship: The **rate** is the decimal form expressing the relationship between the portion and the base. Convert it to a percentage if needed by applying Formula 2.1. This variable can take on any value, whether positive or negative.

Portion is The Part Of The Quantity: The portion represents the part of the whole. Compare it against the base to assess the rate.

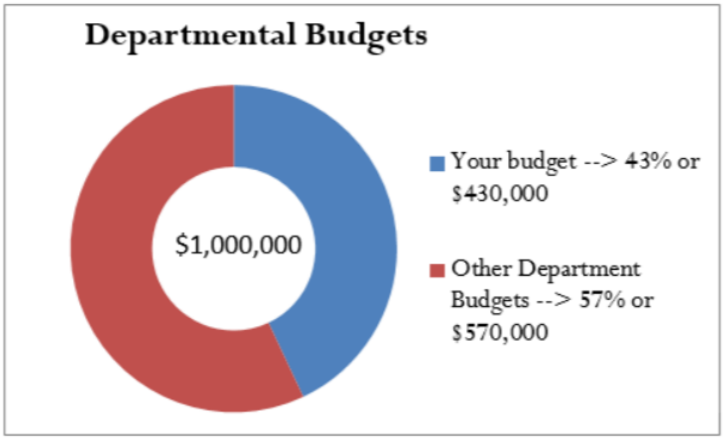
Formula 2.2 – Rate, Portion, Base: $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$

Base is The Entire Quantity: The base is the entire amount or quantity that is of concern. It represents a whole, standard, or benchmark that you assess the portion against.

Formula 2.2

How It Works

Assume that your company has set a budget of $1,000,000. This is the entire amount of the budget and represents your base. Your department gets $430,000 of the budget—this is your department's part of the whole and represents the portion. You want to know the relationship between your budget and the company's budget. In other words, you are looking for the rate.



- Apply Formula 2.2, where rate $= \frac{\$430,000}{\$1,000,000}$.
- Your budget is 0.43, or 43%, of the company's budget.

Important Notes

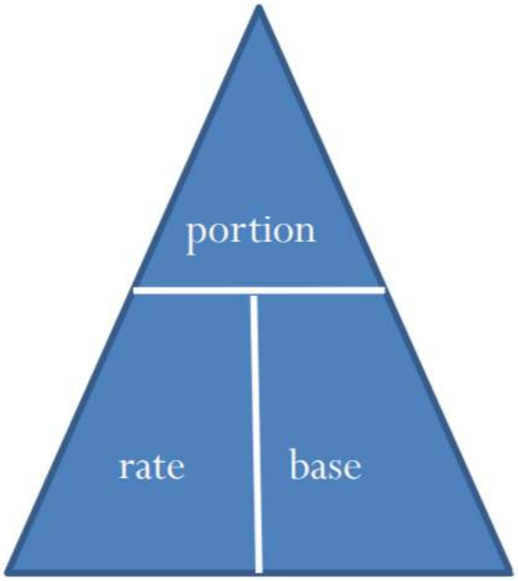
There are three parts to this formula. Mistakes commonly occur through incorrect assignment of a quantity to its associated variable. The table below provides some tips and clue words to help you make the correct assignment.

Variable	Key Words	Example
Base	of	If your department can spend 43% of the company's total budget of $1,000,000, what is your maximum departmental spending?
Portion	is, are	If your department can spend 43% of the company's total budget of $1,000,000, what is your maximum departmental spending?
Rate	%, percent, rate	If your department can spend 43% of the company's total budget of $1,000,000, what is your maximum departmental spending?

Things To Watch Out For

In resolving the rate, you must express all numbers in the same units—you cannot have apples and oranges in the same sequence of calculations. In the above example, both the company's budget and the departmental budget are in the units of dollars. Alternatively, you would not be able to calculate the rate if you had a base expressed in kilometres and a portion expressed in metres. Before you perform the rate calculation, express both in kilometres or both in metres.

Paths To Success



Formula 2.2 is another formula you can use the triangle technique for. You do not need to memorize multiple versions of the formula for each of the variables. The triangle appears to the right.

Be very careful when performing operations involving rates, particularly in summing or averaging rates.

Summing Rates

Summing rates requires each rate to be a part of the same whole or base. If Bob has 5% of the kilometres traveled and Sheila has 6% of the oranges, these are not part of the same whole and cannot be added. If you did, what does the 11% represent? The result has no interpretation. However, if there are 100 oranges of which Bob has 5% and Sheila has 6%, the rates can be added and you can say that in total they have 11% of the oranges.

Averaging Rates

Simple averaging of rates requires each rate to be a measure of the same variable with the same base. If 36% of your customers are female and 54% have high income, the average of 45% is meaningless since each rate measures a different variable. Recall that earlier in this chapter you achieved 68% on your test and Sandhu scored 70%. However, your test involved 19 questions and Sandhu's involved 23 questions. These rates also cannot be simply averaged to 69% on the reasoning that (68% + 70%) / 2 = 69%, since the bases are not the same. When two variables measure the same characteristic but have different bases (such as the math quizzes), you must use a weighted-average technique, which will be discussed in Section 3.1.

When can you average rates? Hypothetically, assume Sandhu achieved his 70% by writing the same test with 19 questions. Since both rates measure the same variable and have the same base, the simple average of 69% is now calculable.

? Exercise $2.3.1$: Give It Some Thought

Consider the following situations and select the best answer without performing any calculations.

1. If the rate is 0.25%, in comparison to the base the portion is
   a. just a little bit smaller than the base.
   b. a lot smaller than the base.
   c. just a little bit bigger than the base.
   d. a lot bigger than the base.
2. If the portion is $44,931 and the base is $30,000, the rate is
   a. smaller than 100%.
   b. equal to 100%.
   c. larger than 100% but less than 200%.
   d. larger than 200%.

3. If the rate is 75% and the portion is $50,000, the base is
    a. smaller than $50,000.
    b. larger than $50,000.
    c. the same as the portion and equal to $50,000.

Answer

1. b (0.25% is 0.0025, resulting in a very small portion)
2. c (the portion is larger than the base, but not twice as large)
3. b (the portion represents 75% of the base, meaning the base must be larger)

✓ Example 2.3.2: Rate, Portion, Base

Solve for the unknown in the following three scenarios.

1. If your total income is $3,000 per month and you can't spend more than 30% on housing, what is the maximum amount of your total income that can be spent on housing?
2. Your manager tells you that 2014 sales are 102% of 2013 sales. The sales for 2014 are $1,487,003. What were the sales in 2013?
3. In Calgary, total commercial real estate sales in the first quarter of 2008 were $1.28 billion. The industrial, commercial, and institutional (ICI) land sector in Calgary had sales of $409.6 million. What percentage of commercial real estate sales is accounted for by the ICI land sector?

Solution

1. You are looking for the maximum amount of your income that can be spent on housing.
2. You need to figure out the sales for 2013.
3. You must determine the percentage of commercial real estate sales accounted for by the ICI land sector in Calgary.

What You Already Know

1. Look for key words in the question: "what is the maximum amount" and "of your total income." The total income is the base, and the maximum amount is the portion.
    - Base = $3,000 Rate = 30% Portion = maximum amount
2. Look for key words in the question: "sales for 2014 are $1,487,003" and "of 2013 sales." The 2014 sales is the portion, and the 2013 sales is the base.
    - Portion = $1,487,003 Rate = 102% Base = 2013 sales
3. Look for key words in the question: "of commercial real estate sales" and "are accounted for by the ICI land sector." The commercial real estate sales are the base, and the ICI land sector sales are the portion.
    - Base=$1.28 billion Portion=$409.6 million Rate=percentage

How You Will Get There

1. Apply Formula 2.2, but rearrange using the triangle technique to have Portion = Rate × Base.
2. Apply Formula 2.2, but rearrange using the triangle technique to have Base=Portion/Rate.
3. Apply Formula 2.2, Rate=Portion/Base.

Perform

$$\begin{aligned} &1.\ \text{Portion} &&= 30\% \times \$3,000 = 0.3 \times \$3,000 = \$900 \\ &2.\ \text{Base} &&= \frac{\$1,487,003}{102\%} = \frac{\$1,487,003}{1.02} = \$1,457,846.08 \\ &3.\ \text{Rate} &&= \frac{\$409.6 \text{ million}}{\$1.28 \text{ billion}} = \frac{\$409,600,000}{\$1,280,000,000} = 0.32 \text{ or } 32\% \end{aligned}$$

Result

1. The maximum you can spend on housing is $900 per month.
2. 2013 sales were $1,457,846.08.
3. The ICI land sector accounted for 32% of commercial real estate sales in Calgary for the first quarter of 2008.

Contributors and Attributions

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2.4: Algebraic Expressions (The Pieces Of The Puzzle)

If you are like most Canadians, your employer pays you biweekly. Assume you earn $12.00 per hour. How do you calculate your pay cheque every pay period? Your earnings are calculated as follows:

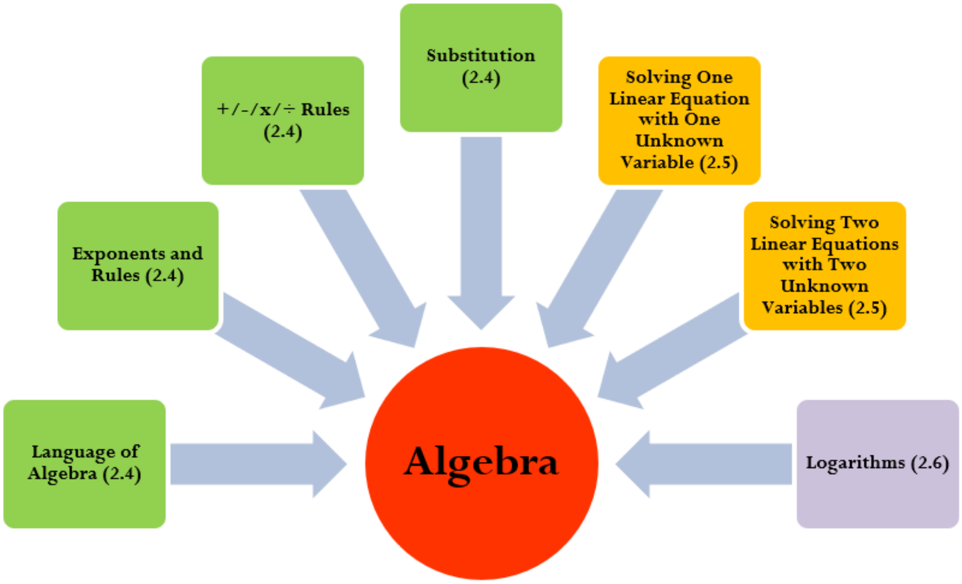
$$\$12.00 \times (\text{Hours worked during the biweekly pay period})$$

The hours worked during the biweekly pay period is the unknown variable. Notice that the expression appears lengthy when you write out the explanation for the variable. Algebra is a way of making such expressions more convenient to manipulate. To shorten the expression, making it easier to read, algebra assigns a letter or group of letters to represent the variable. In this case, you might choose h to represent "hours worked during the biweekly pay period." This rewrites the above expression as follows:

$$\$12.00 \times h \text{ or } \$12h$$

Unfortunately, the word algebra makes many people's eyes glaze over. But remember that algebra is just a way of solving a numerical problem. It demonstrates how the pieces of a puzzle fit together to arrive at a solution.

For example, you have used your algebraic skills if you have ever programmed a formula into Microsoft Excel. You told Excel there was a relationship between cells in your spreadsheet. Perhaps your calculation required cell A3 to be divided by cell B6 and then multiplied by cell F2. This is an algebraic equation. Excel then took your algebraic equation and calculated a solution by automatically substituting in the appropriate values from the referenced cells (your variables).


Substitution (2.4)
+/-/x/÷ Rules (2.4)
Solving One Linear Equation with One Unknown Variable (2.5)
Exponents and Rules (2.4)
Solving Two Linear Equations with Two Unknown Variables (2.5)
Language of Algebra (2.4)
Algebra
Logarithms (2.6)

As illustrated in the figure above, algebra involves integrating many interrelated concepts. This figure shows only the concepts that are important to business mathematics, which this textbook will present piece by piece. Your understanding of algebra will become more complete as more concepts are covered over the course of this book.

This section reviews the language of algebra, exponent rules, basic operation rules, and substitution. In Section 2.5 you will put these concepts to work in solving one linear equation for one unknown variable along with two linear equations with two unknown variables. Finally, in Section 2.6 you will explore the concepts of logarithms and natural logarithms.

The Language of Algebra

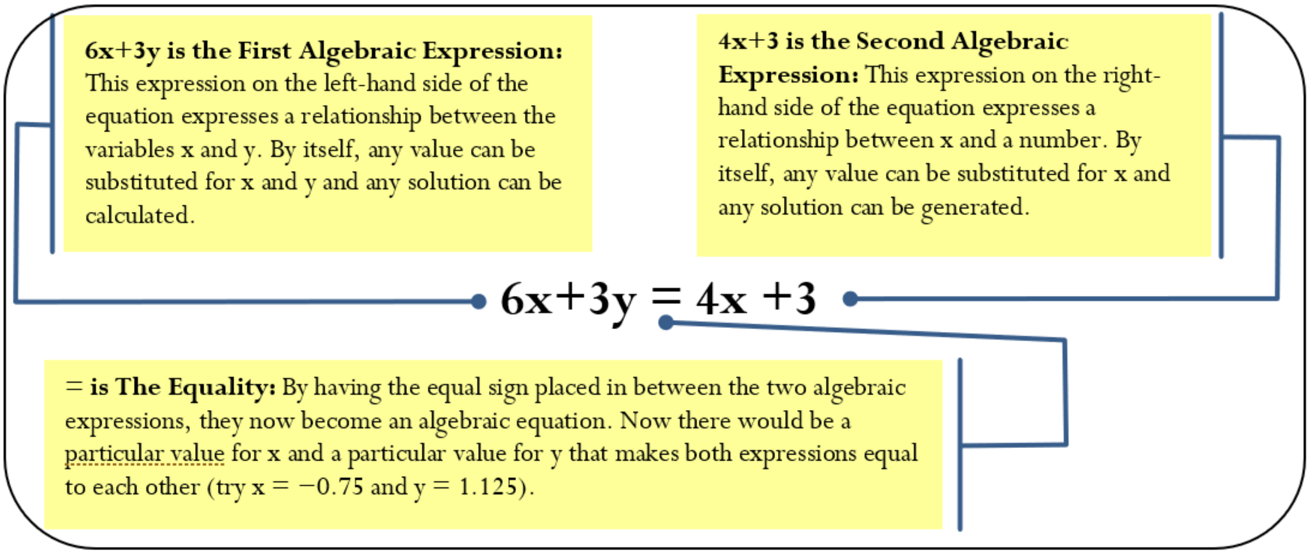
Understanding the rules of algebra requires familiarity with four key definitions.

Algebraic Expression

A mathematical algebraic expression indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables. For example, the expression $12h says that you must take the hourly wage of $12 and multiply it by the hours worked. Note that the expression does not include an equal sign, or "=". It only tells you what to do and requires that you substitute a value in for the unknown variable(s) to solve. There is no one definable solution to the expression.

Algebraic Equation

A mathematical algebraic equation takes two algebraic expressions and equates them. This equation can be solved to find a solution for the unknown variables. Examine the following illustration to see how algebraic expressions and algebraic equations are interrelated.



6x+3y is the First Algebraic Expression: This expression on the left-hand side of the equation expresses a relationship between the variables x and y. By itself, any value can be substituted for x and y and any solution can be calculated.

4x+3 is the Second Algebraic Expression: This expression on the right-hand side of the equation expresses a relationship between x and a number. By itself, any value can be substituted for x and any solution can be generated.

$$6x+3y = 4x+3$$

= is The Equality: By having the equal sign placed in between the two algebraic expressions, they now become an algebraic equation. Now there would be a particular value for x and a particular value for y that makes both expressions equal to each other (try x = −0.75 and y = 1.125).

Term

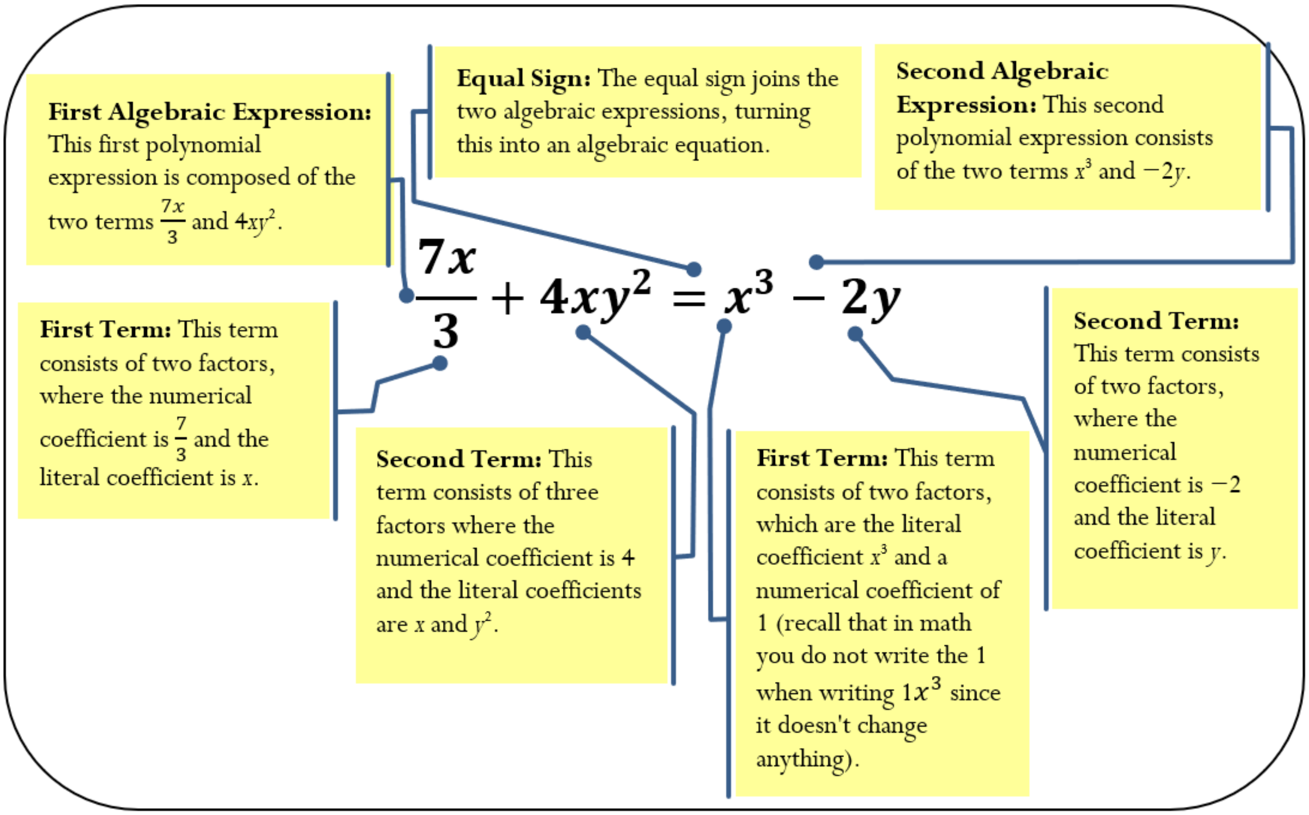
In any algebraic expression, terms are the components that are separated by addition and subtraction. In looking at the example above, the expression $6x+3y$ is composed of two terms. These terms are “$6x$” and “$3y$.” A nomial refers to how many terms appear in an algebraic expression. If an algebraic expression contains only one term, like “$\$12.00h$,” it is called a monomial. If the expression contains two terms or more, such as “$6x+3y$,” it is called a polynomial.

Factor

Terms may consist of one or more factors that are separated by multiplication or division signs. Using the 6x from above, it consists of two factors. These factors are “6” and “x”; they are joined by multiplication.

- If the factor is numerical, it is called the numerical coefficient.
- If the factor is one or more variables, it is called the literal coefficient.

The next graphic shows how algebraic expressions, algebraic equations, terms, and factors all interrelate within an equation.



$$\frac{7x}{3}+4xy^2=x^3-2y$$

First Algebraic Expression: This first polynomial expression is composed of the two terms $\frac{7x}{3}$ and $4xy^2$.

Equal Sign: The equal sign joins the two algebraic expressions, turning this into an algebraic equation.

Second Algebraic Expression: This second polynomial expression consists of the two terms x^3 and $-2y$.

First Term: This term consists of two factors, where the numerical coefficient is $\frac{7}{3}$ and the literal coefficient is x.

Second Term: This term consists of three factors where the numerical coefficient is 4 and the literal coefficients are x and y^2.

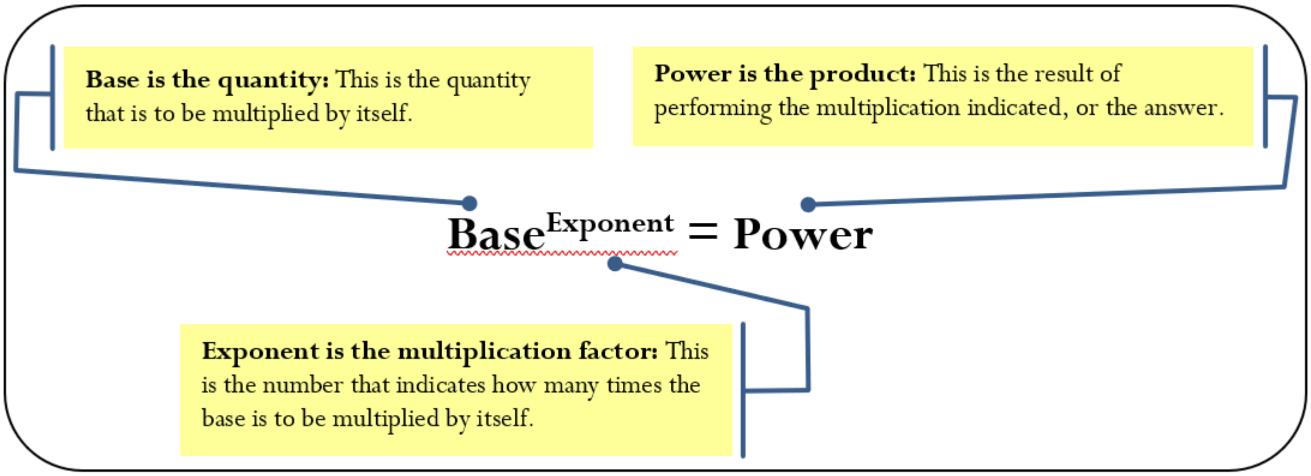
First Term: This term consists of two factors, which are the literal coefficient x^3 and a numerical coefficient of 1 (recall that in math you do not write the 1 when writing $1x^3$ since it doesn't change anything).

Second Term: This term consists of two factors, where the numerical coefficient is -2 and the literal coefficient is y.

Exponents

Exponents are widely used in business mathematics and are integral to financial mathematics. When applying compounding interest rates to any investment or loan, you must use exponents (see Chapter 9 and onwards).

Exponents are a mathematical shorthand notation that indicates how many times a quantity is multiplied by itself. The format of an exponent is illustrated below.



$$\text{Base}^{\text{Exponent}}=\text{Power}$$

Base is the quantity: This is the quantity that is to be multiplied by itself.

Power is the product: This is the result of performing the multiplication indicated, or the answer.

Exponent is the multiplication factor: This is the number that indicates how many times the base is to be multiplied by itself.

Assume you have $2^3=8$. The exponent of 3 says to take the base of 2 multiplied by itself three times, or $2\times2\times2$. The power is 8. The proper way to state this expression is "2 to the exponent of 3 results in a power of 8."

How It Works

Many rules apply to the simplification of exponents, as shown in the table below.

Rule	Illustration	Explanation

Rule	Illustration	Explanation
1. Multiplication	$y^a \times y^b = y^{a+b}$	If the bases are identical, add the exponents and retain the unchanged base.
2. Division	$\frac{y^a}{y^b} = y^{a-b}$	If the bases are identical, subtract the exponents and retain the unchanged base.
3. Raising powers to exponents	$(y^b z^c)^a = y^{b\times a} z^{c\times a}$ or $\left(\frac{y^b}{z^c}\right)^a = \frac{y^{b\times a}}{z^{c\times a}}$	If a single term is raised to an exponent, each factor must retain its base and you must multiply the exponents for each by the raised exponent. Note that if the expression inside the brackets has more than one term, such as $(y^b + z^c)^2$, which has two terms, you cannot multiply the exponent a inside the brackets.
4. Zero exponents	$y^0 = 1$	Any base to a zero exponent will always produce a power of 1. This is explained later in this section when you review the concept of algebraic division.
5. Negative exponents	$y^{-a} = \frac{1}{y^a}$	The negative sign indicates that the power has been shifted between the numerator and denominator. It is commonly used in this textbook to simplify appearances. Note that on the BAII Plus calculator, keying in a negative exponent requires you to key in the value of y, press y^x, enter the value of a, press $\pm$, and then press = to calculate.
6. Fractional exponents	$y^{\frac{a}{b}} = \sqrt[b]{y^a}$	A fractional exponent is a different way of writing a radical sign. Note that the base is first taken to the exponent of a, then the root of b is found to get the power. For example, This is the same as . To key in a fractional exponent on the BAII Plus calculator, input the value of y, press y^x, open a set of brackets, key in $a \div b$, close the brackets, and press = to calculate.

Important Notes

Recall that mathematicians do not normally write the number 1 when it is multiplied by another factor because it doesn't change the result. The same applies to exponents. If the exponent is a 1, it is generally not written because any number multiplied by itself only once is the same number. For example, the number 2 could be written as 21, but the power is still 2. Or take the case of $(yz)^2$. This could be written as $(y^1 z^1)^a$, which when simplified becomes $y^{1\times a} z^{1\times a}$ or $y^a z^a$. Thus, even if you don't see an exponent written, you know that the value is 1.

✓ Example 2.4.1: Exponents in Algebra

Simplify the following expressions:

a. $h^3 \times h^6$
b. $\frac{h^{14}}{h^8}$
c. $\left[\frac{hk^5m^3}{n^4}\right]^3$

d. 1.49268^0
e. $\frac{x^2y^4}{xy^{-2}}$
f. $6^{3/5}$

Solution

You have been asked to simplify the expressions. Notice that expressions (d) and (f) contain only numerical coefficients and therefore can be resolved numerically. All of the other expressions involve literal coefficients and require algebraic skills to simplify.

What You Already Know

You have been provided with the expressions, and you have six rules for simplifying exponents in algebra.

How You Will Get There

a. This expression involves multiplying two powers with the same base. Apply Rule #1.
b. This expression involves dividing two powers with the same base. Apply Rule #2.
c. This expression involves a single term, with products and a quotient all raised to an exponent. Apply Rule #3.
d. This power involves a zero exponent. Apply Rule #4.
e. This expression involves multiplication, division, and negative exponents. Apply Rules #1, #2, and #5.
f. This power involves a fractional exponent. Apply Rule #6.

Perform

a. $h^3 \times h^6 = h^{3+6} = h^9$
b. $\frac{h^{14}}{h^8} = h^{14-8} = h^6$
c. $\left[\frac{hk^5m^3}{n^4}\right]^3 = \frac{h^{1\times3}k^{5\times3}m^{3\times3}}{n^{4\times3}} = \frac{h^3k^{15}m^9}{n^{12}}$
d. $1.49268^0 = 1$
e. $\frac{x^2y^4}{xy^{-2}} = \frac{x^2y^4y^2}{x^1}(\text{ Rule \#5}) = \frac{x^2y^{4+2}}{x^1}(\text{ Rule \#1}) = \frac{x^2y^6}{x^1} = x^{2-1}y^6(\text{ Rule \#2}) = xy^6$
f. $6^{3/5} = 2.930156$

Calculator Instructions

d. $1.49268y^x0 =$

f. $6y^x(3 \div 5) =$

Here are the simplified solutions:

a. h^9
b. h^6
c. $\frac{h^3k^{15}m^9}{n^{12}}$
d. 1
e. xy^6
f. 2.930156

Addition and Subtraction

Simplification of unnecessarily long or complex algebraic expressions is always preferable to increase understanding and reduce the chances of error. For example, assume you are a production manager looking to order bolts for a product that you make. Your company makes three products, all in equal quantity. Product A requires seven bolts, Product B requires four bolts, and Product C requires fourteen bolts. If q represents the quantity of products required, you need to order $7q + 4q + 14q$ bolts. This expression requires four calculations to solve every time (each term needs to be multiplied by q and you then need to add everything together). With the algebra rules that follow, you can simplify this expression to $25q$. This requires only one calculation to solve. So what are the rules?

How It Works

In math, terms with the same literal coefficients are called like terms. Only terms with the identical literal coefficients may be added or subtracted through the following procedure:

Step 1: Simplify any numerical coefficients by performing any needed mathematical operation or converting fractions to decimals. For example, terms such as $\frac{1}{2}y$ should become $0.5y$.

Step 2: Add or subtract the numerical coefficients of like terms as indicated by the operation while obeying the rules of BEDMAS.

Step 3: Retain and do not change the common literal coefficients. Write the new numerical coefficient in front of the retained literal coefficients.

From the previous example, you require $7q+4q+14q$ bolts. Notice that there are three terms, each of which has the same literal coefficient. Therefore, you can perform the required addition.

Step 1: All numerical coefficients are simplified already. Skip to step 2.

Step 2: Take the numerical coefficients and add the numbers: $7+4+14$ equals 25.

Step 3: Retain the literal coefficient of q. Put the new numerical coefficient and literal coefficient together. Thus, $25q$. Therefore $7q+4q+14q$ is the same as $25q$.

Things To Watch Out For

A common mistake in addition and subtraction is combining terms that do not have the same literal coefficient. You need to remember that the literal coefficient must be identical. For example, $7q$ and $4q$ have the identical literal coefficient of q. However, $7q$ and $4q^2$ have different literal coefficients, q and q^2, and cannot be added or subtracted.

Paths To Success

Remember that if you come across a literal coefficient with no number in front of it, that number is assumed to be a 1. For example, x has no written numerical coefficient, but it is the same as $1x$. Another example would be $\frac{x}{4}$ is the same as $\frac{1x}{4}$ or $\frac{1}{4}x$.

On a similar note, mathematicians also don't write out literal coefficients that have an exponent of zero. For example, $7x^0$ is just $7(1)$ or 7. Thus, the literal coefficient is always there; however, it has an exponent of zero. Remembering this will help you later when you multiply and divide in algebra.

Exercise 2.4.1: Give It Some Thought

Examine the following algebraic expressions and indicate how many terms can be combined through addition and subtraction. No calculations are necessary. Do not attempt to simplify.

a. $\frac{3}{2}x+4x^2-10x-2y+\frac{x}{3}$
b. $23g^2-\frac{17g^2}{5}+g^4+g^2-\frac{2}{3}g^2-0.15g+g^3$

Answer

a. Three terms (all of the x)
b. Four terms (all of the g^2)

Example 2.4.2: Addition and Subtraction in Algebra

Simplify the following three algebraic expressions.

a. $9x+3y-\frac{7}{2}x+4y$
b. $P\left(1+0.11\times\frac{121}{365}\right)+\frac{15P}{1+0.11\times\frac{36}{365}}$
c. $x\left(1+\frac{0.1}{4}\right)^3+\frac{x}{\left(1+\frac{0.1}{4}\right)^4}-\frac{3x}{\left(1+\frac{0.1}{4}\right)^2}$

Solution

You have been asked to simplify the three algebraic expressions. Notice that each term of the expressions is joined to the others by addition or subtraction. Therefore, apply the addition and subtraction rules to each expression.

What You Already Know

The three algebraic expressions have already been supplied.

How You Will Get There

Applying the three steps for addition and subtraction you should simplify, combine, and write out the solution.

Perform

a.

$9x+3y-\frac{7}{2}x+4y$	**Step 1**: Simplify the numerical coefficients.
$9x+3y-\boldsymbol{3.5x}+4y$	**Step 2**: Combine the numerical coefficients of like terms. You have two terms with x from which you need to take 9-3.5=5.5. You also have two terms with y from which you need to take 3+4=7
$\boldsymbol{5.5x+7y}$	**Step 3**: Write the numerical coefficients in front of the unchanged literal coefficients.

b.

$P\left(1+0.11\times\frac{121}{365}\right)+\frac{15P}{1+0.11\times\frac{36}{365}}$	**Step 1**: Simplify the numerical coefficients.
$\boldsymbol{1.036465P+\frac{15}{1.062383}P}$	**Step 1**: Continue to simplify the second term.
$1.036465P+\boldsymbol{14.119194P}$	**Step 2**: Combine the numerical coefficients by performing the addition.
$\boldsymbol{15.15566P}$	**Step 3**: Write the numerical coefficient in front of the unchanged literal coefficient.

c.

$x\left(1+\frac{0.1}{4}\right)^3+\frac{x}{\left(1+\frac{0.1}{4}\right)^4}-\frac{3x}{\left(1+\frac{0.1}{4}\right)^2}$	**Step 1**: Simplify the numerical coefficients.
$\boldsymbol{1.076890x+\frac{1}{1.103812}x-\frac{3}{1.050625}x}$	**Step 1**: Continue to simplify the second and third terms.
$\boldsymbol{1.076890x+0.905950x-2.855443x}$	**Step 2**: Combine the numerical coefficients through the specified operations.
$\boldsymbol{-0.872602x}$	**Step 3**: Write the numerical coefficient in front of the unchanged literal coefficient.

The algebraic expressions simplify as follows:

a. $5.5x+7y$
b. $15.15566P$
c. $-0.872602x$

Notice how much easier these are to work with than the original expressions.

Multiplication

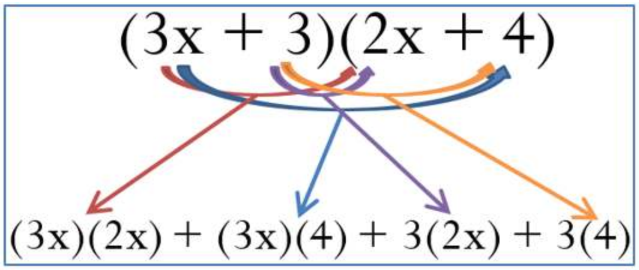
Whether you are multiplying a monomial by another monomial, a monomial by a polynomial, or a polynomial by another polynomial, the rules for multiplication remain the same.

How It Works

Follow these steps to multiple algebraic expressions:

Step 1: Check to see if there is any way to simplify the algebraic expression first. Are there any like terms you can combine? For example, you can simplify $(3x+2+1)(x+x+4)$ to $(3x+3)(2x+4)$ before attempting the multiplication.

Step 2: Take every term in the first algebraic expression and multiply it by every term in the second algebraic expression. This means that numerical coefficients in both terms are multiplied by each other, and the literal coefficients in both terms are multiplied by each other. It is best to work methodically from left to right so that you do not miss anything. Working with the example, in $(3x+3)(2x+4)$ take the first term of the first expression, $3x$, and multiply it by $2x$ and then by 4. Then move to the second term of the first expression, 3, and multiply it by $2x$ and then by 4 (see the figure).



It becomes: $6x^2+12x+6x+12$

Step 3: Perform any final steps of simplification by adding or subtracting like terms as needed. In the example, two terms contain the literal coefficient x, so you simplify the expression to $6x^2+18x+12$.

Important Notes

If the multiplication involves more than two expressions being multiplied by each other, it is easiest to work with only one pair of expressions at a time starting with the leftmost pair. For example, if you are multiplying $(4x+3)(3x)(9y+5x)$, resolve $(4x+3)(3x)$ first. Then take the solution, keeping it in brackets since you have not completed the math operation, and multiply it by $(9y+5x)$. This means you are required to repeat step 2 in the multiplication procedure until you have resolved all the multiplications.

Things To Watch Out For

The negative sign causes no end of grief for a lot of people when working with multiplication. First, if a numerical coefficient is not written explicitly, it is assumed to be 1. For example, look at $2(4a+6b)-(2a-3b)$. This is the same as $2(4a+6b)+(-1)(2a-3b)$.

When you multiply a negative through an expression, all signs in the brackets will change. Continuing with the second term in the above example, $-(2a-3b)$ becomes $-2a+3b$. The expression then looks like $2(4a+6b)-2a+3b$.

Paths To Success

The order in which you write the terms of an algebraic expression does not matter so long as you follow all the rules of BEDMAS. For example, whether you write 3×4 or 4×3, the answer is the same because you can do multiplication in any order. The same applies to $4+3-1$ or $3-1+4$. Now let’s get more complex. Whether you write $3x^2+5x-4$ or $5x-4+3x^2$, the answer is the same as you have not violated any rules of BEDMAS. You still multiply first and add last from left to right.

Although the descending exponential format for writing expressions is generally preferred, for example, $3x^2+5x+4$, which lists literal coefficients with higher exponents first, it does not matter if you do this or not. When checking your solutions against those given in this textbook, you only need to ensure that each of your terms matches the terms in the solution provided.

✓ Example 2.4.3: Multiplication in Algebra

Simplify the following algebraic expression: $(6x+2+2)(3x–2)$

Solution

You have been asked to simplify the expression.

What You Already Know

You are multiplying two expressions with each other. Each expression contains two or three terms.

How You Will Get There

Applying the three steps for multiplication of algebraic expressions, you simplify, multiply each term, and combine.

Perform

Step 1: Simplify the expression first.

$$\begin{gathered}(6x+2+2)(3x-2)\\(6x+4)\mathbf{(3x-2)}\end{gathered}$$

Step 2: Multiply all terms in each expression with all terms in the other expression.

$$\mathbf{(6x)(3x)+(6x)(-2)+(4)(3x)+(4)(-2)}$$

Step 2 (continued): Resolve the multiplications.

$$18x^2-12x+12x-8$$

Step 3: Perform the final simplifications. You can combine the middle two terms.

$$18x^2-8$$

This is the final solution.

The simplified algebraic expression is $18x^2-8$.

✓ Example 2.4.4: More Challenging Algebraic Multiplication

Simplify the following algebraic expression: $–(3ab)(a^2+4b–2a)–4(3a+6)$

Solution

You have been asked to simplify the expression.

What You Already Know

You have been provided with the expression. Notice that the first term consists of three expressions being multiplied together. The second term involves two expressions being multiplied together. Apply the rules of multiplication.

How You Will Get There

Applying the three steps for multiplication of algebraic expressions, you will simplify, multiply each term, and combine.

Perform

Step 1: You cannot simplify anything. Be careful with the negatives and write them out.

$$-(3ab)\left(a^2+4b-2a\right)-4(3a+6)$$

Step 2: Work with the first pair of expressions in the first term and multiply.

$$\mathbf{(-1)(3ab)\left(a^2+4b-2a\right)+(-4)(3a+6)}$$

Multiply the resulting pair of expressions in the first term.

$$\mathbf{(-3ab)\left(a^2+4b-2a\right)+(-4)(3a+6)}$$

Work with the pair of expressions in the original second term and multiply.

$$(\mathbf{-3a^3b - 12ab^2 + 6a^2b}) + (\mathbf{-4})(\mathbf{3a+6})$$

Step 3: Drop the brackets to simplify.

$$-3a^3b - 12ab^2 + 6a^2b + [\mathbf{-12a - 24}]$$

There are no like terms. You cannot simplify this expression any further.

$$-3a^3b - 12ab^2 + 6a^2b - 12a - 24$$

The simplified algebraic expression is $-3a^3b - 12ab^2 + 6a^2b - 12a - 24$.

Division

You are often required to divide a monomial into either a monomial or a polynomial. In instances where the denominator consists of a polynomial, it is either not possible or extremely difficult to simplify the expression algebraically. Only division where denominators are monomials are discussed here.

How It Works

To simplify an expression when its denominator is a monomial, apply these rules:

Step 1: Just as in multiplication, determine if there is any way to combine like terms before completing the division. For example, with $\frac{3ab + 3ab - 3a^2b + 9ab^2}{3ab}$ you can simplify the numerator to $\frac{6ab - 3a^2b + 9ab^2}{3ab}$.

Step 2: Take every term in the numerator and divide it by the term in the denominator. This means that you must divide both the numerical and the literal coefficients. As with multiplication, it is usually best to work methodically from left to right so that you do not miss anything. So in our example we get:

$$\frac{6ab}{3ab} - \frac{3a^2b}{3ab} + \frac{9ab^2}{3ab}$$

$$(2)(1)(1) - (1)(a)(1) + (3)(1)(b)$$

$$2 - a + 3b$$

Step 3: Perform any final simplification by adding or subtracting the like terms as needed. As there are no more like terms, the final expression remains $2 - a + 3b$.

Things To Watch Out For

You may have heard of an outcome called "cancelling each other out." For example, in resolving the division $\frac{4a}{4a}$ many people would say that the terms cancel each other out. Many people will also mistakenly interpret this to mean that the quotient is zero and say that $\frac{4a}{4a} = 0$. In fact, when terms cancel each other out the quotient is one, not zero. The numerical coefficient is $\frac{4}{4} = 1$. The literal coefficient is $\frac{a}{a} = 1$. Thus, $\frac{4a}{4a} = (1)(1) = 1$. This also explains why a zero exponent equals one: $\frac{a^1}{a^1} = a^{1-1} = a^0 = 1$.

Paths To Success

A lot of people dislike fractions and find them hard to work with. Remember that when you are simplifying any algebraic expression you can transform any fraction into a decimal. For example, if your expression is , you can convert the fraction into decimals: $0.4x + 0.75x$. In this format, it is easier to solve.

✓ Example 2.4.5: A Monomial Division

Simplify the following algebraic expression: $\frac{30x^6 + 5x^3 + 10x^3}{5x}$

Solution

You have been asked to simplify the expression.

What You Already Know

Notice that the provided expression is a polynomial divided by a monomial. Therefore, apply the division rules.

How You Will Get There

Applying the three steps for division of an algebraic expression, you will simplify, divide each term, and combine.

Perform

Step 1: The numerator has two terms with the same literal coefficient (x^3). Combine these using the rules of addition.

$$\frac{30x^6 + 5x^3 + 10x^3}{5x}$$

Step 2: Now that the numerator is simplified, divide each of its terms by the denominator.

$$\frac{30x^6 + \mathbf{15x^3}}{5x}$$

Resolve the divisions by dividing both numerical and literal coefficients.

$$\frac{30x^6}{5x} + \frac{\mathbf{15x^3}}{5x}$$

Step 3: There are no like terms, so this is the final solution.

$$6x^5 + 3x^2$$

The simplified algebraic expression is $6x^5 + 3x^2$.

Example 2.4.6: A More Challenging Division

Simplify the following algebraic expression: $\frac{15x^2y^3 + 25xy^2 - xy + 10x^4y + 5xy^2}{5xy}$

Solution

You have been asked to simplify the expression.

What You Already Know

Notice that the provided expression is a polynomial divided by a monomial. Therefore, apply the division rules.

How You Will Get There

Applying the three steps for division of algebraic expressions, you will simplify, divide each term, and combine.

Perform

Step 1: The numerator has two terms with the same literal coefficient (xy^2). Combine these by the rules of addition.

$$\frac{15x^2y^3 + 25xy^2 - xy + 10x^4y + 5xy^2}{5xy}$$

Step 2: Now that the numerator is simplified, divide each of its terms by the denominator.

$$\frac{15x^2y^3 + \mathbf{30xy^2 - xy + 10x^4y}}{5xy}$$

Resolve the divisions by dividing both numerical and literal coefficients.

$$\frac{15x^2y^3}{5xy} + \frac{30xy^2}{5xy} - \frac{xy}{5xy} + \frac{10x^4y}{5xy}$$

Step 3: Simplify and combine any like terms.

$$3xy^2 + 6(1)(y) - 0.2(1)(1) + 2\left(x^3\right)(1)$$

There are no like terms, so this is the final solution.

$$3xy^2 + 6y - 0.2 + 2x^3$$

The simplified algebraic expression is $3xy^2 + 6y - 0.2 + 2x^3$.

Substitution

The ultimate goal of algebra is to represent a relationship between various variables. Although it is beneficial to simplify these relationships where possible and shorten the algebraic expressions, in the end you want to calculate a solution. Substitution involves replacing the literal coefficients of an algebraic expression with known numerical values. Once the substitution has taken place, you solve the expression for a final value.

How It Works

Follow these steps to perform algebraic substitution:

Step 1: Identify the value of your variables. Assume the algebraic equation is $PV = \dfrac{FV}{1+rt}$. You need to calculate the value of PV. It is known that $FV = \$5,443.84$, $r = 0.12$, and $t = \dfrac{270}{365}$.

Step 2: Take the known values and insert them into the equation where their respective variables are located, resulting in $PV = \dfrac{\$5,443.84}{1+(0.12)\left(\dfrac{270}{365}\right)}$.

Step 3: Resolve the equation to solve for the variable. Calculate $PV = \dfrac{\$5,443.84}{1.088767} = \$5,000.00$.

Things To Watch Out For

It is common in algebra to represent a variable with more than just one letter. As you can see from the example above, FV is a variable, and it represents future value. This should not be interpreted to be two variables, F and V. Similarly, PMT represents annuity payment. When you learn new formulas and variables, take careful note of how a variable is represented.

As well, some literal coefficients have subscripts. For example, you could see d_1 and d_2 in the same formula. What sometimes happens is that there is more than one value for the same variable. As you will learn in Chapter 6 about merchandising, when you buy an item you may receive more than one discount rate (what d stands for). Therefore, the first discount gets a subscript of 1, or d_1, and the second discount gets a subscript of 2, or d_2. This allows you to distinguish between the two values in the equation and substitute the correct value in the correct place.

Paths To Success

If you are unsure whether you have simplified an expression appropriately, remember that you can make up your own values for any literal coefficient and substitute those values into both the original and the simplified expressions. If you have obeyed all the rules and simplified appropriately, both expressions will produce the same answer. For example, assume you simplified $2x + 5x$ into $7x$, but you are not sure if you are right. You decide to let $x = 2$. Substituting into $2x + 5x$, you get $2(2) + 5(2) = 14$. Substituting into your simplified expression you get $7(2) = 14$. Since both expressions produced the same answer, you have direct confirmation that you have simplified correctly.

✓ Example $2.4.7$: Substitution

Substitute and solve the following equation:

$$N = L \times (1 - d_1) \times (1 - d_2) \times (1 - d_3)$$

Where $L = \$1,999.99$, $d_1 = 35\%$, $d_2 = 15\%$, $d_3 = 5\%$

Solution

You need to get a dollar value for the literal coefficient N.

What You Already Know

You are provided with the equation and the values of four literal coefficients.

How You Will Get There

Step 1: The values of L, d_1, d_2, and d_3 are known.

Step 2: Substitute those values into the equation.

Step 3: Solve for N.

Perform

Step 1:

$L = \$1,999.99$, $d_1 = 0.35$, $d_2 = 0.15$, $d_3 = 0.05$

Step 2:

$$N = \$1,999.99 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.05)$$

Step 3:

$$N = \$1,999.99 \times 0.65 \times 0.85 \times 0.95$$

$N = \$1,049.74$

The value of N is $1,049.74.

Contributors and Attributions

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2.5: Linear Equations - Manipulating and Solving (Solving the Puzzle)

You are shopping at Old Navy for seven new outfits. The price points are $10 and $30. You really like the $30 outfits; however, your total budget can't exceed $110. How do you spend $110 to acquire all the needed outfits without exceeding your budget while getting as many $30 items as possible?

This is a problem of linear equations, and it illustrates how you can use them to make an optimal decision. Let L represent the quantity of clothing at the low price point of $10, and H represent the quantity of clothing at the high price point of $30. This results in the following algebraic equations:

$$L+H=7 \text{ (the total number of outfits you need)}$$

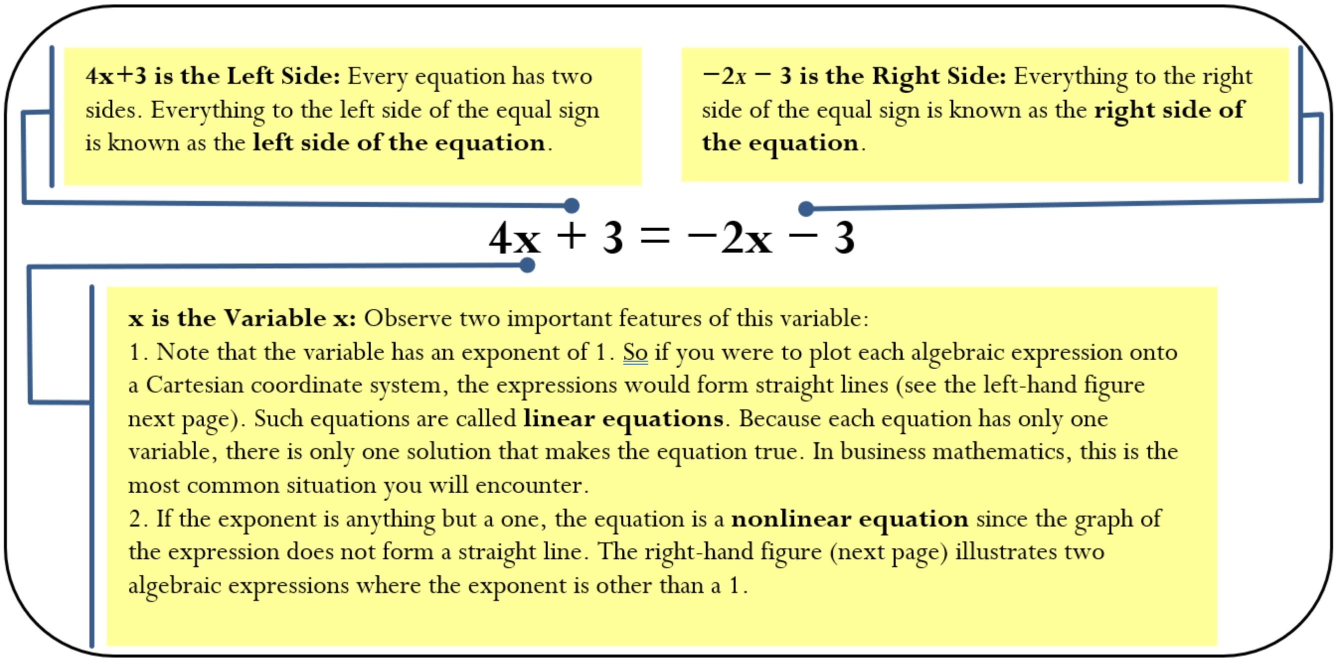
$$\$10L+\$30H=\$110 \text{ (your total budget)}$$

By simultaneously solving these equations you can determine how many outfits at each price point you can purchase.

You will encounter many situations like this in your business career, for example, in making the best use of a manufacturer's production capacity. Assume your company makes two products on the same production line and sells all its output. Each product contributes differently to your profitability, and each product takes a different amount of time to manufacture. What combination of each of these products should you make such that you operate your production line at capacity while also maximizing the profits earned? This section explores how to solve linear equations for unknown variables.

Understanding Equations

To manipulate algebraic equations and solve for unknown variables, you must first become familiar with some important language, including linear versus nonlinear equations and sides of the equation.



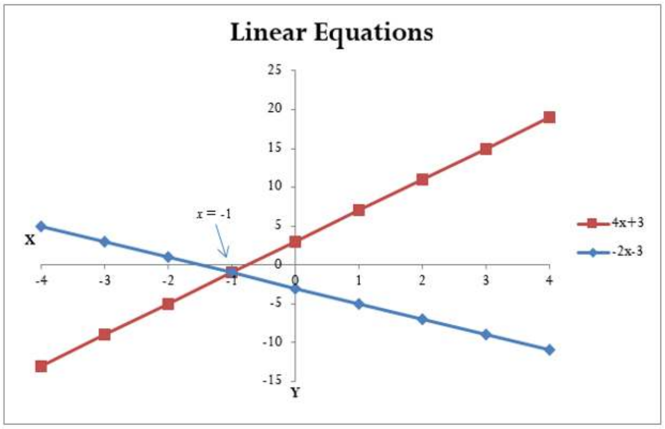
4x+3 is the Left Side: Every equation has two sides. Everything to the left side of the equal sign is known as the **left side of the equation**.

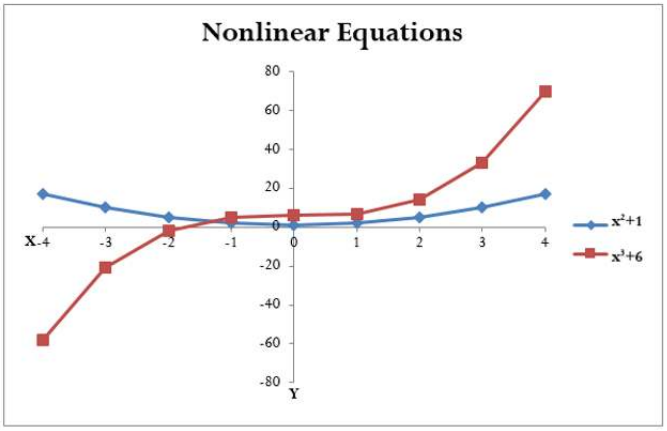
−2x − 3 is the Right Side: Everything to the right side of the equal sign is known as the **right side of the equation**.

$$4x+3=-2x-3$$

x is the Variable x: Observe two important features of this variable:
1. Note that the variable has an exponent of 1. So if you were to plot each algebraic expression onto a Cartesian coordinate system, the expressions would form straight lines (see the left-hand figure next page). Such equations are called **linear equations**. Because each equation has only one variable, there is only one solution that makes the equation true. In business mathematics, this is the most common situation you will encounter.
2. If the exponent is anything but a one, the equation is a **nonlinear equation** since the graph of the expression does not form a straight line. The right-hand figure (next page) illustrates two algebraic expressions where the exponent is other than a 1.

The goal in manipulating and solving a linear equation is to find a value for the unknown variable that makes the equation true. If you substitute a value of $x=-1$ into the above example, the left-hand side of the equation equals the right-hand side of the equation (see Figure below). The value of $x=-1$ is known as the root, or solution, to the linear equation.





Solving One Linear Equation with One Unknown Variable

In your study of solving linear equations, you need to start by manipulating a single equation to solve for a single unknown variable. Later in this section you will extend from this foundation to the solution of two linear equations with two unknowns.

How It Works

To determine the root of a linear equation with only one unknown variable, apply the following steps:

Step 1: Your first goal is to separate the terms containing the literal coefficient from the terms that only have numerical coefficients. Collect all of the terms with literal coefficients on only one side of the equation and collect all of the terms with only numerical coefficients on the other side of the equation. It does not matter which terms go on which side of the equation, so long as you separate them.

To move a term from one side of an equation to another, take the mathematical opposite of the term being moved and add it to both sides. For example, if you want to move the +3 in $4x+3=-2x-3$ from the left-hand side to the right-hand side, the mathematical opposite of +3 is −3. When you move a term, remember the cardinal rule: What you do to one side of an equation you must also do to the other side of the equation. Breaking this rule breaks the equality in the equation.

Step 2: Combine all like terms on each side and simplify the equation according to the rules of algebra.

Step 3: In the term containing the literal coefficient, reduce the numerical coefficient to a 1 by dividing both sides of the equation by the numerical coefficient.

Important Notes

When you are unsure whether your calculated root is accurate, an easy way to verify your answer is to take the original equation and substitute your root in place of the variable. If you have the correct root, the left-hand side of the equation equals the right-hand side of the equation. If you have an incorrect root, the two sides will be unequal. The inequality typically results from one of the three most common errors in algebraic manipulation:

1. The rules of BEDMAS have been broken.
2. The rules of algebra have been violated.
3. What was done to one side of the equation was not done to the other side of the equation.

Things To Watch Out For

When you move a term from one side of the equation to another using multiplication or division, remember that this affects each and every term on both sides of the equation. To remove the x from the denominator in the following equation, multiply both sides of the equation by x:

$\frac{5}{x}+\frac{1}{x}=\frac{2}{x}+2$ becomes $x\left(\frac{5}{x}+\frac{1}{x}\right)=\left(\frac{2}{x}+2\right)x$ which then becomes $5+1=2+2x$

Multiplying every term on both sides by x maintains the equality.

Paths To Success

Negative numbers can cause some people a lot of grief. In moving terms from a particular side of the equation, many people prefer to avoid negative numerical coefficients in front of literal coefficients. Revisiting $4x+3=-2x-3$, you could move the $4x$ from the left side to the right side by subtracting $4x$ from both sides. However, on the right side this results in $-6x$. The negative is easily overlooked or accidentally dropped in future steps. Instead, move the variable to the left side of the equation, yielding a positive coefficient of $6x$.

Example $2.5.1$: How to Solve the Opening Example

Take the ongoing example in this section and solve it for x: $4x+3=-2x-3$

Solution

This is a linear equation since the exponent on the variable is 1. You are to solve the equation and find the root for x.

What You Already Know

The equation has already been provided.

How You Will Get There

Apply the three steps for solving linear equations. To arrive at the root, you must follow the rules of algebra, BEDMAS, and equality.

Perform

Step 1: Move terms with literal coefficients to one side and terms with only numerical coefficients to the other side. Let's collect the literal coefficient on the left-hand side of the equation. Move $-2x$ to the left-hand side by placing $+2x$ on both sides.

$$4x+3=-2x – 3$$

On the right-hand side, the $-2x$ and $+2x$ cancel out to zero.

$$4x+3\mathbf{(+2x)}=\mathbf{-2x-3(+2x)}$$

Step 1 (continued): All of the terms with the literal coefficient are now on the left. Let's move all of the terms containing only numerical coefficients to the right-hand side. Move the +3 to the right-hand side by placing −3 on both sides.

$$4x+3+2x=-3$$

On the left-hand side, the +3 and −3 cancel out to zero.

$$4x+3+2x\mathbf{(-3)}=\mathbf{-3(-3)}$$

Step 2: The terms are now separated. Combine like terms according to the rules of algebra.

$$4x+2x=-3 – 3$$

Step 3: The term with the literal coefficient is being multiplied by the numerical coefficient of 6. Therefore, divide both sides by 6.

$$\mathbf{6x=-6}$$

The left-hand side numerical coefficients will divide to 1. Resolve the numerical coefficients on the right-hand side.

$$\frac{6x}{\mathbf{6}}=\frac{-6}{\mathbf{6}}$$

This is the root of the equation.

$$x=-1$$

The root of the equation is $x=-1$. To verify the accuracy of your manipulation, take the root of $x=-1$ and substitute it into the original equation:

$$4(-1)+3=-2(-1)-3$$

$$-4+3=2-3$$

$$-1=-1$$

The left-hand side equals the right-hand side, so the root is correct.

✓ Example $2.5.2$: Solving a Linear Equation with One Unknown Variable

Solve the following equation for m: $\frac{3m}{4}+2m=4m-15$

Solution

This is a linear equation since the exponent on the variable is a 1. You are to solve the equation and find the root for m.

What You Already Know

The equation has already been provided.

How You Will Get There

Simplify the equations first and then apply the three steps for solving linear equations. To arrive at the root you must follow the rules of algebra, BEDMAS, and equality. You can use an approach that avoids negatives.

Perform

First, simplify all fractions to make the equation easier to work with.

$$\frac{3m}{4}+2m=4m-15$$

Still simplifying, collect like terms where possible.

$$\mathbf{(0.75m)+2m=4m-15}$$

Step 1: Collect all terms with the literal coefficient on one side of the equation. Move all terms with literal coefficients to the right-hand side.

$$\mathbf{(2.75m)=4m-15}$$

Step 1 (continued): Combine like terms and move all terms with only numerical coefficients to the left-hand side.

$$2.75m\mathbf{(-2.75m)=4m-15(-2.75m)}$$

On the left-hand side, the $+2.75m$ and $-2.75m$ cancel each other out. Now move the numerical coefficients to the left-hand side.

$$\mathbf{(0)=4m-15(-2.75m)}$$

On the right-hand side, the −15 and +15 cancel each other out.

$$0\mathbf{(+15)=4m-15-2.75m(+15)}$$

Step 2: Combine like terms on each side.

$$0\mathbf{(+15)=4m-2.75m}$$

Step 3: Divide both sides by the numerical coefficient that accompanies the literal coefficient.

$$\mathbf{15=1.25m}$$

Simplify.

$$\frac{15}{\mathbf{1.25}}=\frac{1.25m}{\mathbf{1.25}}$$

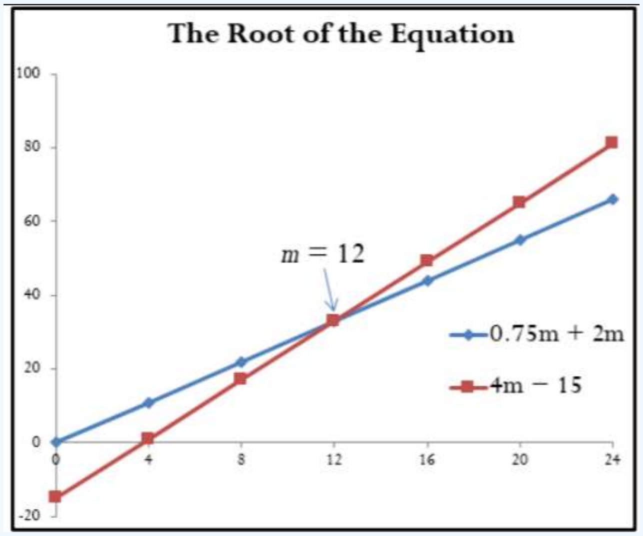
This is the root of the equation.

$$12=m$$

The root of the equation is $m=12$.

This makes both sides of the equation,

$\frac{3m}{4} + 2m$ and $4m - 15$, equal 33.



✓ Example $2.5.3$: Solving a Linear Equation with One Unknown Variable Containing Fractions

Solve the following equation for b and round your answer to four decimals: $\frac{5}{8}b + \frac{2}{5} = \frac{17}{20} - \frac{b}{4}$

Solution

This is a linear equation since the exponent on the variable is a 1. You are to solve the equation and find the root for b.

What You Already Know

The equation has already been provided. Although you could attempt to clear each and every fraction or try to find a common denominator, recall that you can eliminate fractions by converting them to decimals.

How You Will Get There

Simplify the fractions into decimal form. Then apply the three steps for solving linear equations. To arrive at the root, you must follow the rules of algebra, BEDMAS, and equality.

Perform

Simplify the fractions and convert to decimals.

$$\frac{5}{8}b + \frac{2}{5} = \frac{17}{20} - \frac{b}{4}$$

Step 1: Move the literal coefficient terms to the left-hand side.

$$(\mathbf{0.625})\mathbf{b}(+\mathbf{0.4}) = (\mathbf{0.85} - \mathbf{0.25})\mathbf{b}$$

The literal coefficients on the right-hand side cancel each other out.

$$0.625b + 0.4 + (\mathbf{0.25b}) = \mathbf{0.85} - \mathbf{0.25b} + (\mathbf{0.25b})$$

Move the numerical coefficient terms to the right-hand side.

$$0.625b + 0.4 + 0.25b = 0.85$$

The numerical coefficients on the left-hand side cancel each other out.

$$0.625b + 0.4 + 0.25b(-\mathbf{0.4}) = \mathbf{0.85}(-\mathbf{0.4})$$

Step 2: Combine like terms on each side.

$$0.625b + 0.25b = 0.85 - 0.4$$

Step 3: Divide both sides by the numerical coefficient that accompanies the literal coefficient.

$$\mathbf{0.875b = 0.45}$$

Simplify.

$$\frac{0.875b}{\mathbf{0.875}} = \frac{0.45}{\mathbf{0.875}}$$

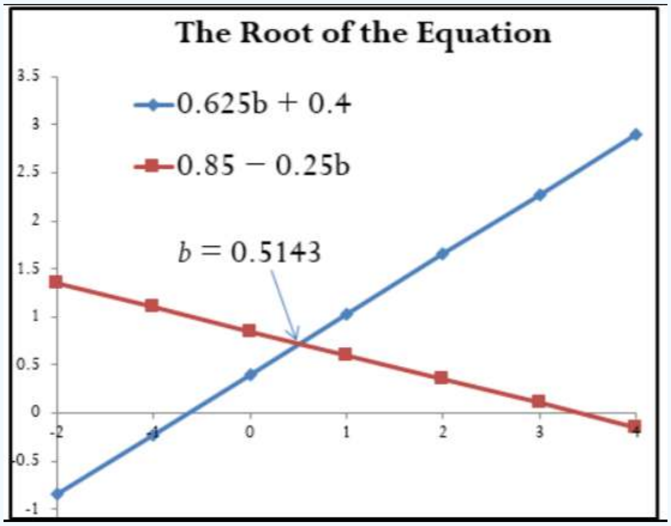
Round to four decimals as instructed.

$$b = 0.514285$$

This is the root.

$$b = 0.5143$$

The root of the equation, rounded to four decimals, is $b = 0.5143$.



Solving Two Linear Equations with Two Unknown Variables

The manipulation process you have just practiced works well for solving one linear equation with one variable. But what happens if you need to solve two linear equations with two variables simultaneously? Remember when you were at Old Navy purchasing seven outfits earlier in this chapter (equation 1)? You needed to stay within a pricing budget (equation 2). Each equation had two unknown variables representing the number of lower-priced and higher-priced outfits.

The goal is to reduce two equations with two unknowns into a single linear equation with one unknown. Once this transformation is complete, you then identify the unknown variable by applying the three-step procedure for solving one linear equation, as just discussed.

When you work with two linear equations with two unknowns, the rules of algebra permit the following two manipulations:

1. What you do to one side of the equation must be done to the other side of the equation to maintain the equality. Therefore, you can multiply or divide any equation by any number without changing the root of the equation. For example, if you multiply all terms of $x + y = 2$ by 2 on both sides, resulting in $2x + 2y = 4$, the equality of the equation remains unchanged and the same roots exist.
2. Terms that are on the same side of an equation can be added and subtracted between equations by combining like terms. Each of the two equations has a left side and right side. This rule permits taking the left side of the first equation and either adding or subtracting like terms on the left side of the second equation. When you perform this action, remember the first rule above. If you add the left sides of the equations together, you then must add the right side of both equations together to maintain equality.

How It Works

Follow these steps to solve two linear equations with two unknown variables:

Step 1: Write the two equations one above the other, vertically lining up terms that have the same literal coefficients and terms that have only the numerical coefficient. If necessary, the equations may need to be manipulated such that all of the literal coefficients are on one side with the numerical coefficients on the other side.

Step 2: Examine your two equations. Through multiplication or division, make the numerical coefficient on one of the terms containing a literal coefficient exactly equal to its counterpart in the other equation.

Step 3: Add or subtract the two equations as needed so as to eliminate the identical term from both equations.

Step 4: In the new equation, solve for the last literal coefficient.

Step 5: Substitute the root of the known literal coefficient into either of the two original equations. If one of the equations takes on a simpler structure, pick that equation.

Step 6: Solve your chosen equation for the other literal coefficient.

Paths To Success

Sometimes it is unclear exactly how you need to multiply or divide the equations to make two of the terms identical. For example, assume the following two equations:

$$4.9x + 1.5y = 38.3$$

$$2.7x - 8.6y = 17.8$$

If the goal is to make the terms containing the literal coefficient x identical, there are two alternative solutions:

1. Take the larger numerical coefficient for x and divide it by the smaller numerical coefficient. The resulting number is the factor for multiplying the equation containing the smaller numerical coefficient. In this case, $4.9 \div 2.7 = 1.\overline{814}$. Multiply all terms in the second equation by $1.\overline{814}$ to make the numerical coefficients for x equal to each other, resulting in this pair of equations:

$$4.9x + 1.5y = 38.3$$

$$4.9x - 15.6\overline{074}y = 32.3\overline{037} \text{ (every term multiplied by } 1.\overline{814})$$

2. Take the first equation and multiply it by the numerical coefficient in the second equation. Then take the second equation and multiply it by the numerical coefficient in the first equation. In this case, multiply all terms in the first equation by 2.7. Then multiply all terms in the second equation by 4.9.

$$13.23x + 4.05y = 103.41 \text{ (every term multiplied by } 2.7)$$

$$13.23x - 42.14y = 87.22 \text{ (every term multiplied by } 4.9)$$

Note that both approaches successfully result in both equations having the same numerical coefficient in front of the literal coefficient x.

Paths To Success

Ultimately, every pairing of linear equations with two unknowns can be converted into a single equation through substitution. To make the conversion, do the following:

1. Solve either equation for one of the unknown variables.
2. Take the resulting algebraic expression and substitute it into the other equation. This new equation is solvable for one of the unknown variables.
3. Substitute your newfound variable into one of the original equations to determine the value for the other unknown variable.

Take the following two equations:

$$a + b = 4 \qquad 2a + b = 6$$

1. Solving the first equation for a results in $a = 4 - b$.
2. Substituting the expression for a into the second equation and solving for b results in $2(4 - b) + b = 6$, which solves as $b = 2$.

3. Finally, substituting the root of b into the first equation to calculate a gives $a+2=4$ resulting in $a=2$. Therefore, the roots of these two equations are $a=2$ and $b=2$.

✓ Example $2.5.4$: Buying Those Outfits

Recall from the section opener that in shopping for outfits there are two price points of $10 and $30, your budget is $110, and that you need seven articles of clothing. The equations below represent these conditions. Identify how many low-priced outfits (L) and high-priced outfits (H) you can purchase.

$$L+H=7 \ \$10L+\$30H=\$110$$

Solution

You need to determine the quantity of low-price-point items, or L, and high-price-point items, or H, that are within your limited budget. Note that the exponents on the variables are 1 and that there are two unknowns. So there are two linear equations with two unknowns.

What You Already Know

You require seven articles of clothing and only have a budget of $110. The equations express the relationships of quantity and budget.

How You Will Get There

Apply the six-step procedure for solving two linear equations with two unknowns.

Step 1:

Write the equations one above the other and line them up.

$$\begin{array}{lllll} L & + & H & = & 7 \\ \$10L & + & \$30H & = & \$110 \end{array}$$

Step 2:

Multiply all terms in the first equation by 10 so that L has the same numerical coefficient in both equations.

$$\begin{array}{lllll} 10L & + & 10H & = & 70 \\ \$10L & + & \$30H & = & \$110 \end{array}$$

Step 3:

Subtract the equations by subtracting all terms on both sides.

$$\begin{array}{llllll} & 10L & + & 10H & = & 70 \\ \text{subtract} & \$10L & + & \$30H & = & \$110 \\ & & - & \$20H & = & -\$40 \end{array}$$

Step 4:

Solve for H by dividing both sides by −20.

$$\frac{-\$20H}{-\$20}=\frac{-\$40}{-\$20} \quad H=2$$

Step 5:

Substitute the known value for H into one of the original equations. The first equation is simple, so choose that one.

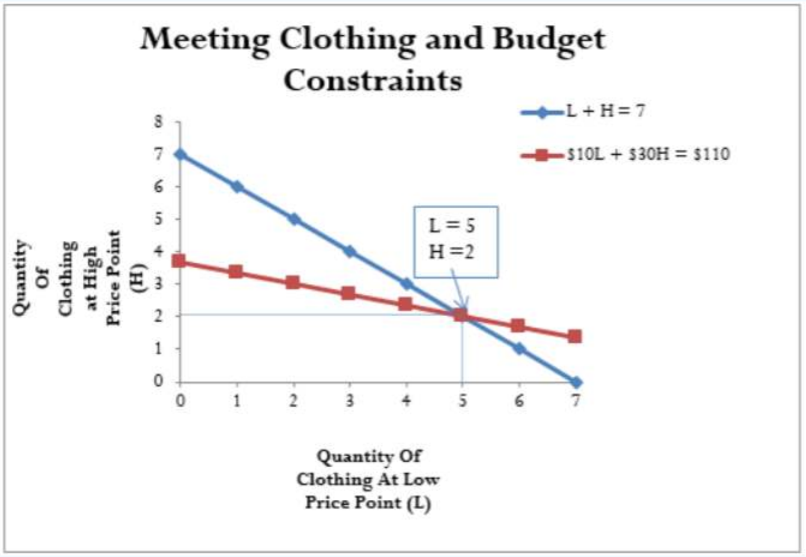
$$\begin{array}{lllll} L & + & H & = & 7 \\ L & + & 2 & = & 7 \end{array}$$

Step 6:

Solve for L by subtracting 2 from both sides. You now have the roots for L and H.

$$\begin{array}{lllllllll} L & + & 2 & - & 2 & = & 7 & - & 2 \\ & & & & L & = & 5 & & \end{array}$$

You can purchase five articles of clothing at the low price point and two articles of clothing at the high price point. This allows you to purchase seven articles of clothing and stay within your budget of $110.



Paths To Success

One of the most difficult areas of mathematics involves translating words into mathematical symbols and operations. To assist in this translation, the table below lists some common language and the mathematical symbol that is typically associated with the word or phrase.

Language			Math Symbol
Sum Addition	In addition to In excess	Increased by Plus	+
Subtract Decreased by Diminished by	Less Minus	Difference Reduced by	-
Multiplied by Times	Percentage of	Product of Of	×
Divide Division	Divisible Quotient	Per	÷
Becomes Is/Was/Were	Will be	Results in Totals	=
More than	Greater than		>
Less than	Lower than		<
Greater than or equal to			≥
Less than or equal to			≤
Not equal to			≠

✓ Example 2.5.5: Solving Two Linear Equations with Two Unknowns for an Amusement Park

Tinkertown Family Fun Park charges $15 for a child wrist band and $10.50 for an adult wrist band. On a warm summer day, the amusement park had total wrist band revenue of $15,783 from sales of 1,279 wrist bands. How many adult and child wrist bands did the park sell that day?

Solution

You need the number of both adult and child wrist bands sold on the given day. Therefore, you must identify two unknowns.

What You Already Know

The price of the wrist bands,total quantity, and sales are known:

Child wrist band price = $15

Adult wrist band price = $10.50

Total revenue = $15,783

Total unit sales = 1,279

The quantity of adult wrist bands sold and the quantity of child wrist bands sold are unknown:

Adult wrist bands quantity = a

Child wrist bands quantity = c

How You Will Get There

1. Work with the quantities first. Calculate the total unit sales by adding the number of adult wrist bands to the number of child wrist bands:

$$\# \text{ of adult wrist bands } + \# \text{ of child wrist bands } = \text{ total unit sales}$$

$$a+c=1,279$$

2. Now consider the dollar figures. Total revenue for any company is calculated as unit price multiplied by units sold. In this case, you must sum the revenue from two products to get the total revenue.

$$\text{Total adult revenue } + \text{ Total child revenue } = \text{ Total revenue}$$

$$(\text{Adult price } \times \text{ Adult guantity })+(\text{Child price } \times \text{ Child quantity}) = \text{ Total revenue}$$

$$\$10.50a+\$15c=\$15,783$$

3. Apply the six-step procedure for solving two linear equations with two unknowns.

Perform

Step 1:

Write the equations one above the other and line them up.

$$\begin{array}{lllll} a & + & c & = & 1,279 \\ \$10.50a & + & \$15c & = & \$15,783 \end{array}$$

Step 2:

Multiply all terms in the first equation by 10.5, resulting in a having the same numerical coefficient in both equations.

$$\begin{array}{lllll} \mathbf{10.50a} & + & \mathbf{10.50c} & = & \mathbf{13,429.50} \\ \$10.50a & + & \$15c & = & \$15,783 \end{array}$$

Step 3:

Subtract the equations by subtracting all terms on both sides.

$$\begin{array}{llllll} & \mathbf{10.50a} & + & \mathbf{10.50c} & = & \mathbf{13,429.50} \\ \text{Subtract} & \underline{\$10.50a} & \underline{+} & \underline{\$15c} & = & \underline{\$15,783} \end{array}$$

$$\mathbf{-4.5c \quad = \quad -2,353.50}$$

Step 4:

Solve for c by dividing both sides by −4.5.

$$\frac{-4.5c}{-4.5} = \frac{-2,353.50}{-4.5} \qquad c = 523$$

Step 5:

Substitute the known value for c into one of the original equations. The first equation is simple, so choose that one.

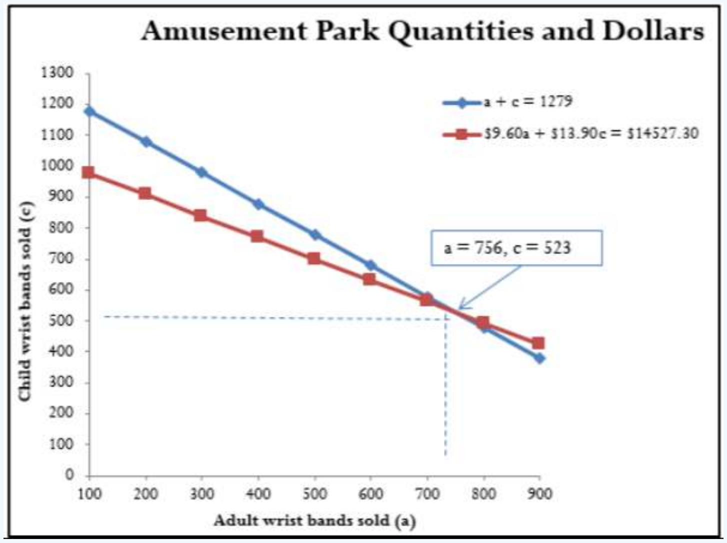
$$\begin{array}{ccccc} a & + & c & = & 1,279 \\ a & + & \mathbf{523} & = & 1,279 \end{array}$$

Step 6:

Solve for a by subtracting 523 from both sides. You now have the roots for a and c.

$$a + 523 — \mathbf{523} = 1,279 — \mathbf{523}$$
$$a = 756$$

Tinkertown Family Fun Park sold 523 child wrist bands and 756 adult wrist bands.



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2.6: Natural Logarithms (How Can I Get That Variable Out of the Exponent?)

Your investment is earning compound interest of 6% per year. You wonder how long it will take to double in value. Answering this question requires you to solve an equation in which the unknown variable is located in the exponent instead of the base position. To find the answer, you need logarithms, which are reviewed in this section and will be applied in Chapter 9 and Chapter 11.

Logarithms

A logarithm is defined as the exponent to which a base must be raised to produce a particular power. Although the base can assume any value, two values for the base are most often used.

1. **A Base Value of 10**. This is referred to as a common logarithm. Thus, if you have $10^2 = 100$, then 2 is the common logarithm of 100 and this is written as $\log_{10}(100) = 2$, or just $\log(100) = 2$ (you can assume the 10 if the base is not written). The format for a common logarithm can then be summarized as

$$\log\ (\text{power})\ =\ \text{exponent}$$

$$\text{If you have } 10^x = y,\ \text{then } \log(y) = x.$$

2. **A Base Value of e**. This is referred to as a natural logarithm. In mathematics, there is a known constant e, which is a nonterminating decimal and has an approximate value of $e = 2.71828182845$. If you have $e^3 = 20.085537$, then 3 is the natural logarithm of 20.085537 and you write this as $ln(20.085537) = 3$. The format for a natural logarithm is then summarized as:

$$\ln(\ \text{power}\ ) =\ \text{exponent}$$

$$\text{If you have } e^4 = 54.59815,\ \text{then } \ln(54.59815) = 4.$$

Common logarithms have been used in the past when calculators were not equipped with power functions. However, with the advent of computers and advanced calculators that have power functions, the natural logarithm is now the most commonly used format. From here on, this textbook focuses on natural logarithms only.

Properties of Natural Logarithms

Natural logarithms possess six properties:

1. **The natural logarithm of 1 is zero**. For example, if 1 is the power and 0 is the exponent, then you have $e^0 = 1$. This obeys the laws of exponents discussed in Section 2.4 of this chapter.
2. **The natural logarithm of any number greater than 1 is a positive number**. For example, the natural logarithm of 2 is 0.693147, or $e^{0.693147} = 2$.
3. **The natural logarithm of any number less than 1 is a negative number**. For example, the natural logarithm of 0.5 is -0.693147, or $e^{-0.693147} = 0.5$. Similar to property 1, this obeys the laws of exponents discussed in Section 2.4, where $e^{-0.693147} = \frac{1}{e^{0.693147}}$, always producing a proper fraction.
4. **A natural logarithm cannot be less than or equal to zero**. Since e is a positive number with an exponent, there is no value of the exponent that can produce a power of zero. As well, it is impossible to produce a negative number when the base is positive.
5. **The natural logarithm of the quotient of two positive numbers is** $\ln\left(\frac{\mathrm{x}}{\mathrm{y}}\right) = \ln(\mathrm{x}) - \ln(\mathrm{y})$. For example,

$$\ln\left(\frac{\$20,000}{\$15,000}\right) = \ln(\$20,000) - \ln(\$15,000).$$

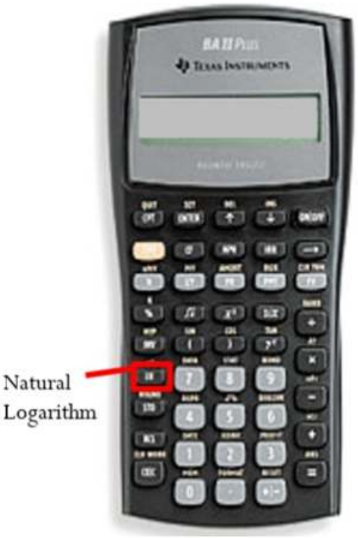
$$\ln(1.\overline{3}) = 9.903487 - 9.615805$$

$$0.287682 = 0.287682$$

6. **The natural logarithm of a power of a positive base is** $\ln\left(\mathrm{x}^{\mathrm{y}}\right) = \mathrm{y}(\ln \mathrm{x})$. This property allows you to relocate the exponent down into the base. You will apply this property especially in Chapter 9 and Chapter 11. Demonstrating this principle,

$$\begin{gathered}\ln\left(x^y\right) = y(\ln x)\\ \ln\left(1.05^6\right) = 6 \times \ln(1.05)\\ \ln(1.340095) = 6 \times 0.048790\\ 0.292741 = 0.292741\end{gathered}$$

Important Notes



Applying natural logarithms on your TI BAII Plus calculator requires two steps.

1. Input the power.
2. With the power still on the display, press the LN key located in the left-hand column of your keypad. The solution on the display is the value of the exponent. If you key in an impossible value for the natural logarithm, an "Error 2" message is displayed on the screen.

If you know the exponent and want to find out the power, remember that $e^x = \text{power}$. This is called the anti-log function. Thus, if you know the exponent is 2, then $e^2 = 7.389056$. On your calculator, the anti-log can be located on the second shelf directly above the LN button. To access this function, key in the exponent first and then press $2^{nd} e^x$.

Paths To Success

You do not have to memorize the mathematical constant value of e. If you need to recall this value, use an exponent of 1 and access the e^x function. Hence, $e^1 = 2.71828182845$.

Exercise 2.6.1: Give It Some Thought

For each of the following powers, determine if the natural logarithm is positive, negative, zero, or impossible.

a. 2.3
b. 1
c. 0.45
d. 0.97
e. −2
f. 4.83
g. 0

Answer

a. positive (property 2)
b. zero (Property 1)
c. negative (property 3)
d. negative (property 3)
e. impossible (Property 4)
f. positive (property 2)
g. impossible (property 4)

✓ Example 2.6.1: Applying Natural Logarithms and Properties

Solve the first two questions using your calculator. For the next two questions, demonstrate the applicable property.

a. $\ln(2.035)$
b. $\ln(0.3987)$
c. $\ln\left(\frac{\$10,000}{\$6,250}\right)$
d. $\ln\left[(1.035)^{12}\right]$

Solution

You need to apply the properties of natural logarithms.

What You Already Know

The properties of natural logarithms are known.

How You Will Get There

a. Apply property 2 and key this through your calculator.
b. Apply property 3 and key this through your calculator.
c. Apply property 5, $\ln\left(\frac{x}{y}\right)=\ln(x)-\ln(y)$.
d. Apply property 6, $\ln\left(x^{y}\right)=y(\ln x)$

Perform

a. $\ln(2.035)=0.710496$
b. $\ln(0.3987)=-0.919546$
c. $$\begin{aligned}\ln\left(\frac{\$10,000}{\$6,250}\right)&=\ln(\$10,000)-\ln(\$6,250)\\ \ln(1.6)&=9.210340-8.740336\\ 0.470004&=0.470004\end{aligned}$$
d. $$\begin{aligned}\ln\left[(1.035)^{12}\right]&=12\times\ln(1.035)\\ \ln(1.511068)&=12\times0.034401\\ 0.412817&=0.412817\end{aligned}$$

Calculator Instructions

			Answer
a.	2.035	LN	0.710496
b.	0.3987	LN	-0.919546
c.	$10000\div6250=$	LN	0.470004
d.	$1.035y^{x}12=$	LN	0.412817

Organizing your answers into a more common format:

a. $e^{0.710496}=2.035$
b. $e^{-0.919546}=0.3987$
c. $\ln\left(\frac{\$10,000}{\$6,250}\right)=0.470004$ (as proven by the property)
d. $\ln\left[(1.035)^{12}\right]=0.412817$ (as proven by the property)

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2.S: Summary

Key Concepts Summary

2.1: Order of Operations (Proceed in an Orderly Manner)

- A review of key mathematical operator symbols
- Rules for order of operations are known as BEDMAS

2.2: Fractions, Decimals, and Rounding (Just One Slice of Pie, Please)

- The language and types of fractions
- Working with equivalent fractions by either solving for an unknown or increasing/reducing the fraction
- Converting any fraction to a decimal format
- Procedures for proper rounding
- The rounding rules that are used throughout this textbook

2.3: Percentages (How Does It All Relate?)

- Converting decimal numbers to percentages
- Working with percentages in the form of rates, portions, and bases

2.4: Algebraic Expressions (The Pieces of the Puzzle)

- Learning about the language of algebra
- The rules for manipulating exponents
- The rules of algebra for addition and subtraction
- The rules of algebra for multiplication
- The rules of algebra for division
- What is substitution and how it is performed?

2.5: Linear Equations: Manipulating and Solving (Solving the Puzzle)

- A review of key concepts about equations
- The procedures required to solve one linear equation for one unknown variable
- The procedures required to solve two linear equations for two unknown variables

2.6: Natural Logarithms (How Can I Get That Variable Out of the Exponent?)

- A review of the logarithm concept
- The rules and properties of natural logarithms

The Language of Business Mathematics

algebraic equation

An equation that takes two algebraic expressions and makes them equal to each other.

algebraic expression

Indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables.

base

The entire amount or quantity of concern.

BEDMAS

An order of operations acronym standing for Brackets, Exponents, Division, Multiplication, Addition, and Subtraction.

common logarithm

A logarithm with a base value of 10.

complex fraction

A fraction that has fractions within fractions, combining elements of compound, proper, and/or improper fractions together.

compound fraction

A fraction that combines an integer with either a proper or improper fraction.

denominator

Any term by which some other term is divided; commonly the number on the bottom of a fraction.

divisor line

The line that separates the numerator and the denominator in a fraction.

equivalent fractions

Two or more fractions of any type that have the same numerical value upon completion of the division.

exponent

A mathematical shorthand notation that indicates how many times a quantity is multiplied by itself

factor

Components of terms that are separated from by multiplication or division signs.

fraction

A part of a whole.

improper fraction

A fraction in which the numerator is larger than the denominator.

left side of the equation

The part of an equation that is to the left of the equal sign.

like terms

Terms that have the same literal coefficient.

linear equation

An algebraic expression in which the variable has an exponent of 1; when plotted, it will form a straight line.

literal coefficient

A factor that is a variable.

logarithm

The exponent to which a base must be raised to produce a particular power.

monomial

An algebraic expression with only one term.

natural logarithm

A logarithm with a base value of the mathematical constant e.

nomial

The number of terms that appear in an algebraic expression.

nonlinear equation

An algebraic expression in which the variable has an exponent other than 1; when plotted, it will not form a straight line.

numerator

Any term into which some other term is being divided; commonly the number on the top in a fraction.

numerical coefficient

A factor that is numerical.

percentage

A part of a whole expressed in hundredths.

polynomial

An algebraic expression with two or more terms.

portion

Represents part of a whole or base.

proper fraction

A fraction in which the numerator is smaller than the denominator.

rate

Expresses a relationship between a portion and a base.

right side of the equation

The part of an equation that is to the right of the equal sign.

root

The value of the unknown variable that will make a linear equation true.

substitution

Replacing the literal coefficients of an algebraic expression with their numerical values.

term

In any algebraic expression, the components that are separated by addition and subtraction.

triangle technique

A memorization technique that displays simple multiplication formulae in the form of a triangle. Anything on the same line is multiplied, and items above or below each other are divided to arrive at various solutions.

The Formulas You Need to Know

Symbols Used

$\%$ = percentage

$\boldsymbol{dec}$ = any number in decimal format

$\ln$= natural logarithm

Rate = the relationship between the portion and base

Portion = the part of the whole

Base = the whole quantity

Formulas Introduced

Formula 2.1 Percentage Conversion: $\% = \boldsymbol{dec} \times 100$

Formula 2.2 Rate, Portion, Base: $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$

Technology

Calculator

Formatting Instructions

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0.	Back to regular calculator usage.

Exponents and Signs

x^2 is used for exponents that are squares. 2^2 is keyed in as $2x^2$.

y^x is used for exponents that are not squares. 2^3 is keyed in as $2y^x3=$.

± is used to change the sign of a number. To use, key in the number first and then press the ± key.

Memory

STO Store

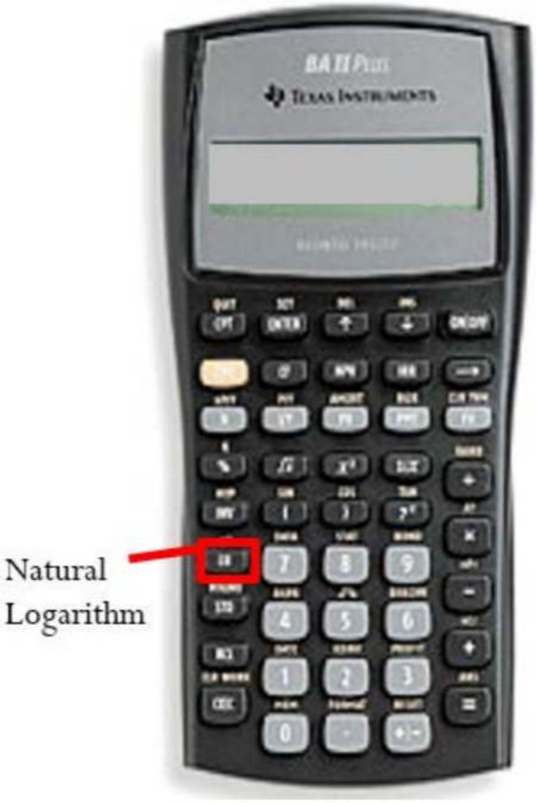
RCL Recall

0–9 Memory cell numbers (10 in total)

To store a number on the display, press STO # (where # is a memory cell number).

To recall a number in the memory, press RCL # (where # is the memory cell where the number is stored).

Natural Logarithms



The natural logarithm function, LN, is located in the left-most column of your calculator. To use this function, input the power and then press the LN button. The solution displayed is the exponent. To calculate the power, input the exponent and then press 2nd ex (called the anti-log).

Recognized Mathematical Symbols

The common mathematical symbols that Excel recognizes are +, −, * (multiplication), / (division), ^ (exponents), and round brackets.

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2.E: Back To The Basics (Exercises)

2.1: Order of Operations

Hint: If a question involves money, round the answers to the nearest cent.

Mechanics

1. $81 \div 27 + 3 \times 4$
2. $100 \div (5 \times 4 - 5 \times 2)$
3. $3^3 - 9 + (1 + 7 \times 3)$
4. $(6+3)^2 - 17 \times 3 + 70$
5. $100 - (4^2 + 3) - (3 + 9 \times 3 - 4)$

Applications

6. $[(7^2 - \{-41\}) - 5 \times 2] \div (80 - 10)$
7. $\$1,000\left(1 + 0.09 \times \frac{88}{365}\right)$
8. $3\left[\$2,000(1+0.003)^8\right] + \$1,500$
9. $\frac{\$20,000}{1 + 0.07 \times \frac{7}{12}}$
10. $4 \times \left[(5^2+15)^2 \div (13^2 - 9)\right]^2$
11. $\$500\left[\frac{(1+0.00875)^{43} - 1}{0.00875}\right]$
12. $\$1,000\left(1 + \frac{0.12}{6}\right)^{15}$

Challenge, Critical Thinking, & Other Applications

13. $\left(\frac{\$2,500}{1+0.10}\right) + \left(\frac{\$7,500}{(1+0.10)^2}\right) + \left(\frac{-\$1,500}{(1+0.10)^3}\right) + \left(\frac{-\$2,000}{(1+0.10)^4}\right)$
14. $\$175,000(1+0.07)^{15} + \$14,000\left[\frac{(1+0.07)^{20} - 1}{0.07}\right]$
15. $\$5,000\left[\frac{\{1 + [(1+0.08)^{0.5} - 1]\}^{75} - 1}{(1+0.08)^{0.5} - 1}\right]$
16. $\$800\left[\frac{(1+0.07)^{20} - (1+0.03)^{20}}{0.07 - 0.03}\right]$
17. $\$60,000(1+0.0058)^{80} - \$450\left[\frac{(1+0.0058)^{25} - 1}{0.0058}\right]$
18. $\left(\frac{0.08}{2}\right)\$1,000\left[\frac{1 - \frac{1}{\left\{1+\left(\frac{0.08}{2}\right)\right\}^{16}}}{\left(\frac{0.08}{2}\right)}\right] + \$1,000\left[1 + \left(\frac{0.08}{2}\right)\right]^{16}$
19. $\$1,475\left[\frac{\left(1+\frac{0.06}{4}\right)^{16} - 1}{\frac{0.06}{4}}\right]$
20. $\$6,250(1+0.0525)^{10} + \$325\left[\frac{(1+0.0525)^{10} - 1}{0.0525}\right]$

2.2: Fractions, Decimals, and Rounding

Mechanics

1. For each of the following, identify the type of fraction presented.
    a. $\frac{1}{8}$
    b. $3\frac{3}{4}$
    c. $\frac{10}{9}$

d. $\frac{34}{49}$
e. $1\frac{\frac{2}{3}}{9\frac{5}{3}}$
f. $\frac{56}{27}$
g. $\frac{10\frac{1}{5}}{9}$
h. $\frac{6}{11}$

2. In each of the following equations, identify the value of the unknown term.
   a. $\frac{3}{4} = \frac{x}{36}$
   b. $\frac{y}{8} = \frac{16}{64}$
   c. $\frac{2}{z} = \frac{18}{45}$
   d. $\frac{5}{6} = \frac{75}{p}$
3. Take each of the following fractions and provide one example of the fraction expressed in both higher and lower terms.
   a. $\frac{5}{10}$
   b. $\frac{6}{8}$
4. Convert each of the following fractions into decimal format.
   a. $\frac{7}{8}$
   b. $15\frac{5}{4}$
   c. $\frac{13}{5}$
   d. $133\frac{\frac{17}{2}}{3\frac{2}{5}}$
5. Convert each of the following fractions into decimal format and round to three decimals.
   a. $\frac{7}{8}$
   b. $15\frac{3}{4}$
   c. $\frac{10}{9}$
   d. $\frac{15}{32}$
6. Convert each of the following fractions into decimal format and express in repeating decimal notation.
   a. $\frac{1}{12}$
   b. $5\frac{8}{33}$
   c. $\frac{4}{3}$
   d. $\frac{-34}{110}$

Applications

7. Calculate the solution to each of the following expressions. Express your answer in decimal format.
   a. $\frac{1}{5} + 3\frac{1}{4} + \frac{5}{2}$
   b. $\frac{1\frac{3}{8}}{2} - \frac{11}{40} + 19\frac{1}{2} \times \frac{3}{4}$
8. Calculate the solution to each of the following expressions. Express your answer in decimal format with two decimals.
   a. $\left(1 + \frac{0.11}{12}\right)^4$

b. $1-0.05 \times \frac{263}{365}$

c. $200\left[1-\frac{1}{\left(1+\frac{0.10}{4}\right)^{2}}\right]$

9. Calculate the solution to each of the following expressions. Express your answer in repeating decimal notation as needed.

a. $\frac{1}{11}+3 \frac{1}{9}$

b. $\frac{5}{3}-\frac{7}{6}$

Questions 10–14 involve fractions. For each, evaluate the expression and round your answer to the nearest cent.

10. $\$134,000(1+0.14 \times 23/365)$

11. $\$10,000\left(1+\frac{0.0525}{2}\right)^{13}$

12. $\frac{\$535,000}{\left(1+\frac{0.07}{12}\right)^{3}}$

13. $\$2,995\left(1+0.13 \times \frac{90}{365}\right)-\frac{\$400}{1+0.13 \times \frac{15}{365}}$

14. $\frac{\$155,600}{\left(1+\frac{0.06}{12}\right)^{8}}$

Challenge, Critical Thinking, & Other Applications

Questions 15–20 involve more complex fractions and reflect business math equations encountered later in this textbook. For each, evaluate the expression and round your answer to the nearest cent.

15. $\frac{\$648}{0.0575/12}\left[1-\frac{1}{\left(1+\frac{0.0575}{12}\right)^{7}}\right]$

16. $\frac{\$10,000}{\left(1+\frac{0.115}{4}\right)^{2}}+\$68 \frac{\left[1-\frac{1}{\left(1+\frac{0.115}{4}\right)^{2}}\right]}{\frac{0.115}{4}}$

17. $\frac{\$2,000,000}{\left[\frac{\left(1+\frac{0.665}{2}\right)^{12}-1}{\frac{0.065}{2}}\right]}$

18. $\$8,500\left[\frac{1-\left(\frac{1}{1.08}\right)^{4}}{1.08}\right]+\$19,750\left(\frac{1}{1.08}\right)^{4}-\$4,350$

19. $\$15,000\left[\frac{\left(1+\frac{0.058}{4}\right)^{16}-1}{\frac{0.058}{4}}\right]$

20. $\frac{0.08}{2}(\$1,000)\left[\frac{1-\frac{1}{\left(1+\frac{0.07}{2}\right)^{10}}}{\frac{0.07}{2}}\right]+\$1,000 \frac{1}{\left(1+\frac{0.07}{2}\right)^{10}}$

2.3: Percentages

Mechanics

1. Convert the following decimals into percentages.
   1. 0.4638
   2. 3.1579
   3. 0.000138
   4. 0.015
2. Convert the following fractions into percentages.
   1. $\frac{3}{8}$
   2. $\frac{17}{32}$
   3. $\frac{42}{12}$
   4. $2\frac{4}{5}$
3. Convert the following fractions into percentages. Round to four decimals or express in repeating decimal format as needed.
   1. $\frac{46}{12}$
   2. $\frac{2}{9}$
   3. $\frac{3}{11}$
   4. $\frac{48}{93}$
4. Convert the following percentages into decimal form.
   1. 15.3%
   2. 0.03%
   3. 153.987%
   4. 14.0005%
5. What percentage of $40,000 is $27,000?
6. What is $\frac{1}{2}$% of $500,000?
7. $0.15 is 4,900% of what number?

Applications

8. In February 2009, 14,676 mortgages were in arrears in Canada, which represented 0.38% of all mortgages. How many total mortgages were in the Canadian market at that time?
9. In 2009, medical experts predicted that one out of two Manitobans would contract some form of the H1N1 virus. If the population of Manitoba in 2009 was 1,217,200, how many Manitobans were predicted to become ill?
10. In August 2004, Google Inc. offered its stocks to the public at $85 per share. In October 2007, the share price had climbed to $700.04. Express the 2007 share price as a percentage of the 2004 price.
11. During Michael Jordan's NBA career (1984–2003), he averaged a free throw completion percentage of 83.5% in regular season play. If Jordan threw 8,772 free throws in his career, how many completed free throws did he make?
12. If total advertising expenditures on television advertising declined 4.1% to $141.7 billion in the current year, how much was spent on television advertising in the previous year? Round your answer to one decimal.
13. If the new minimum wage of $8.75 per hour is 102.9412% of the old minimum wage, what was the old minimum wage?
14. A machine can produce 2,500 products per hour. If 37 of those products were defective, what is the defect rate per hour for the machine?

Challenge, Critical Thinking, & Other Applications

15. In 2011, Manitoba progressive income tax rates were 10.8% on the first $31,000, 12.75% on the next $36,000, and 17.4% on any additional income. If your gross taxable earnings for the year were $85,000, what percentage of your earnings did you pay in taxes?
16. In 2011, the maximum amount that you could have contributed to your RRSP (Registered Retirement Savings Plan) was the lesser of $22,450 or 18% of your earned income from the previous year. How much income do you need to claim a $22,450

contribution in 2011?

17. Maria, a sales representative for a large consumer goods company, is paid 3% of the total profits earned by her company. Her company averages 10% profit on sales. If Maria's total income for the year was \$60,000, what total sales did her company realize?
18. A house was purchased six years ago for \$214,000. Today it lists at a price that is 159.8131% of the original purchase price. In dollars, how much has the price of the house increased over the six years?
19. An investor buys 1,000 shares of WestJet Airlines at \$10.30 per share. A few months later, the investor sells the shares when their value hits 120% of the original share price. What is the price of a WestJet share when the investor sells these shares? How much money did the investor make?
20. A Honda Insight has fuel economy of 3.2 litres consumed per 100 kilometres driven. It has a fuel tank capacity of 40 litres. A Toyota Prius is rated at 4.2 L per 100 km driven. It has a fuel tank capacity of 45 L. What percentage is the total distance driveable (rounded to the nearest kilometre before calculating) of a Honda Insight compared to that of a Toyota Prius?

2.4: Algebraic Expressions

Mechanics

For questions 1–4, simplify the algebraic expressions.

1. $2a-3a+4+6a-3$
2. $5b(4b+2)$
3. $\frac{6x^3+12x^2+13.5x}{3x}$
4. $(1+i)^3\times(1+i)^{14}$
5. Evaluate the power $8^{2/3}$
6. Substitute the known variables and solve for the unknown variable:

$$I=Prt \text{ where } P=\$2,500,\ r=0.06, \text{ and } t=\frac{135}{365}$$

Applications

For questions 7–11, simplify the algebraic expressions.

7. $(6r^2+10-6r+4r^2-3)-(-12r-5r^2+2+3r)$
8. $\left[\frac{5x^9+3x^9}{2x}\right]^5$
9. $\frac{t}{2}+0.75t-t^3+\frac{5t^4}{t}-\frac{2\left(t+t^3\right)}{4}$
10. $\frac{14(1+i)+21(1+i)^4-35(1+i)^7}{7(1+i)}$
11. $\frac{R}{1+0.08\times\frac{183}{365}}+3R\left(1+0.08\times\frac{52}{365}\right)$
12. Evaluate the power: $\left[\left(\frac{2}{5}\right)^2\right]^2$

For questions 13 and 14, substitute the known variables and solve for the unknown variable.

13. $PV=\frac{FV}{(1+i)^N}$ where $FV=\$3,417.24$, $i=0.05$, and $N=6$
14. $PMT=\frac{FV}{\left[\frac{(1+i)^N-1}{i}\right]}$ where $FV=\$10,000$, $N=17$, and $i=0.10$

Challenge, Critical Thinking, & Other Applications

For questions 15–17, simplify the algebraic expressions.

15. $\left[\frac{10a^2b^3c^4}{5b^3c^4}\right]^2+6\left(a^8\right)^{1/2}-\left(3a^2+6\right)\left(3a^2-3\right)$
16. $\frac{-(5x+4y+3)(2x-5y)-(10x-2y)(2y+3)}{5}$
17. $\frac{(-3z)^3\left(3z^2\right)^2}{\left(2z^3\right)^{-4}}$

18. Substitute the known variables and solve for the unknown variable.

$$FV_{ORD} = PMT(1+\Delta\%)^{N-1}\left[\frac{\left[\frac{(1+i)^{\frac{CY}{PY}}}{(1+\Delta\%)}\right]^{N}-1}{\frac{(1+i)^{\frac{CY}{PY}}}{(1+\Delta\%)}-1}\right]$$ where $PMT = \$500$, $i = 0.05$, $\Delta\% = 0.02$, $CY = 2$, $PY = 4$, and $N = 20$

For questions 19-20, evaluate the expression.

19. $\$50,000 \times \left(1+\frac{0.10}{12}\right)^{-27}$
20. $\$995\left[\frac{1-(1+0.02)^{13}\left(1+\frac{0.09}{4}\right)^{-13}}{\frac{0.09}{4}-0.02}\right]$

2.5: Linear Equations - Manipulating and Solving

Mechanics

Solve the following equations for the unknown variable.

1. $3(x-5) = 15$
2. $12b-3 = 4+5b$
3. $0.75(4m+12)+15-3(2m+6) = 5(-3m+1)+25$

Solve each of the following pairs of equations for both unknown variables.

4. $x+y=6$
 $3x-2y=8$
5. $4h-7q=13$
 $6h+3q=33$
6. $0.25a+\frac{5b}{2}=3.5$
 $\frac{3a}{4}-\frac{b}{0.2}=3$

Applications

In questions 7 and 8, solve the equation for the unknown variable.

7. $\frac{4y}{1.025^4}+y-2y(1.05)^2 = \$1,500$
8. $\$2,500(1+0.06t)+\$1,000(1+0.04t) = \$3,553.62$

For exercises 9–14, read each question carefully and solve for the unknown variable(s).

9. Pamela is cooking a roast for a 5:30 p.m. dinner tonight. She needs to set a delay timer on her oven. The roast takes 1 hour and 40 minutes to cook. The time right now is 2:20 p.m. How long of a delay must she set the oven for (before it automatically turns on and starts to cook the roast)?
10. In 2010, 266 million North Americans were using the Internet, which represented a 146.3% increase in Internet users over the year 2000. How many North American Internet users were there in 2000?
11. A human resource manager is trying to estimate the cost of a workforce accident. These costs usually consist of direct costs (such as medical bills, equipment damage, and legal expenses) and indirect costs (such as decreased output, production delays, and fines). From past experience, she knows that indirect costs average six times as much as direct costs. If she estimates the cost of an accident to be $21,000, what is the direct cost of the accident?
12. In 2011, Canadian federal tax rates were 0% on the first $10,527 of gross income earned, 15% on the next $31,017, 22% on the next $41,544, 26% on the next $45,712, and 29% on anything more. If a taxpayer paid $28,925.35 in federal tax, what was her gross annual income for 2011?

13. St. Boniface Hospital raises funds for research through its Mega Lottery program. In this program, 16,000 tickets are available for purchase at a price of one for \$100 or three for \$250. This year, the lottery sold out with sales of \$1,506,050. To better plan next year's lottery, the marketing manager wants to know how many tickets were purchased under each option this year.
14. An accountant is trying to allocate production costs from two different products to their appropriate ledgers. Unfortunately, the production log sheet for last week has gone missing. However, from other documents he was able to figure out that 1,250 units in total were produced last week. The production machinery was run for 2,562.5 minutes, and he knows that Product A takes 1.5 minutes to manufacture while Product B takes 2.75 minutes to manufacture. How many units of each product were produced last week?

Challenge, Critical Thinking, & Other Applications

15. Jacob owns 15,000 shares in a corporation, which represents 2% of all issued shares for the company. He sold $\frac{2}{5}$ of his shares to another investor for \$7,800. What is the total value for all of the shares issued by the company?
16. Two cell phone companies are offering different rate plans. Rogers is offering \$19.99 per month, which includes a maximum of 200 weekday minutes plus \$0.35 for every minute above the maximum. TELUS is offering \$39.99 for a maximum 300 weekday minutes, but it charges \$0.10 for every minute above the maximum. Above how many minutes would TELUS be the better choice?
17. Marianne, William, Hendrick, and Charlotte have all decided to go into business together. They need \$175,000 in initial capital funding. William was able to contribute 20% less than Marianne, Hendrick contributed 62.5% more than William, and Charlotte contributed \$5,000 less than half as much as Marianne. How much did each partner contribute to the initial funds?
18. A mall is being constructed and needs to meet the legal requirements for parking availability. Parking laws require one parking stall for every 100 square feet of retail space. The mall is designed to have 1,200,000 square feet of retail space. Of the total parking stalls available, 2% need to be handicap accessible, there need to be three times as many small car spaces as handicap spaces, RV spaces need to be one-quarter of the number of small car spaces, and the rest of the spaces are for regular stalls. How many of each type of parking space does the mall require?
19. Simplify the following equation into the format of $\#z=\#$ and find the root. Verify the solution through substitution.

$$z\left(1+0.073\times\frac{280}{365}\right)-\frac{z}{1+0.073\times\frac{74}{365}}+\$1,000=\$2,764.60$$

20. Find the roots for the following pairs of equations. Verify the solution through substitution into both equations.

$$3\frac{4}{5}q+0.18r=12.2398$$
$$-5.13q-\frac{13r}{5}=-38.4674$$

2.6: Natural Logarithms

Mechanics

Using your financial calculator, evaluate each of the following. Write your answer using the $e^x=$ power format. Round your answers to six decimals.

1. $\ln(3.9243)$
2. $\ln(0.7445)$
3. $\ln(1.83921)$
4. $\ln(13.2746)$
5. $\ln(0.128555)$

Applications

For each of the following, demonstrate the fifth property of natural logarithms, where $\ln\left(\frac{x}{y}\right)=\ln(x)-\ln(y)$.

6. $\ln\left(\frac{\$28,500}{\$19,250}\right)$
7. $\ln\left(\frac{\$100,000}{\$10,000}\right)$
8. $\ln\left(\frac{\$75,000}{\$12,255}\right)$

For each of the following, demonstrate the sixth property of natural logarithms, where $\ln(x^y) = y(\ln x)$.

9. $\ln[(1.02)^{23}]$
10. $\ln[(1.01275)^{41}]$
11. $\ln[(1.046)^{34}]$

Using $\dfrac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+i)}$, substitute the known values and evaluate the expression.

	FV	PV	i
12.	\$78,230	\$25,422	0.0225
13.	\$233,120	\$91,450	0.0425
14.	\$18,974	\$8,495	0.02175

Challenge, Critical Thinking, & Other Applications

Using $FV = PV(1+i)^N$, substitute in the known values and solve for N.

	FV	PV	i
15.	\$18,302.77	\$14,000.00	0.015
16.	\$58,870.20	\$36,880	0.01

Review Exercises

Mechanics

1. Solve the following expression: $\left(1+\frac{0.0725}{12}\right)^{\frac{2}{4}}-1$
2. Convert the following into a decimal format and express it in an appropriate manner: $4\frac{12}{37}$
3. In Canadian five-pin bowling, the maximum score per game is 450. Alexandria just finished bowling a game of 321. What percentage of a perfect score does this represent?
4. Simplify the following algebraic expression: $\frac{16x^2}{4x}-\frac{21x^3y^2}{7x^2y^2}$
5. Solve the following equation for r: $-2r-2.25=3.75-8r$
6. Demonstrate that $3\times\ln(2)=\ln(2^3)$.

Applications

7. Solve the following expression: $(0.06)(\$1,725)\left[\frac{1-\left(1+\frac{0.14}{4}\right)^{-7}}{\frac{0.14}{4}}\right]$
8. Evaluate the following: $\left[\frac{1+\frac{\frac{0.0975}{4}\times\$3,225}{\$255}}{1+\frac{0.0975}{4}}\right]$
9. A 240 mL bottle of an orange drink claims that it is made with real fruit juice. Upon examination of the ingredient list, only 15% of the contents is actually fruit juice. How many millilitres of "other ingredients" are in the bottle?
10. On average across all industries, full-time employed Canadians worked 39.5 hours per week. If the typical adult requires eight hours of sleep per day, what percentage of the hours in a single week are left over for personal activities?
11. Substitute the given values in the following equation and solve: $a\left[\frac{(1+b)^c-1}{b}\right]$ where $a=\$535$, $b=0.025$, and $c=6$
12. Simplify the following algebraic expression: $\left(\frac{15a^7b^3c^2}{5a^5c}\right)^3\div\left(\frac{9a^4b^6c^5}{3a^2b^2c^3}\right)^{\frac{3}{2}}$

13. Simplify the following algebraic expression: $PMT\left[\frac{1-(1+0.0225)^{6}(1+0.044)^{-6}}{0.044-0.0225}\right]+PMT\left(\frac{1}{0.044}+1\right)$
14. Housing starts in Winnipeg for the first half of the 2009 year were down 46.733% from 2008, when 1,500 new homes were started. How many housing starts were there in 2009?
15. A local baseball team sells tickets with two price zones. Seats behind the plate from first base to third base are priced at $20 per ticket. All other seats down the base lines and in the outfield are priced at $10 per ticket. At last night's game, 5,332 fans were in attendance and total ticket revenue was $71,750. How many tickets in each zone were sold?

Challenge, Critical Thinking, & Other Applications

16. An industry analyst wants to compare the average salaries of two firms, both to each other and to the industry. Firm A's average salary is 93% of the industry average, Firm B's average salary is $58,000, and the industry average salary is 96% of Firm B's average salary.
    a. Determine the industry average salary.
    b. Determine Firm A's average salary.
    c. Express Firm B's average salary as a percentage of Firm A's average salary. Round the percentage to two decimals.
17. Solve the following expression: $\$500\left[\frac{\left(1+\left\{\left(1+\frac{0.07}{2}\right)^{4/12}-1\right\}\right)^{4}-1}{\left(1+\frac{0.07}{2}\right)^{4/12}-1}\right]$
18. Francesca operates an online flower delivery business. She learned that in 2012 there were an estimated 324,000 online orders in her region, which represented a 1 12 increase over the previous year. Her business received 22,350 online orders in 2012, which represented a 3 14 increase over her previous year. In 2011, what percentage of online orders were placed with Francesca's business?
19. Goodyear Tires just completed a "Buy Three Get One Free" promotion on its ultra-grip SUV tires, regularly priced at $249.99 per tire. Over the course of the promotion, 1,405 tires were sold resulting in sales of $276,238.95. How many tires were sold at the regular price and how many tires were sold at the special promotional price?
20. The current price of $41.99 for a monthly cellphone rate plan is 105% of last year's rate plan. There are currently 34,543 subscribers to the plan, which is 98% of last year. Express the current total revenue as a percentage of last year's total revenue. Round the final answer to one decimal.

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CHAPTER OVERVIEW

3: General Business Management Applications (Get Your Motor Running)

This chapter covers universal business mathematics you will use whether your chosen business profession is marketing, accounting, production, human resources, economics, finance, or something else altogether. To be a successful manager you need to understand percent changes, averages, ratios, proportions, and prorating.

3.0: Introduction
3.1: Percent Change (Are We Up Or Down?)
3.2: Averages (What Is Typical?)
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3.S: Summary
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3.E: General Business Management Applications (Exercises)

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3.0: Introduction

Basic calculations are so much a part of your everyday life that you could not escape them if you tried. While driving to school, you may hear on the radio that today's forecasted temperature of 27°C is 3°C warmer than the historical average. In the news, consumers complain that gas prices rose 11% in three months. A commercial tells you that the Toyota Prius you are driving uses almost 50% less gas than the Honda Civic Coupe Si. After class you head to Anytime Fitness to cancel a gym membership because you are too busy to work out. You have used only four months of an annual membership for which you paid $449, and now you want to know how large a prorated refund you are eligible for.

On a daily basis you use basic calculations such as averages (the temperature), percent change (gas prices), ratios and proportions (energy efficiency), and prorating (the membership refund). To invest successfully, you must also apply these basic concepts. Or if you are an avid sports fan, you need basic calculations to understand your favourite players' statistics.

And it's not hard to see how these calculations would be used in the business world. Retail management examines historical average daily sales to predict future sales and to schedule the employees who will service those sales. Human resource managers continually calculate ratios between accounting and performance data to assess how efficiently labour is being used. Managers proration a company's budget across its various departments.

This chapter covers universal business mathematics you will use whether your chosen business profession is marketing, accounting, production, human resources, economics, finance, or something else altogether. To be a successful manager you need to understand percent changes, averages, ratios, proportions, and prorating.

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3.1: Percent Change (Are We Up Or Down?)

On your way to work, you notice that the price of gasoline is about 10% higher than it was last month. At the office, reports indicate that input costs are down 5.4% and sales are up 3.6% over last year. Your boss asks you to analyze the year-over-year change in industry sales and submit a report. During your coffee break, you look through the day's e-flyers in your inbox. Home Depot is advertising that all garden furniture is 40% off this week; Safeway's ad says that Tuesday is 10% off day; and a Globe and Mail story informs you that the cost of living has risen by 3% since last year. You then log in to your financial services account, where you are happy to find that the mutual funds in your RRSP are up 6.7% from last year. What are you going to do with all this information?

Understanding how data changes from one period to the next is a critical business skill. It allows for quick assessment as to whether matters are improving or getting worse. It also allows for easy comparison of changes in different types of data over time. In this section, the concept of percent change is explored, which allows for the calculation of change between two points in time. Then a rate of change over time is introduced, which allows you to determine the change per period when multiple points in time are involved in the calculation.

Percent Change

It can be difficult to understand a change when it is expressed in absolute terms. Can you tell at a glance how good a deal it is to buy a $359 futon for $215.40? It can also be difficult to understand a change when it is expressed as a percentage of its end result. Are you getting a good deal if that $215.40 futon is 60% of the regular price? What most people do find easier to understand is a change expressed as a percentage of its starting amount. Are you getting a good deal if that $359 futon is on sale at 40% off? A percent change expresses in percentage form how much any quantity changes from a starting period to an ending period.

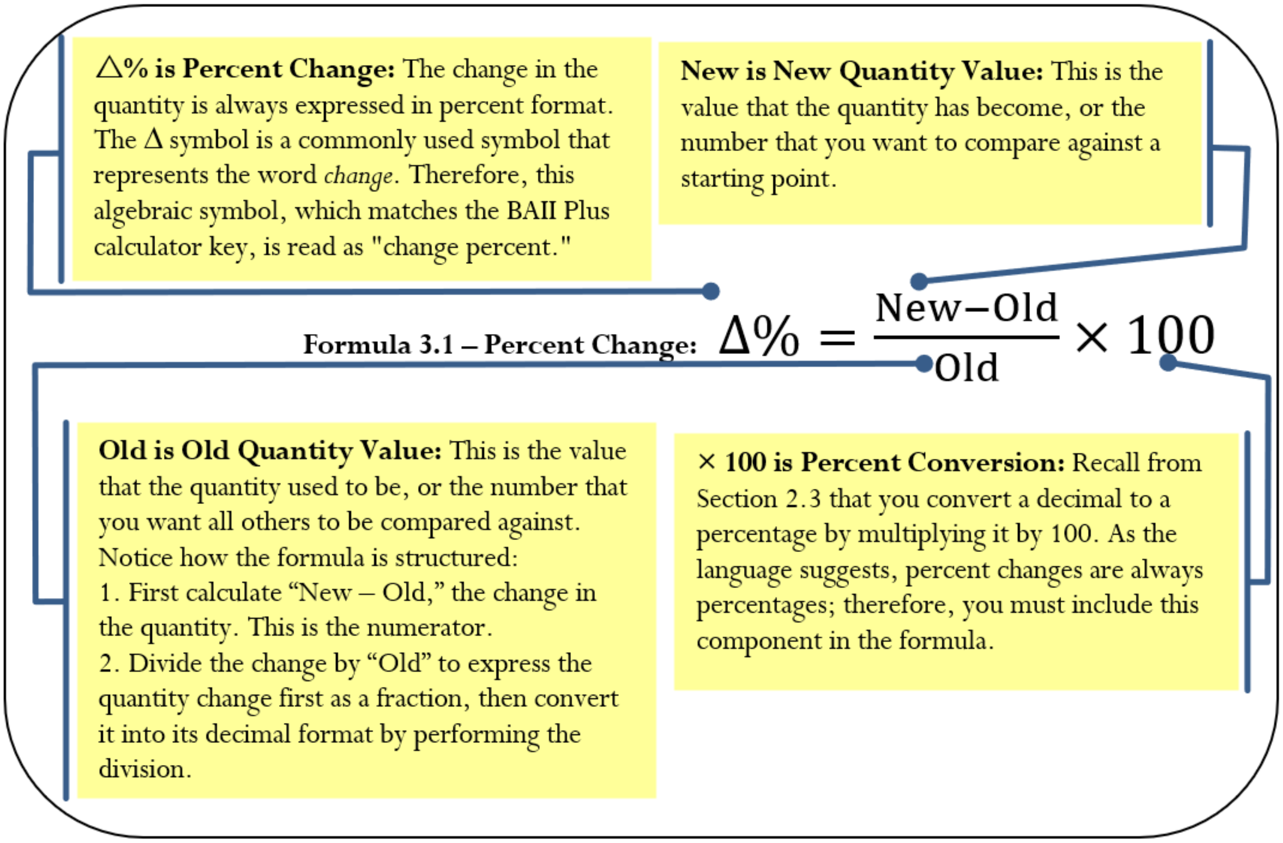
The Formula

To calculate the percent change in a variable, you need to know the starting number and the ending number. Once you know these, Formula 3.1 (see the figure below) represents how to express the change as a percentage.

Remember two critical concepts about percent change:

1. **Do Not Include the Original Quantity in the Change**. Percent change represents the changes in the quantities, not the values of the quantities themselves. The original quantity does not count toward the percent change. Therefore, if any quantity doubles, its percent change is 100%, not 200%. For example, if the old quantity was 25 and the new quantity is 50, note that the quantity has doubled. However, 25 of the final 50 comes from the original amount and therefore does not count toward the change. We subtract it out of the numerator through calculating 50 − 25 = 25. Therefore, the change as a percentage is $\frac{50-25}{25} \times 100 = 100\%$ The same applies to a tripling of quantity. If our new quantity is 75 (which is triple the old quantity of 25), then $\frac{75-25}{25} \times 100 = 200\%$ The original value of 25 is once again subtracted out of the numerator. The original 100% is always removed from the formula.
2. **Negative Changes**. A negative change must be expressed with a negative sign or equivalent wording. For example, if the old quantity was 20 and the new quantity is 15, this is a decrease of 5, or a change of 15 − 20 = −5. The percent change is $\frac{15-20}{20} \times 100 = -25\%$. Be careful in expressing a negative percent change. There are two correct ways to do this properly:
    a. "The change is −25%."
    b. "It has decreased by 25%."

Note in the second statement that the word "decreased" replaces the negative sign. To avoid confusion, do not combine the negative sign with the word "decreased"—recall that two negatives make a positive, so "It has decreased by −25%" would actually mean the quantity has increased by 25%.



∆% is Percent Change: The change in the quantity is always expressed in percent format. The ∆ symbol is a commonly used symbol that represents the word *change*. Therefore, this algebraic symbol, which matches the BAII Plus calculator key, is read as "change percent."

New is New Quantity Value: This is the value that the quantity has become, or the number that you want to compare against a starting point.

Formula 3.1 – Percent Change: $\Delta\% = \frac{New - Old}{Old} \times 100$

Old is Old Quantity Value: This is the value that the quantity used to be, or the number that you want all others to be compared against. Notice how the formula is structured:
1. First calculate "New – Old," the change in the quantity. This is the numerator.
2. Divide the change by "Old" to express the quantity change first as a fraction, then convert it into its decimal format by performing the division.

× 100 is Percent Conversion: Recall from Section 2.3 that you convert a decimal to a percentage by multiplying it by 100. As the language suggests, percent changes are always percentages; therefore, you must include this component in the formula.

Formula 3.1

How It Works

To solve any question about percent change, follow these steps:

Step 1: Notice that there are three variables in the formula. Identify the two known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.1.

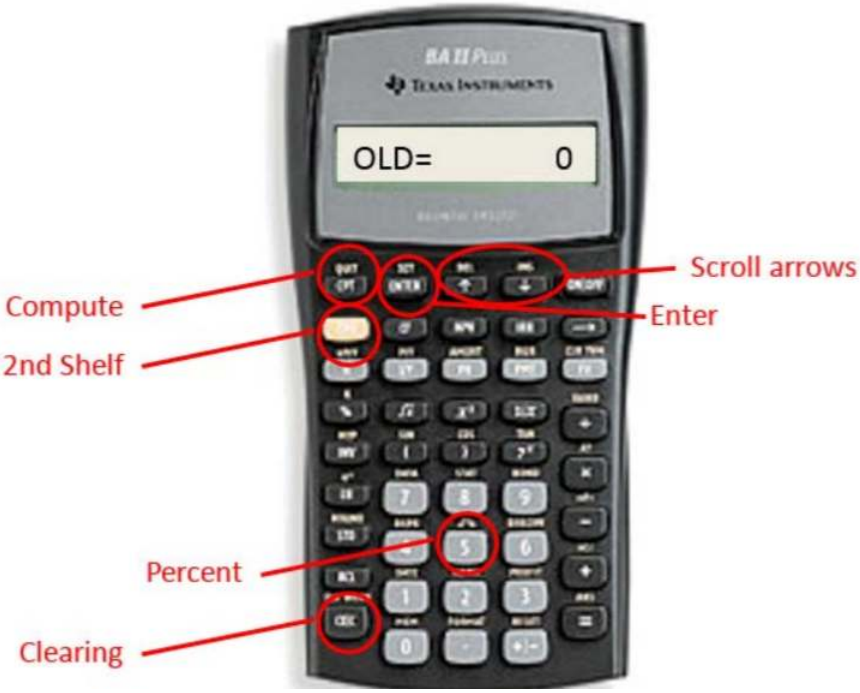
Assume last month your sales were $\$10,000$, and they have risen to $\$15,000$ this month. You want to express the percent change in sales.

Step 1: The known variables are $Old = \$10,000$ and $New = \$15,000$. The unknown variable is percent change, or $\Delta\%$.

Step 2: Using Formula 3.1,

Observe that the change in sales is +$5,000 month-over-month. Relative to sales of $10,000 last month, this month's sales have risen by 50%.

Important Notes



To access the percent change function on your BAII Plus calculator, press 2nd $\Delta\%$ located above the number 5 on your keypad. Always clear the memory of any previous question by pressing 2nd CLR Work once the function is open. Use the $\uparrow$ and $\downarrow$ arrows to scroll through the four lines of this function. To solve for an unknown variable, key in three of the four variables and then press Enter. With the unknown variable on your display, press CPT. Each variable in the calculator is as follows:

- OLD = The old or original quantity; the number that represents the starting point
- NEW = The new or current quantity; the number to compare against the starting point
- %CH = The percent change, or Δ% in its percent format without the % sign
- #PD = Number of consecutive periods for change. By default, it is set to 1. For now, do not touch this variable. Later in this section, when we cover rate of change over time, this variable will be explained.

Things To Watch Out For

Watch out for two common difficulties involving percent changes.

1. **Rates versus Percent Changes**. Sometimes you may be confused about whether questions involve rates (Section 2.3) or percent changes. Recall that a rate expresses the relationship between a portion and a base. Look for some key identifiers, such as "is/are" (the portion) and "of" (the base). For percent change, key identifiers are "by" or "than." For example, "x has increased by $y\%$" and "x is $y\%$ more than last year" both represent a percent change.
2. **Mathematical Operations**. You may imagine that you are supposed to add or subtract percent changes, but you cannot do this. Remember that percentages come from fractions. According to the rules of algebra, you can add or subtract fractions only if they share the same base (denominator). For example, if an investment increases in value in the first year by 10% and then declines in the second year by 6%, this is not an overall increase of $10\% - 6\% = 4\%$. Why? If you originally had $\$100$, an increase of 10% (which is $\$100 \times 10\% = \$10)$ results in $\$110$ at the end of the first year. You must calculate the 6% decline in the second year using the $\$110$ balance, not the original $\$100$. This is a decline of $\$110 \times (-6\%) = -\$6.60$, resulting in a final balance of $\$103.40$. Overall, the percent change is 3.4%.

Paths To Success

A percent change extends the rate, portion, and base calculations introduced in Section 2.3. The primary difference lies in the portion. Instead of the portion being a part of a whole, the portion represents the change of the whole. Putting the two formulas side by side, you can calculate the rate using either approach.

$$\text{Rate } = \frac{\text{Portion}}{\text{Base}} = \frac{\text{New } - \text{ Old}}{\text{Old}}$$

? Exercise 3.1.1: Give It Some Thought

1. It has been five years since Juan went shopping for a new car. On his first visit to a car lot, he had sticker shock when he realized that new car prices had risen by about 20%. What does this situation involve?
   a. Percent change
   b. Rate, portion, base
2. Manuel had his home custom built in 2006 for $300,000. In 2014 he had it professionally appraised at $440,000. He wants to figure out how much his home has appreciated. How would he do so?
   a. The 2006 price is the "New," and the 2014 price is the "Old."
   b. The 2006 price is the "Old," and the 2014 price is the "New."

Answer

1. a (the question involves how car prices have changed; note the key word "by")
2. b (the 2006 price is what the house used to be worth, which is the old quantity; the 2014 price represents the new value of the home)

✓ Example 3.1.1: Price of New Cars in Canada

In 1982, the average price of a new car sold in Canada was $10,668. By 2009, the average price of a new car had increased to $25,683. By what percentage has the price of a new car changed over these years?

Solution

You are trying to find the percent change in the price of the new car, or $\Delta\%$.

What You Already Know

Step 1:

You know the old and new prices for the cars:

Old = $10,668, New = $25,683

How You Will Get There

Step 2:

Apply Formula 3.1.

Perform

Step 2:

$$\begin{aligned}\Delta\% &= \frac{\$25,683 - \$10,668}{\$10,668} \times 100 \\ &= \frac{\$15,015}{\$10,668} \times 100 \\ &= 140.748\%\end{aligned}$$

Calculator Instructions

OLD	NEW	%CH	#PD
10668	25683	Answer: 140.748031	1

From 1982 to 2009, average new car prices in Canada have increased by 140.748%.

✓ Example 3.1.2: Figuring Out a Price after Taxes

When you purchase a retail item, the tax increases the price of the product. If you buy a $799.00 Bowflex Classic Home Gym machine in Ontario, it is subject to 13% HST. What amount do you pay for the Bowflex including taxes?

Solution

You are looking for the price of the Bowflex after increasing it by the sales tax. This is a "New" price for the Bowflex.

What You Already Know

Step 1:

You know the original price of the machine and how much to increase it by:

$\text{Old } = \$799.00 \quad \Delta\% = 13\%$

How You Will Get There

Step 2:

Apply Formula 3.1: $\Delta\% = \dfrac{\text{New } - \text{ Old}}{\text{Old}} \times 100$

Perform

Step 2:

$$\begin{aligned}
13\% &= \frac{\text{New } - \$799}{\$799} \times 100 \\
0.13 &= \frac{\text{New } - \$799}{\$799} \\
\$103.87 &= \text{ New } - \$799 \\
\$902.87 &= \text{ New}
\end{aligned}$$

Calculator Instructions

OLD	NEW	%CH	#PD
799	Answer: 902.87	13	1

The price of a Bowflex, after increasing the price by the taxes of 13%, is $902.87.

✓ Example 3.1.3: Handling Price Changes

Consumers often object to price changes in many daily products, even though inflation and other cost changes may justify these increases. To reduce the resistance to a price increase, many manufacturers adjust both prices and product sizes at the same time. For example, the regular selling price for a bottle of shampoo was $5.99 for 240 mL. To account for cost changes, the manufacturer decided to change the price to $5.79, but also reduce the bottle size to 220 mL. What was the percent change in the price per millilitre?

Solution

You need to find the percent change in the price per millilitre, or Δ%.

What You Already Know

Step 1:

You know the old price and bottle size, as well as the planned price and bottle size:

$$\begin{aligned}
&\text{Old price } = \$5.99 \quad \text{Old size } = 240\text{mL} \\
&\text{New price } = \$5.79 \quad \text{New size } = 220\text{mL}
\end{aligned}$$

How You Will Get There

Step 1 (continued):

Convert the price and size to a price per millilitre by taking the price and dividing by the size.

Step 2:

Apply Formula 3.1.

Perform

Step 1:

$$\frac{\text{Old price}}{\text{Old size}} = \frac{\$5.99}{240\text{mL}} = \$0.024958 \text{ per mL}$$

$$\frac{\text{New price}}{\text{New size}} = \frac{\$5.79}{220\text{mL}} = \$0.026318 \text{ per mL}$$

Step 2:

$$\begin{aligned}\Delta\% &= \frac{\$0.026318 - \$0.024958}{\$0.024958} \times 100\\ &= \frac{\$0.001359}{\$0.024958} \times 100 = 5.4485\%\end{aligned}$$

Calculator Instructions

OLD	NEW	%CH	#PD
$5.99 \div 240 =$	$5.79 \div 200 =$	Answer: 5.4484	1

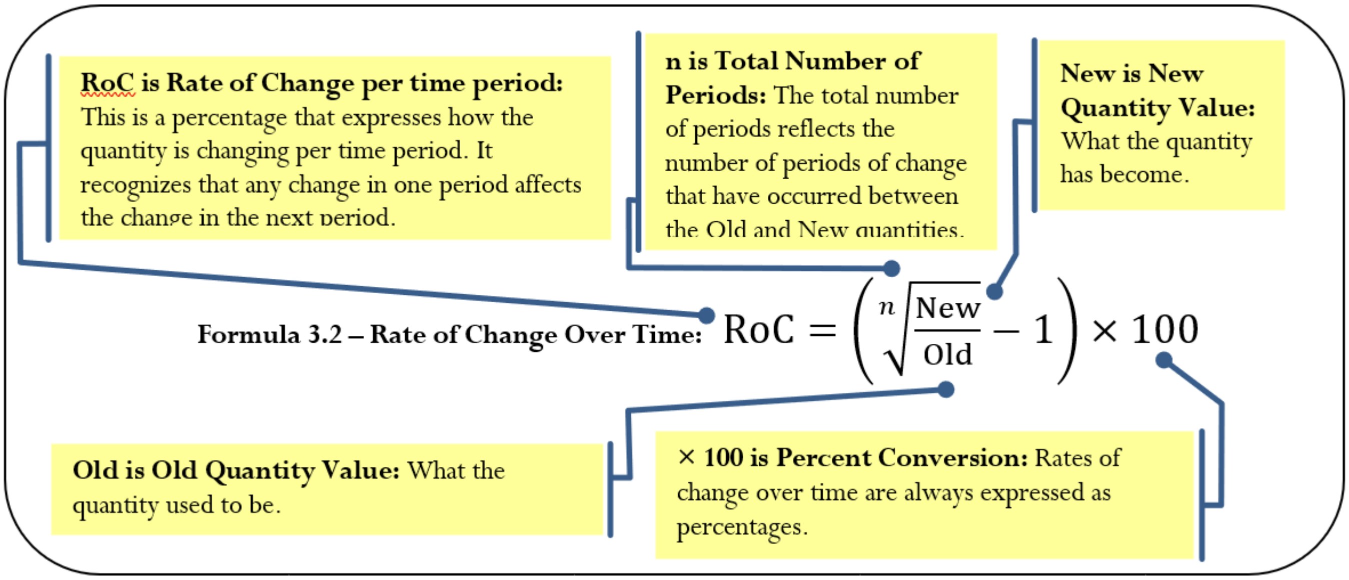
The price per mL has increased by 5.4485%. Note that to the consumer, it would appear as if the price has been lowered from $5.99 to $5.79, as most consumers would not notice the change in the bottle size.

Rate of Change over Time

The percent change measures the change in a variable from start to end overall. It is based on the assumption that only a single change occurs. What happens when the ending number results from multiple changes and you want to know the typical value of each change? For example, the population of the Toronto census metropolitan area (CMA) has grown from 4,263,759 in 1996 to 5,113,149 in 2006. What annual percentage growth in population does this reflect? Notice that we are not interested in calculating the change in population over the 10 years; instead we want to determine the percentage change in each of the 10 years. The rate of change over time measures the percent change in a variable per time period.

The Formula

Calculating the rate of change over time is not as simple as dividing the percent change by the number of time periods involved, because you must consider the change for each time period relative to a different starting quantity. For example, in the Toronto census example, the percent change from 1996 to 1997 is based on the original population size of 4,263,759. However, the percent change from 1997 to 1998 is based on the new population figure for 1997. Thus, even if the same number of people were added to the city in both years, the percent change in the second year is smaller because the population base became larger after the first year. As a result, when you need the percent change per time period, you must use Formula 3.2.



RoC is Rate of Change per time period: This is a percentage that expresses how the quantity is changing per time period. It recognizes that any change in one period affects the change in the next period.

n is Total Number of Periods: The total number of periods reflects the number of periods of change that have occurred between the Old and New quantities.

New is New Quantity Value: What the quantity has become.

Formula 3.2 – Rate of Change Over Time: $\text{RoC} = \left(\sqrt[n]{\frac{\text{New}}{\text{Old}}} - 1\right) \times 100$

Old is Old Quantity Value: What the quantity used to be.

× 100 is Percent Conversion: Rates of change over time are always expressed as percentages.

Formula 3.2

How It Works

When you work with any rate of change over time, follow these steps:

Step 1: Identify the three known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.2.

Important Notes

On your calculator, calculate the rate of change over time using the percent change ($\Delta\%$) function. Previously, we had ignored the #PD variable in the function and it was always assigned a value of 1. In rate of change, this variable is the same as n in our equation. Therefore, if our question involved 10 changes, such as the annual population change of the Toronto CMA from 1996 to 2006, then this variable is set to 10.

Paths To Success

You may find it difficult to choose which formula to use: percent change or rate of change over time. To distinguish between the two, consider the following:

1. If you are looking for the overall rate of change from beginning to end, you need to calculate the percent change.
2. If you are looking for the rate of change per interval, you need to calculate the rate of change over time.

Ultimately, the percent change formula is a simplified version of the rate of change over time formula where $n = 1$. Thus you can solve any percent change question using Formula 3.2 instead of Formula 3.1.

? Exercise 3.1.2: Give It Some Thought

For each of the following, distinguish whether you should solve the question by the percent change formula or the rate of change over time formula.

a. When Peewee started five-pin bowling with the Youth Bowling Canada (YBC) in 1997, his average was 53. In 2011, he finished his last year of the YBC with an average of 248. How did his average change from 1997 to 2011?
b. A stock was priced at $4.34 per share in 2006 and reached $7.15 per share in 2012. What annual return did a shareholder realize?
c. In 2004, total sales reached $1.2 million. By 2010, sales had climbed to $4.25 million. What is the growth in sales per year?

Answer

a. Percent change; looking for overall change

b. Rate of change over time; looking for change per year
c. Rate of change over time; looking for change per year

✓ Example 3.1.4: Population Growth

Using the Toronto CMA information, where the population grew from 4,263,759 in 1996 to 5,113,149 in 2006, calculate the annual percent growth in the population.

Solution

We need to calculate the percent change per year, which is the rate of change over time, or RoC.

What You Already Know

Step 1:

We know the starting and ending numbers for the population along with the time frame involved.

Old = 4,263,759, New = 5,113,149

$n = 2006 - 1996 = 10$ years

How You Will Get There

Step 2:

Apply Formula 3.2.

Perform

Step 2:

$$\begin{aligned} RoC &= \left(\sqrt[10]{\frac{5,113,149}{4,263,759}} - 1\right) \times 100 \\ &= (\sqrt[10]{1.199211} - 1) \times 100 \\ &= (1.018332 - 1) \times 100 \\ &= 0.018332 \times 100 \\ &= 1.8332\% \end{aligned}$$

Calculator Instructions

OLD	NEW	%CH	#PD
4263759	5113149	Answer: 1.833244	10

Over the 10 year span from 1996 to 2006, the CMA of Toronto grew by an average of 1.8332% per year.

✓ Example 3.1.5: Percent Changes and Rate of Change Together

Kendra collects hockey cards. In her collection, she has a rookie year Wayne Gretzky card in mint condition. The book value of the card varies depending on demand for the card and its condition. If the estimated book value of the card fell by $84 in the first year and then rose by $113 in the second year, determine the following:

a. What is the percent change in each year if the card is valued at $1,003.33 at the end of the first year?
b. Over the course of the two years, what was the overall percent change in the value of the card?
c. What was the rate of change per year?

Solution

We need to provide four answers to the questions and find the percent change in year 1, or $\Delta\%_1$, then the percent change in year 2, or $\Delta\%_2$. Using these first two solutions, we calculate both the overall percent change across both years, or $\Delta\%_{\text{overall}}$, and the rate of change per year, or RoC.

What You Already Know

Step 1:

We know the price of the card at the end of the first year as well as how it has changed each year.

Price at end of first year = $1,003.33

Price change in first year = −$84

Price change in second year = $113

How You Will Get There

Step 2: Calculate the price at the beginning of the first year.

Step 3: For the percent change in Year 1, apply Formula 3.1: $\Delta\%_1$

Step 4: Calculate the price at the end of the second year.

Step 5: For the percent change in Year 2, apply Formula 3.1: $\Delta\%_2$

Step 6: For the overall percent change, take the old price at the beginning of the first year and compare it to the new price at the end of the second year.

Apply Formula 3.1: $\Delta\%_{\text{overall}}$

Step 7: Calculate the rate of change over the two years using the same two prices, but apply Formula 3.2.

Perform

Step 2:

$$\text{New} = \text{Old} + \text{ Change}; \quad \$1,003.33 = \text{Old} - \$84.00; \quad \$1,087.33 = \text{Old}$$

Step 3:

$$\Delta\%_1 = \frac{\$1,003.33 - \$1,087.33}{\$1,087.33} \times 100 = -7.7253\%$$

Step 4:

$$\text{New } = \text{Old} + \text{ Change } = \$1,003.33 + \$113 = \$1,116.33$$

Step 5:

$$\Delta\%_2 = \frac{\$1,116\cdot 33 - \$1,003.33}{\$1,003.33} \times 100 = 11.2625\%$$

Step 6:

$$\Delta\%_{\text{overall }} = \frac{\$1,116.33 - \$1,087.33}{\$1,087.33} \times 100 = 2.6671\%$$

Step 7:

$$RoC = (\sqrt[2]{\frac{\$1,116\cdot 33}{\$1,087\cdot 33}} - 1) \times 100 = 1.3248\%$$

Calculator Instructions

Calculation	OLD	NEW	%CH	#PD
Year 1	1087.33	1003.33	Answer: -7.725345	1
Year 2	1003.33	1116.33	Answer: 11.262495	1
Both Years	1087.33	1116.33	Answer: 2.667083	1
RoC	1087.33	1116.33	Answer: 1.324766	2

> The value of the hockey card dropped 7.7253% in the first year and rose 11.2625% in the second year. Overall, the card rose by 2.6671% across both years, which represents a growth of 1.3248% in each year.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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3.2: Averages (What Is Typical?)

No matter where you go or what you do, averages are everywhere. Let's look at some examples:

- Three-quarters of your student loan is spent. Unfortunately, only half of the first semester has passed, so you resolve to squeeze the most value out of the money that remains. But have you noticed that many grocery products are difficult to compare in terms of value because they are packaged in different sized containers with different price points? For example, one tube of toothpaste sells in a 125 mL size for \$1.99 while a comparable brand sells for \$1.89 for 110 mL. Which is the better deal? A fair comparison requires you to calculate the average price per millilitre.
- Your local transit system charges \$2.25 for an adult fare, \$1.75 for students and seniors, and \$1.25 for children. Is this enough information for you to calculate the average fare, or do you need to know how many riders of each kind there are?
- Five years ago you invested \$8,000 in Roller Coasters Inc. The stock value has changed by 9%, −7%, 13%, 4%, and −2% over these years, and you wonder what the average annual change is and whether your investment kept up with inflation.
- If you participate in any sport, you have an average of some sort: bowlers have bowling averages; hockey or soccer goalies have a goals against average (GAA); and baseball pitchers have an earned run average (ERA).

Averages generally fall into three categories. This section explores simple, weighted, and geometric averages.

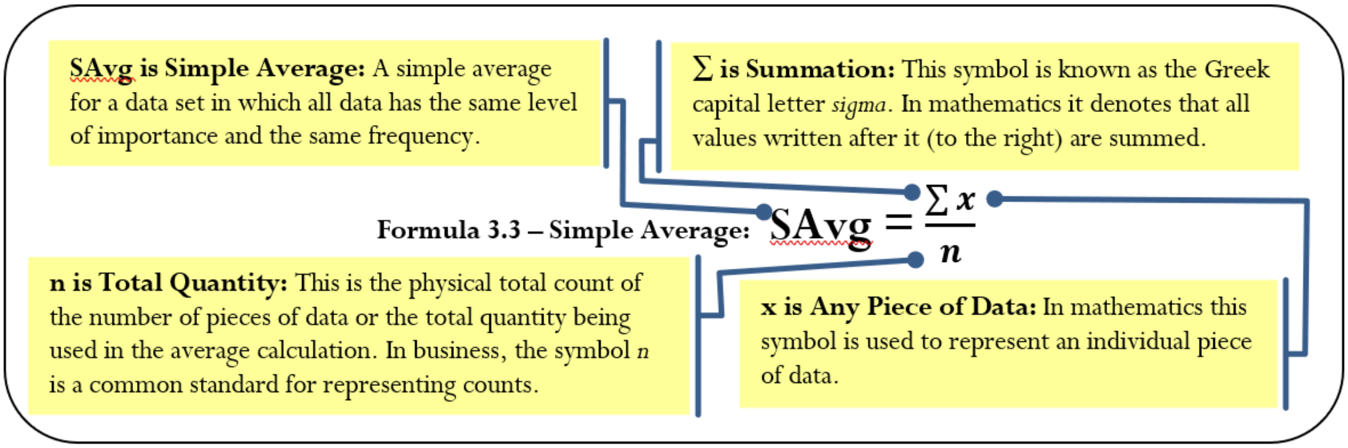
Simple Averages

An average is a single number that represents the middle of a data set. It is commonly interpreted to mean the "typical value." Calculating averages facilitates easier comprehension of and comparison between different data sets, particularly if there is a large amount of data. For example, what if you want to compare year-over-year sales? One approach would involve taking company sales for each of the 52 weeks in the current year and comparing these with the sales of all 52 weeks from last year. This involves 104 weekly sales figures with 52 points of comparison. From this analysis, could you concisely and confidently determine whether sales are up or down? Probably not. An alternative approach involves comparing last year's average weekly sales against this year's average weekly sales. This involves the direct comparison of only two numbers, and the determination of whether sales are up or down is very clear.

In a simple average, all individual data share the same level of importance in determining the typical value. Each individual data point also has the same frequency, meaning that no one piece of data occurs more frequently than another. Also, the data do not represent a percent change. To calculate a simple average, you require two components:

1. The data itself—you need the value for each piece of data.
2. The quantity of data—you need to know how many pieces of data are involved (the count), or the total quantity used in the calculation.

The Formula



SAvg is Simple Average: A simple average for a data set in which all data has the same level of importance and the same frequency.

∑ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

Formula 3.3 – Simple Average: $\text{SAvg} = \dfrac{\sum x}{n}$

n is Total Quantity: This is the physical total count of the number of pieces of data or the total quantity being used in the average calculation. In business, the symbol *n* is a common standard for representing counts.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

Formula 3.3

As expressed in Formula 3.3, you calculate a simple average by adding together all of the pieces of data then taking that total and dividing it by the quantity.

How It Works

The steps required to calculate a simple average are as follows:

Step 1: Sum every piece of data.

Step 2: Determine the total quantity involved.

Step 3: Calculate the simple average using Formula 3.3.

Assume you want to calculate an average on three pieces of data: 95, 108, and 97. Note that the data are equally important and each appears only once, thus having the same frequency. You require a simple average.

Step 1: $\sum X = 95 + 108 + 97 = 300$.

Step 2: There are three pieces of data, or $n = 3$.

Step 3: Apply Formula 3.3: $\text{SAvg} = \dfrac{300}{3} = 100$. The simple average of the data set is 100.

Important Notes

Although mentioned earlier, it is critical to stress that a simple average is calculated only when all of the following conditions are met:

1. All of the data shares the same level of importance toward the calculation.
2. All of the data appear the same number of times.
3. The data does not represent percent changes or a series of numbers intended to be multiplied with each other.

If any of these three conditions are not met, then either a weighted or geometric average is used depending on which of the above criteria failed. We discuss this later when each average is introduced.

? Exercise $3.2.1$: Give It Some Thought

It is critical to recognize if you have potentially made any errors in calculating a simple average. Review the following situations and, without making any calculations, determine the best answer.

1. The simple average of 15, 30, 40, and 45 is:
   a. lower than 20.
   b. between 20 and 40, inclusive.
   c. higher than 40.
2. If the simple average of three pieces of data is 20, which of the following data do not belong in the data set? Data set: 10, 20, 30, 40
   a. 10
   b. 20
   c. 30
   d. 40

Answer

1. b (a simple average should fall in the middle of the data set, which appears spread out between 15 and 45, so the middle would be around 30)
2. d (if the number 40 is included in any average calculation involving the other numbers, it is impossible to get a low average of 20)

✓ Example $3.2.1$: Comparing Average Sales

First quarter sales for Buzz Electronics are as indicated in the table below.

Month	2013 Sales	2014 Sales
January	$413,200	$455,875

Month	2013 Sales	2014 Sales
January	$413,200	$455,876
February	$328,987	$334,582
March	$350,003	$312,777

Martha needs to prepare a report for the board of directors comparing year-over-year quarterly performance. To do this, she needs you to do the following:

a. Calculate the average sales in the quarter for each year.
b. Express the 2014 sales as a percentage of the 2013 sales, rounding your answer to two decimals.

Solution

You need to calculate a simple average, or SAvg, for the first quarter in each of 2013 and 2014. Then convert the numbers into a percentage.

What You Already Know

You know the three quarters annually:

2013: $X_1 = \$413,200 \; X_2 = \$328,986 \; X_3 = \$350,003$

2014: $X_1 = \$455,876 \; X_2 = \$334,582 \; X_3 = \$312,777$

How You Will Get There

For each year, perform steps 1 to 3:

Step 1: Sum the data.

Step 2: Count the total quantity of data involved.

Step 3: Calculate the simple average using Formula 3.3.

Step 4: To calculate the rate expressed as a percentage, apply Formula 2.2 from Chapter 2, $\text{Rate } = \frac{\text{Portion}}{\text{Base}}$, treating the 2013 average sales as the base (note the key word "of" in the question) and the 2014 sales as the portion. Round to two decimals.

Perform

Year 2013

Step 1:

$$\begin{aligned}\sum X &= s413,200 + \$328,986 + \$350,003 \\ &= \$1,092,189\end{aligned}$$

Step 2:

$$n = 3$$

Step 3:

$$\mathrm{SAvg} = \frac{\$1,092,189}{3} = \$364,063$$

Step 4:

$$\% = \frac{\$367,745}{\$364,063} \times 100 = 101.01\%$$

Year 2014

Step 1:

$$\begin{aligned}\sum X &= \$455,876 + \$334,582 + \$312,777 \\ &= \$1,103,235\end{aligned}$$

Step 2:

$$n = 3$$

Step 3:

$$\text{SAvg} = \frac{\$1,103,235}{3} = \$367,745$$

The average monthly sales in 2013 were $364,063 compared to sales in 2014 of $367,745. This means that 2014 sales are 101.01% of 2013 sales.

Weighted Averages

Have you considered how your grade point average (GPA) is calculated? Your business program requires the successful completion of many courses. Your grades in each course combine to determine your GPA; however, not every course necessarily has the same level of importance as measured by your course credits.

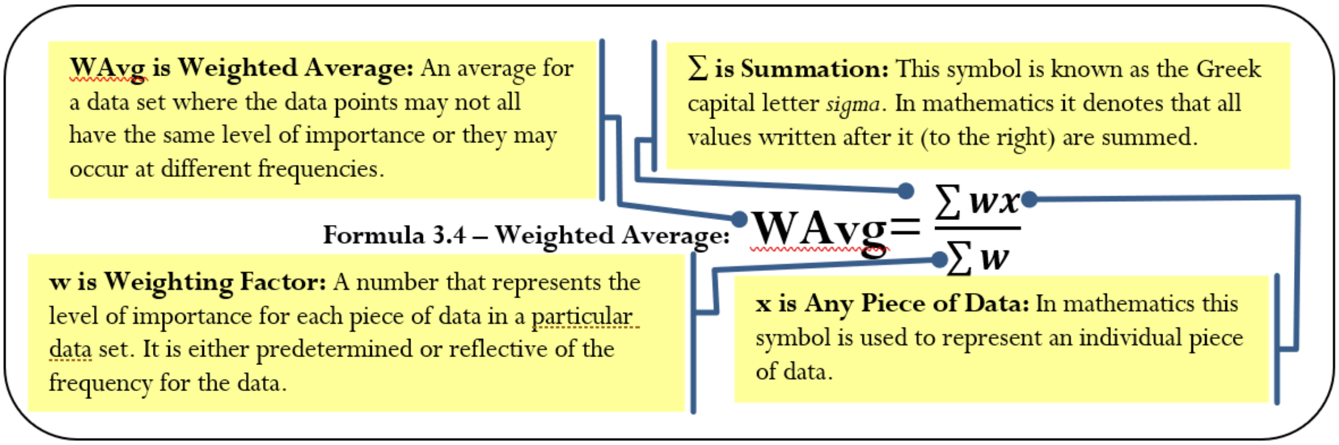
Perhaps your math course takes one hour daily while your communications course is only delivered in one-hour sessions three times per week. Consequently, the college assigns the math course five credit hours and the communications course three credit hours. If you want an average, these different credit hours mean that the two courses do not share the same level of importance, and therefore a simple average cannot be calculated.

In a weighted average, not all pieces of data share the same level of importance or they do not occur with the same frequency. The data cannot represent a percent change or a series of numbers intended to be multiplied with each other. To calculate a weighted average, you require two components:

1. The data itself—you need the value for each piece of data.
2. The weight of the data—you need to know how important each piece of data is to the average. This is either an assigned value or a reflection of the number of times each piece of data occurs (the frequency).

The Formula

As expressed in Formula 3.4, calculate a weighted average by adding the products of the weights and data for the entire data set and then dividing this total by the total of the weights.



WAvg is Weighted Average: An average for a data set where the data points may not all have the same level of importance or they may occur at different frequencies.

$\sum$ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

Formula 3.4 – Weighted Average: $\text{WAvg} = \dfrac{\sum wx}{\sum w}$

w is Weighting Factor: A number that represents the level of importance for each piece of data in a particular data set. It is either predetermined or reflective of the frequency for the data.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

Formula 3.4

How It Works

The steps required to calculate a weighted average are:

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using Formula 3.4.

Let's stay with the illustration of the math and communications courses and your GPA. Assume that these are the only two courses you are taking. You finish the math course with an A, translating into a grade point of 4.0. In the communications course, your C+

translates into a 2.5 grade point. These courses have five and three credit hours, respectively. Since they are not equally important, you use a weighted average.

Step 1: In the numerator, sum the products of each course's credit hours (the weight) and your grade point (the data). This means (math credit hours × math grade point) + (communications credit hours × communications grade point). Numerically, this is $\sum WX = (5 \times 4) + (3 \times 2.5) = 27.5$.

Step 2: In the denominator, sum the weights. These are the credit hours. You have $\sum W = 5 + 3 = 8$.

Step 3: Apply Formula 3.4 to calculate your GPA.

$\text{WAvg} = \frac{27.5}{8} = 3.44$ (GPAs have two decimals).

Note that your GPA is higher than if you had just calculated a simple average of $\frac{4+2.5}{2} = 3.25$. This happens because your math course, in which you scored a higher grade, was more important in the calculation.

Things To Watch Out For

The most common error in weighted averages is to confuse the data with the weight. If you have the two backwards, your numerator is still correct; however, your denominator is incorrect. To distinguish the data from the weight, notice that the data forms a part of the question. In the above example, you were looking to calculate your grade point average; therefore, grade point is the data. The other information, the credit hours, must be the weight.

Paths To Success

The formula used for calculating a simple average is a simplification of the weighted average formula. In a simple average, every piece of data is equally important. Therefore, you assign a value of 1 to the weight for each piece of data. Since any number multiplied by 1 is the same number, the simple average formula omits the weighting in the numerator as it would have produced unnecessary calculations. In the denominator, the sum of the weights of 1 is no different from counting the total number of pieces of data. In essence, you can use a weighted average formula to solve simple averages.

? Exercise $3.2.2$: Give It Some Thought

In each of the following, determine which information is the data and which is the weight.

a. Rafiki operates a lemonade stand during his garage sale today. He has sold 13 small drinks for $0.50, 29 medium drinks for $0.90, and 21 large drinks for $1.25. What is the average price of the lemonade sold?
b. Natalie received the results of a market research study. In the study, respondents identified how many times per week they purchased a bottle of Coca-Cola. Calculate the average number of purchases made per week.

Purchases per Week	# of People
1	302
2	167
3	488
4	256

Answer

a. The price of the drinks is the data, and the number of drinks is the weight.
b. The purchases per week is the data, and the number of people is the weight.

✓ Example $3.2.2$: Calculating Your Weighted Grade Point Average

A mark transcript received by a student at a local college. The chart shows how the grade translates into a grade point.

Course	Grade	Credit hours
Economics 100	B	4

Course	Grade	Credit hours
Math 100	A	5
Marketing 100	B+	3
Communications 100	C	4
Computing 100	A+	3
Accounting 100	D	4

Grade	Grade Point
A+	4.5
A	4.0
B+	3.5
B	3.0
C+	2.5
C	2.0
D	1.0
F	0.0

Calculate the student's grade point average (GPA). Round your final answer to two decimals.

Solution

The courses do not carry equal weights as they have different credit hours. Therefore, to calculate the GPA you must find a weighted average, or WAvg.

What You Already Know

Since the question asked for the grade point average, the grade points for each course are the data, or X. The corresponding credit hours are the weights, or W.

How You Will Get There

You need to convert the grade for each course into a grade point using the secondary table. Then perform the following steps:

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using Formula 3.4.

Perform

Convert the grades:

Course	Grade	Grade Point	Credit hours
Economics 100	B	3.0	4
Math 100	A	4.0	5
Marketing 100	B+	3.5	3
Communications 100	C	2.0	4
Computing 100	A+	4.5	3

Course	Grade	Grade Point	Credit hours
Accounting 100	D	1.0	4
Economics 100	B	3.0	4
Math 100	A	4.0	5
Marketing 100	B+	3.5	3
Communications 100	C	2.0	4
Computing 100	A+	4.5	3
Accounting 100	D	1.0	4

Step 1:

$$\sum WX = (4 \times 3.0) + (5 \times 4.0) + (3 \times 3.5) + (4 \times 2.0) + (3 \times 4.5) + (4 \times 1.0) = 68$$

Step 2:

$$\sum w = 4 + 5 + 3 + 4 + 3 + 4 = 23$$

Step 3:

$$\text{WAvg} = \frac{68}{23} = 2.96$$

The student's GPA is 2.96. Note that math contributed substantially (almost one-third) to the student's grade point because this course was weighted heavily and the student performed well.

✓ Example 3.2.3: Weighted Average Loan Balance

Angelika started the month of March owing $20,000 on her home equity line of credit (HELOC). She made a payment of $5,000 on the fifth, borrowed $15,000 on the nineteenth, and made another payment of $5,000 on the twenty-sixth. Using each day's closing balance for your calculations, what was the average balance in the HELOC for the month of March?

Solution

The balance owing in Angelika’s HELOC is not equal across all days in March. Some balances were carried for more days than others. This means you will need to use the weighted average technique and find WAvg.

What You Already Know

You know the following:

Dates	Number of Days (W)	Balance in HELOC (X)
March 1–March 4	4	$20,000
March 5–March 18	14	$20,000 − $5,000 = $15,000
March 19–March 25	7	$15,000 + $15,000 = $30,000
March 26–March 31	6	$30,000 − $5,000 = $25,000

How You Will Get There

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using Formula 3.4.

Perform

Step 1:

$$\sum WX = (4 \times \$20,000) + (14 \times \$15,000) + (7 \times \$30,000) + (6 \times \$25,000) = \$650,000$$

Step 2:

$$\sum W = 4 + 14 + 7 + 6 = 31$$

Step 3:

$$\text{WAvg} = \frac{\$650,000}{31} = \$20,967.74$$

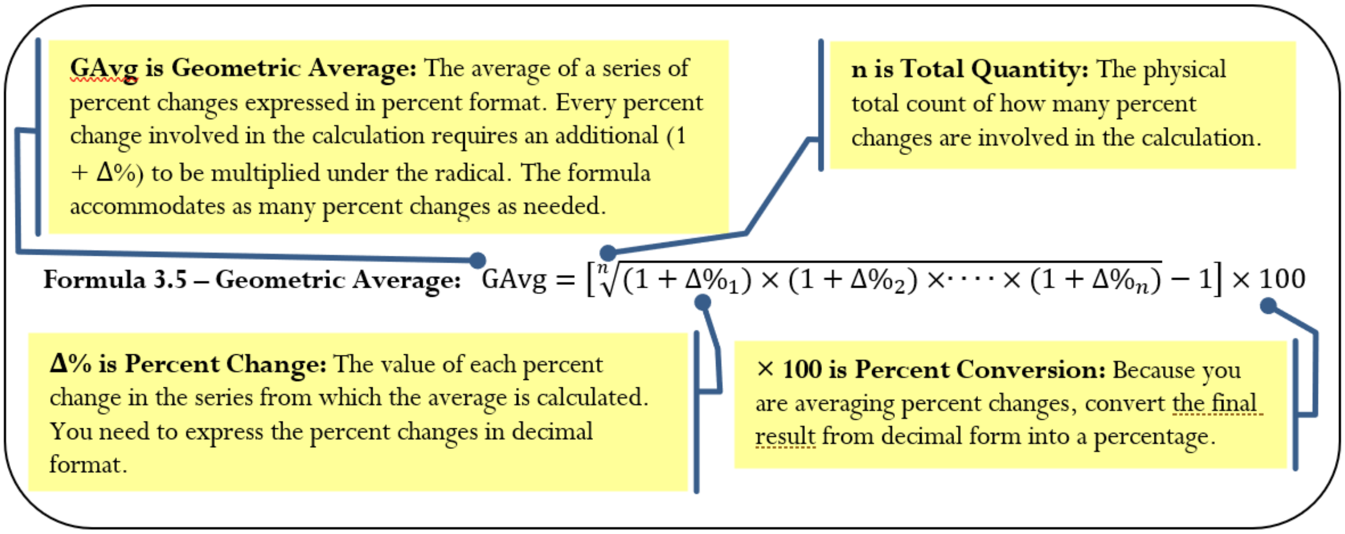
> Over the entire month of March, the average balance owing in the HELOC was $20,967.74. Note that the balance with the largest weight (March 5 to March 18) and the largest balance owing (March 19 to March 25) account for almost two-thirds of the calculated average.

Geometric Averages

How do you average a percent change? If sales increase 100% this year and decrease 50% next year, is the average change in sales an increase of $\frac{(100\% - 50\%)}{2} = 25\%$ per year? The answer is clearly "no." If sales last year were $100 and they increased by 100%, that results in a $100 increase. The total sales are now $200. If sales then decreased by 50%, you have a $100 decrease. The total sales are now $100 once again. In other words, you started with $100 and finished with $100. That is an average change of nothing, or 0% per year! Notice that the second percent change is, in fact, multiplied by the result of the first percent change. A geometric average finds the typical value for a set of numbers that are meant to be multiplied together or are exponential in nature.

The Formula

In business mathematics, you most commonly use a geometric average to average a series of percent changes. Formula 3.5 is specifically written to address this situation.



GAvg is Geometric Average: The average of a series of percent changes expressed in percent format. Every percent change involved in the calculation requires an additional (1 + Δ%) to be multiplied under the radical. The formula accommodates as many percent changes as needed.

n is Total Quantity: The physical total count of how many percent changes are involved in the calculation.

Formula 3.5 – Geometric Average: $\text{GAvg} = \left[\sqrt[n]{(1+\Delta\%_1)\times(1+\Delta\%_2)\times\cdots\times(1+\Delta\%_n)} - 1\right]\times 100$

Δ% is Percent Change: The value of each percent change in the series from which the average is calculated. You need to express the percent changes in decimal format.

× 100 is Percent Conversion: Because you are averaging percent changes, convert the final result from decimal form into a percentage.

Formula 3.5

How It Works

To calculate a geometric average follow these steps:

Step 1: Identify the series of percent changes to be multiplied.

Step 2: Count the total number of percent changes involved in the calculation.

Step 3: Calculate the geometric average using Formula 3.5.

Let's use the sales data presented above, according to which sales increase 100% in the first year and decrease 50% in the second year. What is the average percent change per year?

Step 1: The changes are $\Delta\%_1 = +100\%$ and $\Delta\% = -50\%$.

Step 2: Two changes are involved, or $n = 2$.

Step 3: Apply Formula 3.5:

$$\begin{aligned}\text{GAvg} &= \left[\sqrt[2]{(1+100\%)\times(1-50\%)} - 1\right]\times 100\\ &= \left[\sqrt[2]{2\times 0.5} - 1\right]\times 100\\ &= [\sqrt[2]{1} - 1]\times 100\\ &= 0\times 100 = 0\%\end{aligned}$$

The average percent change per year is 0% because an increase of 100% and a decrease of 50% cancel each other out.

Important Notes

A critical requirement of the geometric average formula is that every $(1+\Delta\%)$ expression must result in a number that is positive. This means that the $\Delta\%$ cannot be a value less than -100% else Formula 3.5 cannot be used.

Paths To Success

An interesting characteristic of the geometric average is that it will always produce a number that is either smaller than (closer to zero) or equal to the simple average. In the example, the simple average of +100% and -50% is 25%, and the geometric average is 0%. This characteristic can be used as an error check when you perform these types of calculations.

? Exercise $3.2.3$: Give It Some Thought

For the first three questions, determine whether you should calculate a simple, weighted, or geometric average.

a. Randall bowled 213, 245, and 187 in his Thursday night bowling league and wants to know his average.
b. Cindy invested in a stock that increased in value annually by 5%, 6%, 3%, and 5%. She wants to know her average increase.
c. A retail store sold 150 bicycles at the regular price of $300 and 50 bicycles at a sale price of $200. The manager wants to know the average selling price.
d. Gonzalez has calculated a simple average of 50% and a geometric average of 60%. He believes his numbers are correct. What do you think?

Answer

a. Simple; each item has equal importance and frequency.
b. Geometric; these are a series of percent changes on the price of stock.
c. Weighted; each item has a different frequency.
d. At least one of the numbers is wrong since a geometric average is always smaller than or equal to the simple average.

✓ Example $3.2.4$: How Are Sales Changing?

From 2006 to 2010, WestJet's year-over-year annual revenues changed by +21.47%, +19.89%, -10.55%, and +14.38%. This reflects growth from sales of $1.751 billion in 2006 to $2.609 billion in 2010.[1] What is the average percent growth in revenue for WestJet during this time frame?

Solution

Note that these numbers reflect percent changes in revenue. Year-over-year changes are multiplied together, so you would calculate a geometric average, or GAvg.

What You Already Know

Step 1:

You know the four percent changes:

$$\Delta\%_1=21.47\% \quad \Delta\%_2=19.89\% \quad \Delta\%_3=-10.55\% \quad \Delta\%_4=14.38\%$$

Step 2:

Four changes are involved, or $n=4$.

How You Will Get There

Step 3:

Express the percent changes in decimal format and substitute into Formula 3.5.

Perform

Step 3:

$$\text{GAvg} = \left[\sqrt[4]{(1+0.2147)\times(1+0.1989)\times(1-0.1055)\times(1+0.1438)}-1\right]\times 100 = 10.483\%$$

On average, WestJet revenues have grown 10.483% each year from 2006 to 2010.

References

1. WestJet, *WestJet Fact Sheet*, www.westjet.com/pdf/investorMedia/investorFactSheet.pdf (accessed May 10, 2011).

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3.3: Ratios, Proportions, and Prorating (It Is Only Fair)

You and your business partner have a good problem: Consumers are snapping up packets of your new eucalyptus loganberry facial scrub as fast as you can produce them. Each packet of the scrub contains 600 mg of loganberry extract and 80 mg of eucalyptus oil, as well as water and clay and other ingredients. Ratios are invaluable in understanding the relationship among different quantities, such as how much of each ingredient you need.

You have no trouble obtaining the water and clay, but the loganberry extract and eucalyptus oil are in short supply because of poor weather. In any time period, your suppliers can provide seven times as much loganberry extract as eucalyptus oil. To figure out which ingredient is limiting your production, you need proportions.

Sometimes you need to relate a proportion to the total of the quantities. This requires prorating skills. For example, once your business has grown, you start using a production line, which follows the common practice of producing more than one product. The papaya facial scrub you have introduced recently and the eucalyptus loganberry scrub require the same amount of production time. Your equipment capacity is 1,000 units per day. How many units of each type of scrub must you produce to meet market demand in the ratio of nine to two?

Years later, you are splitting the profits of your business partnership in proportion to each partner’s total investment. You invested $73,000, while your partner invested $46,000. With total profits of $47,500, what is your share?

Understanding the relationships among various quantities and how the components relate to an overall total emphasizes the need for understanding ratios, proportions, and prorating.

What Is a Ratio?

A ratio is a fixed relationship between two or more quantities, amounts, or sizes of a similar nature. For a ratio to exist, all terms involved in the ratio must be nonzero. Examine the criteria of this definition more closely:

1. **There Must Be Two or More Quantities**. A ratio does not exist if only one quantity is involved. For example, the fuel tank on your Mustang takes 60 litres of gasoline. This is not a ratio, as there is no relationship to any other quantity, amount, or size. On the other hand, if you compare your fuel tank to the fuel tank of your friend's Hummer, you now have two quantities involved and could say that her fuel tank has twice the capacity of yours.
2. **All Terms Must Be of a Similar Nature**. In all of the examples provided note that all quantities, amounts, or sizes are based on the same unit. In the cosmetics formulation it was milligrams to milligrams; in the production line, it was units to units; in the investment scenario, it was dollars to dollars. For a ratio to have meaning and to be properly interpreted, all terms of the ratio must be expressed in a similar nature. When you place different units such as kilometres and metres into the same ratio, the result is confusing and will lead to misinterpretation of the relationship.
3. **All Terms Must Be Nonzero**. The numbers that appear in a ratio are called the terms of the ratio. If we have a recipe with four cups of flour to one cup of sugar, there are two terms: four and one. If any term is zero, then the quantity, amount, or size does not exist. For example, if the recipe called for four cups of flour to zero cups of sugar, there is no sugar! Therefore, every term must have some value other than zero.

Let's continue using the example of four cups of flour to one cup of sugar. Business ratios are expressed in five common formats, as illustrated in the table below.

Format	Ratio Example	Interpretation
To	4 to 1	Four cups of flour to one cup of sugar
: (colon)	4 : 1	Four cups of flour to one cup of sugar
Fraction	$\frac{4}{1}$	Four cups of flour per one cup of sugar
Decimal	4	Four times as much flour than sugar
Percent	400%	Flour is 400% of sugar (Hint: think rate, portion, base)

All of these formats work well when there are only two terms in the ratio. If there are three or more terms, ratios are best expressed in the colon format. For example, if the recipe called for four parts of flour to one part of sugar to two parts of chocolate chips, the ratio is 4 : 1 : 2. The fraction, decimal, and percent forms do not work with three or more terms.

Simplification and Reduction of Ratios

When a ratio is used to express a relationship between different variables, it must be easy to understand and interpret. Sometimes when you set up a ratio initially, the terms are difficult to comprehend. For example, what if the recipe called for 62½ parts flour to 25 parts sugar? That is not very clear. Expressing the same ratio another way, you can say the recipe requires 5 parts flour to 2 parts sugar. Note how the relationship is clearer in the latter expression. Either way, though, both ratios mean the same thing; in decimal format this ratio is expressed as a value of 2.5. Recall that Section 2.2 discussed how fractions are expressed in higher and lower terms. We now apply the same knowledge to ratios to make the relationship as clear as possible.

When you reduce ratios to lower terms, remember two important characteristics involving the cardinal rule and integers:

1. **The Cardinal Rule**. Recall from Section 2.5 that the cardinal rule of algebra states, "What you do to one you must do to the other." In other words, whatever mathematical operation is performed on a term in a ratio must be equally performed on every other term in the ratio. If this rule is violated then the relationship between the terms is broken.
2. **Maintaining Integers**. Integers are easier to understand than decimals and fractions. In reducing a ratio to lower terms, aim to maintain every term as an integer (much as in Section 2.2).

How It Works

The steps involved in reducing ratios to lower terms are listed below. You may not need some steps, so skip them if the characteristic is not evident in the ratio.

Step 1: Clear any fractions that, when divided, produce a nonterminating decimal. Apply the rules of algebra and multiply each term by the denominator being cleared from the ratio. For example, if the ratio is $\frac{1}{3}:2$, the first term when divided produces a nonterminating decimal. Clear the fraction by multiplying every term by the denominator of 3, resulting in a ratio of 1 : 6.

Step 2: Perform division on all fractions that produce a terminating decimal. For example, if the ratio is $\frac{2}{5}:\frac{3}{10}$, both terms convert to terminating decimals, resulting in a ratio of $0.4:0.3$.

Step 3: Eliminate all decimals from the ratio through multiplication. In other words, express the ratio in higher terms by multiplying every term by a power of 10. The power of 10 you choose must be large enough to eliminate all decimals. For example, if the ratio is $0.2:0.25:0.125$, notice that the third term has the most decimal positions. A power of $1,000(10^3)$ is required to move the decimal three positions to the right. Multiply every term by a power of 1,000, resulting in a ratio of $200:250:125$.

Step 4: Find a common factor that divides evenly into every term, thus producing integers. If you find no such factors, then the ratio is in its lowest terms. For example, if the ratio is $10:4:6$ it can be factored by dividing every term by 2, resulting in a ratio of $5:2:3$. There is no common factor that reduces this ratio further; therefore, it is in its lowest terms.

Important Notes

To factor any term, recall your multiplication tables. Assume the ratio is $36:24$. It is always best to pick the lowest term in the ratio when factoring. Looking at the 24 and recall what multiplies together to arrive at 24. In this case, your factors are 1×24, 2×12, 3×8, and 4×6. The goal is to find the largest value among these factors that also divides evenly into the other term of 36. The largest factor that makes this true is 12. Therefore, perform step 4 in our reduction process by dividing every term by a factor of 12, resulting in a ratio of $3:2$.

Things To Watch Out For

Always ensure that before you apply any of the reduction steps your relationship meets the requirements of being a ratio in the first place. For example, the expression of 10 km : 500 m is not a 1 : 50 ratio since it violates the "similar nature" characteristic of the ratio definition. You need to convert the metres into kilometres, producing 10 km : 0.5 km. Now that you have a ratio, the reduction to lowest terms results in a $20:1$ ratio. This is very different than $1:50$! The lesson learned is to make sure you are working with a proper ratio before you manipulate it.

Paths To Success

In the fourth step of the procedure, do not feel compelled to find what is called the "magic factor." This is the single factor that reduces the ratio to its lowest terms in a single calculation. Although this is nice if it happens, you may as well find any factor that will make the ratio smaller and easier to work with.

- For example, assume a ratio of $144:72:96$. It may not be very apparent what factor goes into each of these terms on first glance. However, they are all even numbers, meaning they can all be divided by 2. This produces $72:36:48$.
- Once again, the "magic factor" may not be clear, but every term is still even. Divide by 2 again, producing $36:18:24$.
- If no magic factor is apparent, again note that all numbers are even. Divide by 2 yet again, producing $18:9:12$.
- The common factor for these terms is 3. Divided into every term you have $6:3:4$. There is no common factor for these terms, and the ratio is now in its lowest terms.

In the above example it took four steps to arrive at the answer—and that is all right. Did you notice the "magic factor" that could have solved this in one step? You can find it if you multiply all of the factors you have: $2\times2\times2\times3=24$. Dividing every term in the original ratio by 24 produces the solution in one step. If you did not notice this "magic factor," there is nothing wrong with taking four steps (or more!) to get the answer. In the end, both methods produce the same solution.

✓ Example $3.3.1$: Reducing Ratios to Lowest Terms

Reduce the following ratios to their lowest terms:

a. $49:21$
b. $0.33:0.066$
c. $\frac{9}{2}:\frac{3}{11}$
d. $5^{1/8}:6^{7/8}$

Solution

You have been asked to reduce the ratios to their lowest terms.

What You Already Know

The four ratios to be reduced have been provided in colon format.

How You Will Get There

Step 1: Clear all fractions that produce nonterminating decimals.

Step 2: Perform division on all fractions that produce terminating decimals.

Step 3: Eliminate all decimals through multiplication.

Step 4: Divide every term by a common factor, reducing it to the lowest terms.

Perform

Step 1:

a. No fractions
b. No fractions
c. Multiply both terms by 11: $\frac{99}{2}:3$
d. No fractions that produce nonterminating decimals

Step 2:

a. No fractions
b. No fractions
c. Perform division: $49.5:3$
d. Perform division: $5.125:6.875$

Step 3:

a. No decimals
b. Clear decimals by multiplying by 1,000: $330:66$

c. Clear decimals by multiplying by 10: $495 : 30$
d. Clear decimals by multiplying by 1,000: $5,125 : 6,875$

Step 4:

a. Divide by 7: $7 : 3$
b. Divide by 66: $5 : 1$
c. Divide by 15: $33 : 2$
d. Divide by 125: $41 : 55$

In lowest terms, the ratios are $7 : 3$, $5 : 1$, $33 : 2$, and $41 : 55$, respectively.

Reducing a Ratio to the Smallest Term of One

Your goal in reducing a ratio is to make it easier to understand. Sometimes, even after you have applied the techniques for reducing the ratio, the end result is still hard to grasp. Look at part (d) of Example $3.3.1$. You arrived at a final solution of $41 : 55$, with no further reduction possible. Think of this as 41 cups of flour to 55 cups of sugar. The relationship is not very clear.

In these circumstances, although integers are preferable in general, you must reduce the ratio further, reintroducing decimals to make the relationship more comprehensible. This means you apply a technique called “reducing the ratio to the smallest term of one.” In this technique, the smallest term in the ratio will have a value of 1 once you perform the ratio reduction and simplification.

How It Works

Follow these steps to reduce a ratio to the smallest term of one:

Step 1: Locate the smallest term in the ratio. (Do not just pick the first term.)

Step 2: Divide every term in the ratio by the selected smallest term. Every other term becomes a decimal number, for which either a clear rounding instruction is provided or you must obey the rounding rules used in this textbook. The smallest term by nature of the division equals one.

Let’s continue with part (d) of Example $3.3.1$, in which the reduced ratio is $41 : 55$.

Step 1: Locate the smallest term in the ratio. It is the first term and has a value of 41.

Step 2: Take every term and divide it by 41 to arrive at $1 : 1.\overline{34146}$. The decimal number allows you to roughly interpret the ratio as "one cup of flour to a touch over $1^{1/3}$ cups of sugar.” Although not perfect, this is more understandable than $41 : 55$.

Important Notes

You may have to make a judgment call when you decide whether to leave a reduced ratio alone or to reduce it to a ratio where the smallest term is one. There is no clear definitive rule; however, keep the following two thoughts in mind:

1. For purposes of this textbook, you are provided with clear instructions on how to handle the ratio, allowing everyone to arrive at the same solution. Your instructions will read either “Reduce the ratio to its smallest terms” or “Reduce the ratio to the smallest term of one.”
2. In the real world, no such instructions exist. Therefore you should always base your decision on which format is easier for your audience to understand. If the reduced fraction leaves your audience unable to understand the relationship, then use the reduction to a ratio with the smallest term of one instead.

✓ Example $3.3.2$: Maintaining Inventory

The inventory management system at your company requires you to keep a ratio of 27 to 47 Nintendo Wiis to Sony PlayStations on the shelf. There are 38 Nintendo Wiis on the shelf.

a. Express the ratio where the smallest term is one. Round your solution to two decimals.
b. Using the ratio, determine how many PlayStations should be in your inventory.

Solution

These terms have no common factor (note that 47 is a prime number). The terms in this ratio cannot be reduced in any way, which makes the relationship difficult to assess. Therefore, reduce the ratio to the smallest term of one. Once the relationship is evident, determine how many PlayStations need to be on the shelf.

What You Already Know

The ratio of Wiis to PlayStations is 27 : 47.

38 Wiis are currently on the shelf.

How You Will Get There

Step 1: Locate the smallest term.

Step 2: Divide all terms by the smallest term to get the ratio.

Step 3: Use the ratio to calculate the number of PlayStations needed on the shelf.

Perform

Step 1:

The smallest term is 27.

Step 2:

Divide every term by 27. $\frac{27}{27} : \frac{47}{27}$ becomes $1 : 1.74$

The inventory management requires 1.74 units of PlayStations to be on the shelf for every unit of Wiis.

Step 3:

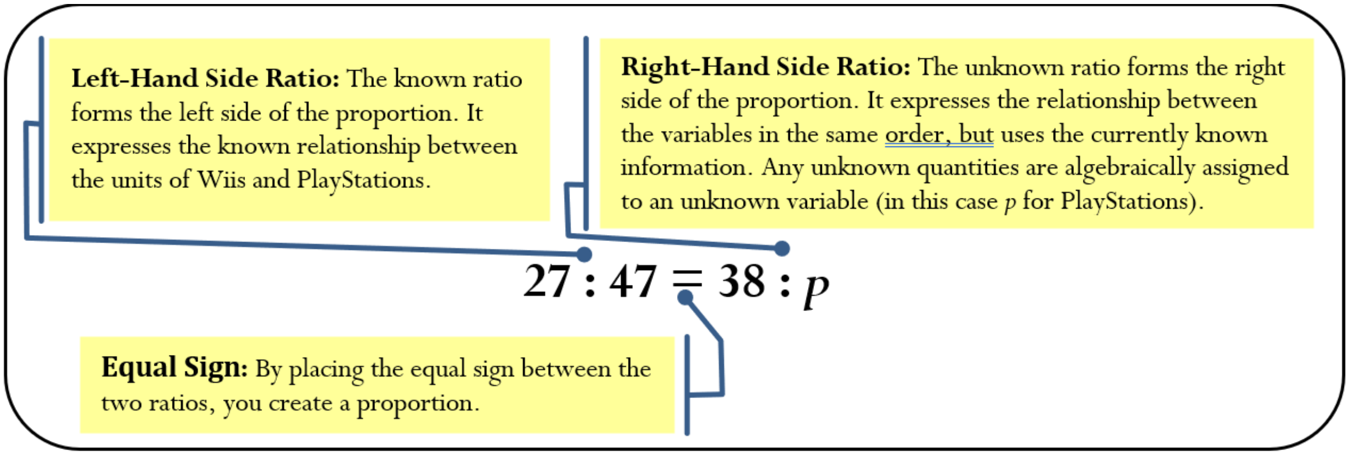
$38 \times 1.74 = 66.12$ units. Since only whole units are possible, round to the nearest integer of 66.

The inventory management system requires approximately 1.74 PlayStations for every Nintendo Wii on the shelf. Therefore, with 38 Wiis on the shelf you require 66 PlayStations to also be on the shelf.

What Are Proportions?

Knowing the relationship between specific quantities is helpful, but what if your quantity happens not to match the specific quantity expressed in the existing ratio? Example 3.3B illustrated that you need to learn how to apply the ratio to meet current conditions. The ratio in the inventory system was expressed in terms of 27 Nintendo Wiis; however, the shelf displayed 38 units. How can we relate an existing ratio to a needed ratio?

A proportion is a statement of equality between two ratios. Just as we have both algebraic expressions and algebraic equations, there are ratios and proportions. With algebraic expressions, only simplification was possible. When the expression was incorporated into an algebraic equation, you solved for an unknown. The same is true for ratios and proportions. With ratios, only simplification is possible. Proportions allow you to solve for any unknown variable. The Wii and PlayStation example could have been set up as follows:



Left-Hand Side Ratio: The known ratio forms the left side of the proportion. It expresses the known relationship between the units of Wiis and PlayStations.

Right-Hand Side Ratio: The unknown ratio forms the right side of the proportion. It expresses the relationship between the variables in the same order, but uses the currently known information. Any unknown quantities are algebraically assigned to an unknown variable (in this case p for PlayStations).

$$27 : 47 = 38 : p$$

Equal Sign: By placing the equal sign between the two ratios, you create a proportion.

A proportion must adhere to three characteristics, including ratio criteria, order of terms, and number of terms.

- **Characteristic #1:** ***Ratio Criteria Must Be Met***. By definition, a proportion is the equality between two ratios. If either the left side or the right side of the proportion fails to meet the criteria for being a ratio, then a proportion cannot exist.

- **Characteristic #2: *Same Order of Terms***. The order of the terms on the left side of the proportion must be in the exact same order of terms on the right side of the proportion. For example, if your ratio is the number of MP3s to CDs to DVDs, then your proportion is set up as follows:

$$\mathrm{MP3} : \mathrm{CD} : \mathrm{DVD} = \mathrm{MP3} : \mathrm{CD} : \mathrm{DVD}$$

- **Characteristic #3: *Same Number of Terms***. The ratios on each side must have the same number of terms such that every term on the left side has a corresponding term on the right side. A proportion of $\mathrm{MP3} : \mathrm{CD} : \mathrm{DVD} = \mathrm{MP3} : \mathrm{CD}$ is not valid since the DVD term on the left side does not have a corresponding term on the right side.

When you work with proportions, the mathematical goal is to solve for an unknown quantity or quantities. In order to solve any proportion, always obey the following four rules:

1. **Rule #1**: At Least One Value for Any Term Is Known. At least one of the left side or right side values for each term must be known. For example, $x : 5 = y : 10$ is not a solvable proportion since the corresponding first terms on both sides are unknown. However, $15 : 5 = y : 10$ is solvable since at least one of the first corresponding terms (the 15 and the y) is known.
2. **Rule #2**: One Pair of Corresponding Terms Must Be Known. At least one pair of corresponding terms on the left side and right side must have both quantities known. For example, $3 : x : 6 = y : 4 : z$ is not a solvable proportion since there is no pair of first terms (3 and y), second terms (x and 4), or third terms (6 and z) that produces a pair of known values. However, $3 : x : 6 = 9 : 4 : z$ is a solvable proportion since the first terms on both sides (the 3 and 9) are known.
3. **Rule #3**: Obey BEDMAS and Perform Proper Algebraic Manipulation. To manipulate a proportion, you must satisfy the rules of BEDMAS (Section 2.1) and all of the rules of algebra (Section 2.4 and Section 2.5). Violating any of these rules breaks the equality of the ratios and produces an incorrect proportion.
4. **Rule #4**: Use the Fractional Format. The fractional format for ratios is recommended for solving a proportion. The other four formats generally make solving the proportion much more difficult, and the mathematical operations required become unclear.

How It Works

Follow these steps in solving any proportion for an unknown variable or variables:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Work with only two terms at a time, and express the two terms in fractional format. This is not a problem if the proportion has only two terms on each side of the equation, for example, $27 : 47 = 38 : p$. This is expressed as $\frac{27}{47} = \frac{38}{p}$. If the proportion has three or more terms on each side, you can pick any two terms from each side of the proportion so long as you pick the same two terms on each side. In making your selection, aim to have a pair of terms on one side of the equation where both values are known while the other side of the equation is made up of one known term and one unknown term. For example, assume the proportion $6 : 5 : 4 = 18 : 15 : y$. The selection of the first and third terms only on each side of the equation produces $6 : 4 = 18 : y$. Notice that this is now a proportion with only two terms on each side, which you can express as $\frac{6}{4} = \frac{18}{y}$.

Step 3: Solve for the unknown variable. Obey the rules of BEDMAS and perform proper algebraic manipulation.

Step 4: If the proportion contained more than one unknown variable, go back to step 2 and select another pairing that isolates one of the unknown variables. Although one of the unknown variables is now known as a result of step 3, do not use this known value in making your selection. The danger in using a solved unknown variable is that if an error has occurred, the error will cascade through all other calculations. For example, assume the original proportion was $3 : 7 : 6 : 8 = x : y : z : 28$. From step 2 you may have selected the pairing of $3 : 8 = x : 28$, and in step 3 you may have erroneously calculated $x = 11.5$ (the correct answer is $x = 10.5$). In returning to step 2 to solve for another unknown variable, do not involve x in your next pairing. To isolate y, use $7 : 8 = y : 28$ and not $3 : 7 = 11.5 : y$, which will at least ensure that your calculation of y is not automatically wrong based on your previous error.

Things To Watch Out For

You must always pick terms from the same positions on both sides of the proportion. Otherwise, you will violate the equality of the proportion, since the terms are no longer in the same order on both sides and Characteristic #2 is not satisfied. For example, in working with the proportion $6 : 5 : 4 = 18 : 15 : y$ from above, you cannot select the first and third terms on the left side and also select the second and third terms on the right side. In other words, $6 : 4 \neq 15 : y$ since 1^{st} term : 3^{rd} term $\neq 2^{nd}$ term : 3^{rd} term.

Paths To Success

It is always easiest to solve a proportion when the unknown variable is in the numerator. This characteristic requires minimal algebra and calculations to isolate the variable. If you find yourself with an unknown variable in the denominator, you can mathematically invert the fraction on both sides, since this obeys the cardinal rule of “what you do to one, you must do to the other.” When you invert, the numerator becomes the denominator and vice versa. For example, if the proportion is , inversion then produces a proportion of . Notice that isolating the unknown variable in the inverted proportion requires only a multiplication of 12 on both sides. This is a lot less work!

Exercise $3.3.1$: Give It Some Thought

1. Some of the following proportions violate the characteristics or rules of proportions. Examine each and determine if all the rules and characteristics are met. If not, identify the problem.
   a. $4 : 7 = 6 : y$
   b. $5 : 3 = 6 : a : b$
   c. $6km : 3m = 2m : 4km$
   d. $6 : k = 18 : 12$
   e. $4 : 0 = 8 : z$
   f. $9 = p$
   g. $4 : 7 : 10 = d : e : f$
   h. $y : 10 : 15 = x : 30 : z$
2. In the following solved proportions, which person properly executed step 2 of the proportion steps?

$$6 : 5 : 4 : 3 = x : y : z : 9$$

   a. Person A: $6 : 5 = x : y$
   b. Person B: $4 : 3 = y : 9$
   c. Person C: $4 : 3 = z : 9$

Answer

1. a. OK
   b. Not the same number of terms on each side (Characteristic #3)
   c. Terms are not in same units; terms are not in the same order (Characteristics #1 and #2)
   d. OK
   e. Does not meet ratio criteria; there is a term of zero (Characteristic #1)
   f. Does not meet ratio criteria; must have at least two terms on each side (Characteristic #1)
   g. There is no corresponding known term on both sides; every pair of terms contains an unknown (Rule #2)
   h. The first corresponding term is not known on both sides; at least one needs to be known (Rule #1)
2. Person C did it right. Person A failed to isolate a variable, and different terms were extracted by Person B (3rd term : 4th term ≠ 2nd term : 4th term).

Example $3.3.3$: Estimating Competitor Profits

A recent article reported that companies in a certain industry were averaging an operating profit of $23,000 per 10 full-time employees. A marketing manager wants to estimate the operating profitability for one of her company's competitors, which employs 87 full-time workers. What is the estimated operating profit for that competitor?

Solution

There is a relationship between the number of full-time employees and the operating profit. This is a ratio. If you set up a similar ratio for the competitor, you create a proportion that you can solve for the competitor's estimated operating profit, or p.

What You Already Know

You know the industry and competitor information:

Industry operating profit = $23,000

Industry full-time employees = 10

Competitor operating profit = p

Competitor full-time employees = 87

How You Will Get There

Step 1: Set up the proportion.

Step 2: Express in fractional format.

Step 3: Isolate the unknown variable, p.

Step 4: Since there is only one unknown, this step is not needed.

Perform

Step 1:

$$\text{industry profit} : \text{industry employees} = \text{competitor profit} : \text{competitor employees}$$

$$\$23,000 : 10 = p : 87$$

Step 2:

$$\frac{\$23,000}{10} = \frac{p}{87}$$

Step 3:

$$\frac{\$23,000 \times 87}{10} = p$$

$$p = \$200,100$$

If the industry is averaging $23,000 in operating profits per 10 full-time employees, then the competitor with 87 full-time employees has an estimated $200,100 in operating profits.

Example $3.3.4$: Making the Party Punch

In preparing for a party, Brendan needs to mix a punch consisting of fruit juice, vodka, and 7UP in the ratio of $5:3:2$, respectively. If Brendan has a 2 L bottle of vodka, how much fruit juice and 7UP must he mix with it?

Solution

The relationship between the punch recipe ingredients of fruit juice, vodka, and 7UP forms a ratio. If you set up a similar ratio for what Brendan actually has and needs, you can calculate the exact amounts of fruit juice, vodka, and 7UP that are required.

What You Already Know

You know the punch recipe ratio of $\text{fruit juice} : \text{vodka} : \text{7UP}$ is $5:3:2$. Brendan has 2 L of vodka.

How You Will Get There

Step 1: Set up the proportion.

Step 2: Pick two terms containing one unknown variable and express in fractional format.

Step 3: Isolate the unknown variable.

Step 4: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable.

Perform

Step 1:

$$5:3:2 = f:2:s$$

Step 2:

$$\begin{aligned}\frac{5}{3} &= \frac{f}{2} \\ \frac{10}{3} &= f \\ 3.\overline{3} &= f\end{aligned}$$

Step 3:

$$\begin{aligned}\frac{3}{2} &= \frac{2}{s} \\ \frac{2}{3} &= \frac{s}{2} \\ \frac{4}{3} &= s \\ 1.\overline{3} &= s\end{aligned}$$

Maintaining the ratio of $5 : 3 : 2$ between the fruit punch, vodka, and 7UP, Brendan should mix $3^{1/3}$ L of fruit juice and $1^{1/3}$ L of 7UP with his 2 L of vodka.

What Is Prorating?

Ratios and proportions are commonly used in various business applications. But there will be numerous situations where your business must allocate limited funds across various divisions, departments, budgets, individuals, and so on. In the opener to this section, one example discussed the splitting of profits with your business partner, where you must distribute profits in proportion to each partner’s total investment. You invested $73,000 while your partner invested $46,000. How much of the total profits of $47,500 should you receive?

The process of prorating is the taking of a total quantity and allocating or distributing it proportionally. In the above example, you must take the total profits of $47,500 and distribute it proportionally with your business partner based on the investment of each partner. The proportion is:

$$\text{your investment : your partner's investment } = \text{ your profit share : your partner's profit share}$$

This proportion has two major concerns:

1. You don't know either of the terms on the right side. As per the rules of proportions, this makes the proportion unsolvable.
2. There is a piece of information from the situation that you didn't use at all! What happened to the total profits of $47,500?

Every prorating situation involves a hidden term. This hidden term is usually the sum of all the other terms on the same side of the proportion and represents a total. In our case, it is the $47,500 of total profits. This quantity must be placed as an extra term on both sides of the proportion to create a proportion that can actually be solved.

How It Works

Prorating represents a complex proportion. As such, the steps involved in prorating are similar to the steps for solving any proportion:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Insert the hidden term on both sides of the proportion. Usually this term represents the total of all other terms on the same side of the proportion.

Step 3: Working with only two terms at a time, express the two terms in fractional format. Ensure that only one unknown variable appears in the resulting proportion.

Step 4: Solve for the unknown variable. Ensure that the rules of BEDMAS are obeyed and proper algebraic manipulation is performed.

Step 5: If the prorating contains more than one unknown variable, go back to step 3 and select a pairing that isolates another one of the unknown variables.

To solve your profit-splitting scenario, let y represent your profit share and p represent your partner's share:

Step 1: your investment : your partner’s investment = your profit share : your partner’s profit share

$$\$73,000 : \$46,000 = y : p$$

Step 2: Insert the hidden total term on both sides:

$$\begin{array}{c}\text{your investment} \quad : \quad \text{your partner's investment} \quad : \quad \text{total investment} \quad = \\ \text{your profit share} \quad : \quad \text{your partner's profit share} : \quad \text{total profits}\end{array}$$

$$\$73,000 : \$46,000 : \$119,000 = y : p : \$47,500$$

Step 3:

Set up one proportion:

$$\frac{\$73,000}{\$119,000} = \frac{y}{\$47,500}$$

Step 4:

Solve for y. Calculate $y = \$29,138.66$.

Step 5:

Set up the other proportion:

$$\frac{\$46,000}{\$119,000} = \frac{p}{\$47,500}$$

Solve for p. Calculate $p = \$18,361.34$.

Your final proportion is:

$$\$73,000 : \$46,000 : \$119,000 = \$29,138.66 : \$18,361.34 : \$47,500.00$$

So you will receive $\$29,138.66$ of the total profits and your partner will receive $\$18,361.34$.

Important Notes

To make prorating situations easier to solve, it is always best to insert the hidden term as the last term on both sides of the equation. This forces the unknown variables into the numerator when you select pairs of terms. It then takes less algebra for you to isolate and solve for the unknown variable.

Things To Watch Out For

Ensure that when you insert the hidden term you put it in the same position on both sides of the proportion. A common error is to put the total on the "outsides" of the proportion:

$$A : B = C : D \text{ when prorated becomes}$$

$$\text{ Total } : A : B = C : D : \text{ Total }$$

This violates the proportion characteristics, since the terms are not in the same order on both sides. The correct insertion of the hidden total term makes the proportion look like this:

$$A : B : \text{ Total } = C : D : \text{ Total }$$

Paths To Success

Another way of approaching prorating is to calculate a "per unit" basis and then multiply each term in the proportion by this base. For example, assume you are distributing $100 across three people, who have shares of three, five, and two. This is a total of 10 shares (the hidden term). Therefore, $100 divided by 10 shares is $10 per share. If the first person has three shares, then $\$10 \times 3 = \$30$. The second and third people get $\$10 \times 5 = \$50$ and $\$10 \times 2 = \$20$, respectively. Therefore,

$$3 : 5 : 2 : 10 = \$30 : \$50 : \$20 : \$100$$

✓ Example $3.3.5$: A Refund on Your Car Insurance Premiums

You paid your annual car insurance premium of $1,791 on a Ford Mustang GT. After five complete months, you decide to sell your vehicle and use the money to cover your school expenses. Assuming no fees or other deductions from your insurance agency, how much of your annual insurance premium should you receive as a refund?

Solution

You need to figure out how much of your annual insurance premium should be refunded. This will be based on the amount of time that you did not require the insurance. Let's denote this amount as r, the refund.

What You Already Know

You have some information on premiums and time frames:

Total insurance paid = $1,791

Total months paid for = 12

Months used = 5

How You Will Get There

Step 1: Set up the proportion.

Step 2: Insert the hidden term.

Step 3: Pick two terms containing one unknown variable and express in fractional format.

Step 4: Isolate the unknown variable.

Step 5: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable.

Perform

Step 1:

$$\text{insurance used} : \text{refund} \;=\; \text{months used} : \text{months not used}$$

$$\text{insurance used } : r = 5 : \text{ months not used}$$

Step 2:

$$\text{insurance used } : r : \$1,791 = 5 : \text{ months not used} : 12$$

Two variables remain undefined. We deduce that you did not use your car for 7 months (12 − 5). We do not know the insurance used and assign it the variable u.

$$u : r : \$1,791 = 5 : 7 : 12$$

Step 3:

$$\frac{r}{\$1,791} = \frac{7}{12}$$

Step 4:

$$r = \$1,044.75$$

Step 5:

$$\frac{u}{\$1,791} = \frac{5}{12} \quad u = \$746.25$$

You have used $746.25 of the $1,791 annual insurance premium. With seven months left, you are entitled to a refund of $1,044.75.

✓ Example $3.3.6$: Distribution of Costs across Product Lines

An accountant is trying to determine the profitability of three different products manufactured by his company. Some information about each is below:

	Chocolate	Wafers	Nougat
Direct Costs	$743,682	$2,413,795	$347,130

Although each product has direct costs associated with its manufacturing and marketing, there are some overhead costs (costs that cannot be assigned to any one product) that must be distributed. These amount to $721,150. A commonly used technique is to assign these overhead costs in proportion to the direct costs incurred by each product. What is the total cost (direct and overhead) for each product?

Solution

You need to calculate the sum of the direct and overhead costs for each product. The direct costs are known. The individual overhead costs are unknown but can be calculated using prorating of the total overhead costs. You have three unknowns, namely, chocolate overhead (Co), wafers overhead (Wo), and nougat overhead (No).

What You Already Know

You have some information on direct costs and overhead costs:

Chocolate direct (Cd) = $743,682

Wafers direct (Wd) = $2,413,795

Nougat direct (Nd) = $347,130

Total overhead costs = $721,150

How You Will Get There

Step 1: Set up the proportion.

Step 2: Insert the hidden term.

Step 3: Pick two terms containing one unknown variable and express in fractional format. You can start with chocolate overhead, or Co.

Step 4: Isolate the unknown variable.

Step 5: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the wafers overhead, or Wo.

Step 6: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the nougat overhead, or No.

Step 7: Sum the direct and overhead costs for each product.

Perform

Step 1:

$$Cd : Wd : Nd = Co : Wo : No$$

$$\$743,682 : \$2,413,795 : \$347,130 = Co : Wo : No$$

Step 2:

$$\$743,682 : \$2,413,795 : \$347,130 : \text{ Total direct costs } = Co : Wo : No : \$721,150$$

Calculate the total direct costs as a sum of all three direct costs:

$$\text{Total direct costs } = \$743,682 + \$2,413,795 + \$347,130 = \$3,504,607$$

The proportion is now:

$$\$743,682 : \$2,413,795 : \$347,130 : \$3,504,607 = Co : Wo : No : \$721,150$$

Step 3:

$$\frac{\$743,682}{\$3,504,607} = \frac{Co}{\$721,150}$$

Step 4:

$$\$153,028.93 = Co$$

Step 5:

$$\begin{aligned}\frac{\$2,413,795}{\$3,504,607} &= \frac{Wo}{\$721,150}\\ \$496,691.43 &= Wo\end{aligned}$$

Step 6:

$$\begin{aligned}\frac{\$347,130}{\$3,504,607} &= \frac{No}{\$721,150}\\ \$71,429.64 &= No\end{aligned}$$

Step 7:

$$\text{Chocolate total cost } = \$743,682.00 + \$153,028.93 = \$896,710.93$$

$$\text{Wafer total cost } = \$2,413,795.00 + \$496,691.43 = \$2,910,486.43$$

$$\text{Nougat total cost } = \$347,130.00 + \$71,429.64 = \$418,559.64$$

Putting together the direct costs with the allocated overhead costs, the total cost for chocolate is \$896,710.93, wafers is \$2,910,486.43, and nougat is \$418,559.64.

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3.S: Summary

Key Concepts Summary

3.1: Percent Change (Are We Up or Down?)

- Measuring the percent change in a quantity from one value to another
- Measuring the constant rate of change over time in a quantity

3.2: Averages (What Is Typical?)

- The calculation of simple averages when everything is equal
- The calculation of weighted averages when not everything is equal
- The calculation of geometric averages when everything is multiplied together

3.3: Ratios, Proportions, and Prorating (It Is Only Fair)

- The characteristics of a ratio
- How to simplify and reduce a ratio to its lowest terms
- How to simplify a ratio by reducing its smallest term to a value of one
- How to equate two ratios into the form of a proportion
- How to use proportions to prorate

The Language of Business Mathematics

average

A single number that represents the middle of a data set.

geometric average

A typical value for a set of numbers that are meant to be multiplied together or are exponential in nature. This may take the form of a series of percent changes.

hidden term

In prorating, this is the sum of all the other terms on the same side of the proportion and represents a total.

percent change

An expression in percent form of how much any quantity changes from a starting period to an ending period.

proportion

A statement of equality between two ratios.

prorating

The process of taking a total quantity and allocating or distributing it proportionally.

rate of change over time

A measure of the percent change in a variable per time period.

ratio

A fixed relationship between two or more quantities, amounts, or sizes of a similar nature where all terms are nonzero.

simple average

An average where each piece of data shares the same level of importance and frequency but does not represent percent changes or numbers that are intended to be multiplied with each other.

terms of the ratio

The numbers appearing in a ratio.

weighted average

An average where not all pieces of data share the same level of importance or they do not occur with the same frequency; the data cannot represent percent changes or numbers that are intended to be multiplied with each other

The Formulas You Need to Know

Symbols Used

$\Delta\%$= percent change

Σ= summation symbol, which means you add everything up

GAvg = geometric average

n=a count of something, whether a total number of periods or a total quantity

New= the value that a quantity has become; the number that is being compared

Old= the value that a quantity used to be; the number to compare to

RoC= rate of change per time period

SAvg= simple average

X= any individual piece of data

W= weighting factor

WAvg = weighted average

Formulas Introduced

Formula 3.1 Percent Change: $\Delta\% = \dfrac{\text{New} - \text{Old}}{\text{Old}} \times 100$

Formula 3.2 Rate of Change over Time: $RoC = \left(\sqrt[n]{\dfrac{\text{New}}{\text{old}}} - 1\right) \times 100$

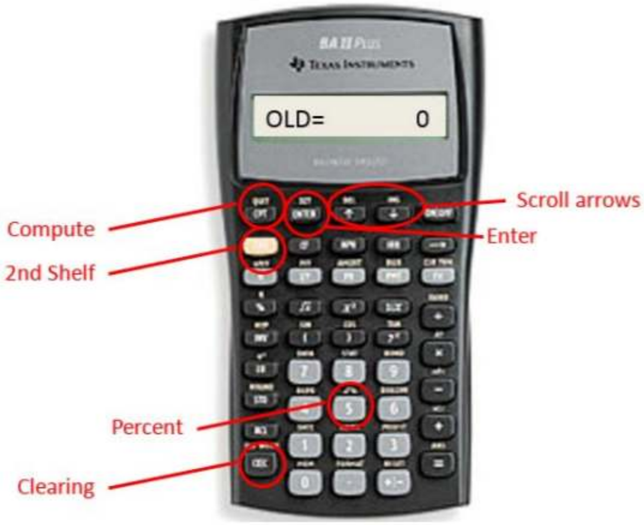
Formula 3.3 Simple Average: $\text{SAvg} = \dfrac{\Sigma X}{n}$

Formula 3.4 Weighted Average: $\text{WAvg} = \dfrac{\Sigma WX}{\Sigma W}$

Formula 3.5 Geometric Average: $\text{GAvg} = [\sqrt[n]{(1+\Delta\%_1) \times (1+\Delta\%_2) \times \cdots \times (1+\Delta\%_n)} - 1] \times 100$

Technology

Calculator



Percent Change

2nd Δ% (located above the 5 key) to access and enter three of the following. Press CPT on the unknown to compute. It is recommended to clear a previous question from memory by then pressing 2nd CLR Work.

OLD = The old or original quantity; the number to compare to

NEW = The new or current quantity; the number wanting to be compared

%CH = The percent change; in percent format

#PD = Number of consecutive periods for the change. By default, it is set to 1. However, if you want to increase a number by the same percentage (a constant rate of change) consecutively, enter the number of times the change occurs. For example, if increasing a number by 10% three times in a row, set this variable to 3.

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3.CS: Case Study (Forecasting Future Sales)

The Situation

Lightning Wholesale needs to predict dollar sales for each month in 2014. A variety of techniques are used in business to forecast sales. One technique examines year-to-year change in sales. Thus, the management at Lightning Wholesale has decided to perform a historical three-year analysis on both average monthly sales and average yearly sales to grasp how these figures have changed. Once the change is understood, the averages will be used to project the sales for each month in 2014.

The Data

Month	2011 Sales	2012 Sales	2013 Sales
January	$1,337	$1,928	$1,798
February	$2,198	$2,122	$2,407
March	$2,098	$2,503	$2,568
April	$2,540	$2,843	$2,985
May	$2,751	$2,765	$3,114
June	$2,885	$3,152	$3,242
July	$2,513	$3,128	$3,306
August	$3,784	$3,513	$3,852
September	$4,200	$4,700	$4,815
October	$6,079	$6,888	$7,222
November	$10,878	$11,120	$12,839
December	$8,136	$10,226	$9,630

(All figures are in thousands of dollars)

Your Tasks

1. Calculate the average monthly sales for each year.
2. Calculate the year-over-year percent changes in average monthly sales.
    - Determine the percent changes from year to year using your answers to question 1.
    - Using your answers to question 2(a), calculate the annual rate of change.
3. Establish monthly average ratios to determine how each month compares to the yearly average.
    - For each month, calculate the average sales across all three years. Also average the yearly averages calculated in question 1. Round your answers to whole numbers.
    - For each month, establish a ratio between the average sales in the month (from question 3(a)) and the average of the yearly sales (also from question 3(a)). Express the ratio so that the average yearly sales is a term of one. Round your answers to two decimals.
4. Project sales into the future to forecast monthly sales in 2014 along with total yearly sales.
    - Project the average monthly dollar volume sales for 2014 using the annual rate of change from question 2(b). Round your answer to a whole number.
    - Convert the projection from question 4(a) into a monthly projection for 2014 using the monthly ratio established in question 3(b). Total the projected sales for 2014. Round all of your answers to whole numbers.

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3.E: General Business Management Applications (Exercises)

3.1: Percent Change (Are We Up Or Down?)

Mechanics

For questions 1–3, solve for the unknown (?) using Formula 3.1 (percent change).

	Old	New	Δ%
1.	\$109.95	\$115.45	?
2.	?	\$622.03	13.25%
3.	5.94%	?	−10%

4. If \$9.99 is changed to \$10.49, what is the percent change?
5. \$19.99 lowered by 10% is what dollar amount?
6. What amount when increased by 40% is \$3,500?
7. If 10,000 grows to 20,000 over a period of 10 years, what is the annual rate of change?

Applications

8. How much, including taxes of 12%, would you pay for an item with a retail price of \$194.95?
9. From September 8, 2007 to November 7, 2007, the Canadian dollar experienced a rapid appreciation against the US dollar, going from \$0.9482 to \$1.1024. What was the percent increase in the Canadian dollar?
10. From 1996 to 2006, the “big three” automakers in North America (General Motors, Ford, and Chrysler) saw their market share drop from 71.5% to 52.7%. What is the overall change and the rate of change per year?
11. The average price of homes in Calgary fell by \$10,000 to \$357,000 from June 2009 to July 2009. The June 2009 price was 49% higher than the June 2005 price.
    a. What was the percent change from June 2009 to July 2009?
    b. What was the average price of a home in June 2005?
    c. What was the annual rate of change from June 2005 to June 2009?
12. On October 28, 2006, Saskatchewan lowered its provincial sales tax (PST) from 7% to 5%. What percent reduction does this represent?
13. A local Superstore sold 21,983 cases of its President's Choice cola at \$2.50 per case. In the following year, it sold 19,877 cases at \$2.75 per case.
    a. What is the percent change in price year-over-year?
    b. What is the percent change in quantity year-over-year?
    c. What is the percent change in total revenue year-over-year? (Hint: revenue = price × quantity)
14. A bottle of liquid laundry detergent priced at \$16.99 for a 52-load bottle has been changed to \$16.49 for a 48-load bottle. By what percentage has the price per load changed?

Challenge, Critical Thinking, & Other Applications

15. At a boardroom meeting, the sales manager is happy to announce that sales have risen from \$850,000 to \$1,750,000 at a rate of 4.931998% per year. How many years did it take for the sales to reach \$1,750,000?
16. The Nova Scotia Pension Agency needs to determine the annual cost of living adjustment (COLA) for the pension payments made to its members. To do this, it averages the consumer price index (CPI) for both the previous fiscal year and the current fiscal year. It then calculates the percent change between the two years to arrive at the COLA. If CPI information is as follows, determine the COLA that the pensioners will receive.

	Previous Fiscal Year	Current Fiscal Year
November	109.2	111.9
December	109.4	112.0

	Previous Fiscal Year	Current Fiscal Year
January	109.4	111.8
February	110.2	112.2
March	111.1	112.6
April	111.6	113.5
May	112.1	114.6
June	111.9	115.4
July	112.0	115.8
August	111.7	115.6
September	111.9	115.7
October	111.6	114.5

17. During The Bay's warehouse clearance days, it has reduced merchandise by 60%. As a bonus, today is Scratch 'n' Save day, where you can receive up to an additional 25% off the reduced price. If you scratched the maximum of 25% off, how many dollars would you save off an item that is regularly priced at $275.97? What percent savings does this represent?
18. Federal Canadian tax rates for 2010 and 2011 are listed below. For example, you pay no tax on income within the first bracket, 15% on income within the next bracket, and so on. If you earned $130,000 in each year, by what percentage did your federal tax rate change? In dollars, what was the difference?

2010 Tax Brackets	Taxed at	2011 Tax Brackets
$0–$10,382	0%	$0–$10,527
$10,383–$40,970	15%	$10,528–$41,544
$40,971–$81,941	22%	$41,545–$83,088
$81,942–$127,021	26%	$83,089–$128,800
$127,022+	29%	$128,801+

19. Melina is evaluating two colour laser printers for her small business. A Brother model is capable of printing 21 colour pages per minute and operates 162.5% faster than a similar Hewlett-Packard model. She needs to print 15,000 pages for a promotion. How much less time (stated as a percentage) will it take on the Brother model?
20. A chocolate bar has been priced at $1.25 for a 52 gram bar. Due to vending machine restrictions, the manufacturer needs to keep the price the same. To adjust for rising costs, it lowers the weight of the bar to 48 grams.
    a. By what percentage has the price per gram changed?
    b. If this plan is implemented over two periods, what rate of change occurs in each period?

3.2: Averages (What Is Typical?)

Mechanics

Calculate a simple average for questions 1 and 2.

1. 8, 17, 6, 33, 15, 12, 13, 16
2. $1,500 $2,000 $1,750 $1,435 $2,210

Calculate a weighted average for questions 3 and 4.

3. 4, 4, 4, 4, 12, 12, 12, 12, 12, 12, 12, 15, 15
4.

Data	$3,600	$3,300	$3,800	$2,800	$5,800
Weight	2	5	3	6	4

Calculate a geometric average for exercises 5 and 6. Round all percentages to four decimals.

5. 5.4%, 8.7%, 6.3%
6. 10%, −4%, 17%, −10%

Applications

7. If a 298 mL can of soup costs $2.39, what is the average price per millilitre?
8. Kerry participated in a fundraiser for the Children's Wish Foundation yesterday. She sold 115 pins for $3 each, 214 ribbons for $4 each, 85 coffee mugs for $7 each, and 347 baseball hats for $9 each. Calculate the average amount Kerry raised per item.
9. Stephanie’s mutual funds have had yearly changes of 9.63%, -2.45%, and 8.5%. Calculate the annual average change in her investment.
10. In determining the hourly wages of its employees, a company uses a weighted system that factors in local, regional, and national competitor wages. Local wages are considered most important and have been assigned a weight of 5. Regional and national wages are not as important and have been assigned weights of 3 and 2, respectively. If the hourly wages for local, regional, and national competitors are $16.35, $15.85, and $14.75, what hourly wage does the company pay?
11. Canadian Tire is having an end-of-season sale on barbecues, and only four floor models remain, priced at $299.97, $345.49, $188.88, and $424.97. What is the average price for the barbecues?
12. Calculate the grade point average (GPA) for the following student. Round your answer to two decimals.

Course	Grade	Credit hours
Economics 100	D	5
Math 100	B	3
Marketing 100	C	4
Communications 100	A	2
Computing 100	A+	3
Accounting 100	B+	4

Grade	Grade Point
A+	4.5
A	4.0
B+	3.5
B	3.0
C+	2.5
C	2.0
D	1.0
F	0.0

13. An accountant needs to report the annual average age (the length of time) of accounts receivable (AR) for her corporation. This requires averaging the monthly AR averages, which are listed below. Calculate the annual AR average.

Month	Monthly AR Average
January	$45,000

Month	Monthly AR Average
February	$70,000
March	$85,000
April	$97,000
May	$145,000
June	$180,000
July	$260,000
August	$230,000
September	$185,000
October	$93,000
November	$60,000
December	$50,000

14. From January 2007 to January 2011, the annual rate of inflation has been 2.194%, 1.073%, 1.858%, and 2.346%. Calculate the average rate of inflation during this period.

Challenge, Critical Thinking, & Other Applications

15. Gabrielle is famous for her trail mix recipe. By weight, the recipe calls for 50% pretzels, 30% Cheerios, and 20% peanuts. She wants to make a 2 kg container of her mix. If pretzels cost $9.99/kg, Cheerios cost $6.99/kg, and peanuts cost $4.95/kg, what is the average cost per 100 g rounded to four decimals?
16. Caruso is the marketing manager for a local John Deere franchise. He needs to compare his average farm equipment sales against his local Case IH competitor's sales. In the past three months, his franchise has sold six $375,000 combines, eighteen $210,000 tractors, and fifteen $120,000 air seeders. His sales force estimates that the Case IH dealer has sold four $320,000 combines, twenty-four $225,000 tractors, and eleven $98,000 air seeders. Express the Case IH dealer's average sales as a percentage of the John Deere dealer's average sales.
17. You are shopping for shampoo and consider two brands. Pert is sold in a bundle package of two 940 mL bottles plus a bonus bottle of 400 mL for $13.49. Head & Shoulders is sold in a bulk package of three 470 mL bottles plus a bonus bottle of 280 mL for $11.29.
    a. Which package offers the best value?
    b. If the Head & Shoulders increases its package size to match Pert at the same price per mL, how much money do you save by choosing the lowest priced package?
18. The following are annual net profits (in millions of dollars) over the past four years for three divisions of Randy's Wholesale:

$$\text{Cosmetics: } \$4.5, \$5.5, \$5.65, \$5.9$$

$$\text{Pharmaceutical: } \$15.4, \$17.6, \$18.5, \$19.9$$

$$\text{Grocery: } \$7.8, \$6.7, \$9.87, \$10.75$$

Rank the three divisions from best performing to worst performing based on average annual percent change.

19. You are shopping for a Nintendo Wii gaming console and visit www.shop.com, which finds online sellers and lists their prices for comparison. Based on the following list, what is the average price for a gaming console (rounded to two decimals)?

NothingButSoftware.com	$274.99
eComElectronics	$241.79
NextDayPC	$241.00
Ecost.com	$249.99

Amazon	$169.99
eBay	$165.00
Buy.com	$199.99
HSN	$299.95
Gizmos for Life	$252.90
Toys ‘R’ Us	$169.99
Best Buy	$169.99
The Bay	$172.69
Walmart	$169.00

20. Juanita receives her investment statement from her financial adviser at Great-West Life. Based on the information below, what is Juanita’s average rate of return on her investments?

Investment Fund	Proportion of Entire Portfolio Invested in Fund	Fund Rate of Return
Real Estate	0.176	8.5%
Equity Index	0.073	36.2%
Mid Cap Canada	0.100	−1.5%
Canadian Equity	0.169	8.3%
US Equity	0.099	−4.7%
US Mid Cap	0.091	−5.7%
North American Opportunity	0.063	2.5%
American Growth	0.075	−5.8%
Growth Equity	0.085	26.4%
International Equity	0.069	−6.7%

3.3: Ratios, Proportions, and Prorating (It Is Only Fair)

Mechanics

For questions 1–4, reduce the ratios to their smallest terms.

1. a. $66 : 12$
   b. $48 : 112 : 80$
   c. $24 : 36 : 84 : 108$
2. a. $7.2 : 6$
   b. $0.03 : 0.035 : 0.02$
   c. $0.27 : 0.18 : 0.51 : 0.15$
3. a. $5\frac{1}{4} : 6$
   b. $\frac{3}{2} : \frac{9}{5} : \frac{21}{10}$
   c. $2\frac{1}{8} : \frac{9}{8} : 3\frac{3}{8} : 1$
4. a. $\frac{8}{3} : \frac{5}{9}$
   b. $\frac{5}{6} : \frac{1}{3} : 2$

   c. $\frac{1}{7} : \frac{1}{3} : \frac{1}{4} : \frac{1}{6}$

5. Reduce the following ratios to the smallest term of one. Round to two decimals as needed.
   a. $48 : 53$
   b. $5\frac{7}{8} : 2\frac{3}{4} : 3\frac{1}{12}$
   c. $\frac{2}{3} : 11\frac{1}{11} : 5\frac{5}{9} : 2.08$
6. Solve the following proportions for the unknown variable. Round to two decimals as needed.
   a. $7 : 12 = 109 : y$
   b. $3.23 : 4.07 : 2.12 = x : 55.9625 : 29.15$
   c. $q : \frac{5}{3} = 13.75 : 25$
7. Solve the following proportions for all of the unknown variables.
   a. $\$12.15 : \$38.30 : r = x : \$59,184.53 : \$26,998.93$
   b. $16 : 5 : 9 : 30 = g : h : i : \$397,767$

For questions 8 and 9, prorate the total as indicated by the ratio.

8. $\text{Ratio } = 3 : 2, \ \text{Total } = \$11,368.25$
9. $\text{Ratio } = 7 : 5 : 3, \ \text{Total } = \$46,923.90$

Applications

10. EB Games sells gaming consoles. Last month, it sold $22,500 worth of Nintendo Wii consoles, $31,500 worth of Microsoft Xbox consoles, and $18,000 worth of Sony PlayStation consoles. Express the ratio of Wii to Xbox to PlayStation sales in its lowest terms.
11. The manufacturing cost of a deluxe candle is made up of $2.40 in paraffin, $1.20 in dye, $1.60 in overhead, and $4.40 in direct labour. Express the ratio between these costs respectively in their lowest terms.
12. If one Canadian dollar buys $1.0385 of US dollars, how much US money can you spend so that you do not exceed the maximum duty-free amount of $800 Canadian on your next US vacation?
13. For every $100 of retail spending, the average Canadian spends approximately $19.12 at motor vehicle dealerships and $18.35 at grocery stores. If the average Canadian spends $49,766 per year on retail spending, how much more does a Canadian spend annually at motor vehicle dealerships than at grocery stores?
14. Marina has a three-sevenths interest in a partnership. If she sells one-quarter of her interest for $8,250, what is the implied value of the partnership?
15. You are making a punch for an upcoming party. The recipe calls for 2½ parts ginger ale to ⅘ parts grenadine to 1⅜ parts vodka. If your punch bowl can hold 8.5 litres, how much (in litres) of each ingredient is needed? Round the answers to two decimals as needed.

Challenge, Critical Thinking, & Other Applications

16. A manufacturing facility requires one member of the board of directors for every 10 executives. There are five managers for every executive and eight workers for every manager. The average salary paid to directors is $105,000 along with $80,000 to executives, $55,000 to managers, and $30,000 to workers. If the facility has 1,383 employees, what is the total labour cost?
17. It is common for strip mall owners to allocate to their tenants general overhead costs such as snow clearing, security, and parking lot maintenance in one of two ways. The first method involves allocating these costs on the basis of the square metres each tenant operates. The second method involves allocating these costs on the basis of the number of annual transactions made by each tenant. At Charleswood Square, there are three tenants. The 7-Eleven takes up 150 square metres, Tim Hortons takes up 235 square metres, and Quiznos takes up 110 square metres. Last year, 7-Eleven had 350,400 transactions, Tim Hortons had 657,000 transactions, and Quiznos had 197,100 transactions. Total overhead costs for Charleswood Square last year were $25,980. Determine which method of allocation each tenant would prefer and how much of a savings in percentage its preferred method would represent relative to the other allocation method.
18. The average annual Canadian cellphone plan covers 780 minutes of voice time, 600 text messages, and 8 video messages with a total annual cost of $500.63. However, some Canadians are high-usage consumers and average 439 minutes per month.
    a. Assuming the same proportion of usage and that each component shares the cost equally on a per-unit basis, how many text messages and video messages do these high-usage Canadian consumers average per year? How much should they pay

annually for their cellphone usage?

   b. A similar typical cellphone plan in the Netherlands has a total annual cost of $131.44. How much would a high-usage plan cost annually in the Netherlands?

19. Procter & Gamble (P&G) is analyzing its Canadian sales by region: 23% of sales were on the West coast; 18% were in the Prairies, 45% were in central Canada, and 14% were in Atlantic Canada. Express and reduce the ratio of P&G's respective sales to the smallest term of one. Round to two decimals.
20. A company has 14 managers, 63 supervisors, and 281 workers. After a record profit year, the company wants to distribute an end-of-year bonus to all employees. Each worker is to get one share, each supervisor gets twice as many shares as workers, and each manager gets twice as many shares as supervisors. If the bonus to be distributed totals $1,275,000, how much does each manager, supervisor, and worker receive?

Review Exercises

Mechanics

1. Compute an average for the following data set: $1,368.95 $747.45 $5,362.25 $699.99 $2,497.97
2. Compute an average mark for the following quiz results.

Mark	0	1	2	3	4	5	6	7	8
# of Students	0	1	0	3	4	12	12	6	2

3. Solve each of the following for the unknown.

	Last Year's Sales	This Year's Sales	Percent Change
a.	$325,000	$377,000	?
b.	?	$5,460,000	225%
c.	$12,675	?	−38%

4. The following problems involve a series of percent changes. Calculate the annual rate of change.

	Initial Value	Number of % Changes	Final Value
a.	$4,300	2	$4,568.75
b.	$300	3	$114

5. Reduce the following ratios to their smallest terms.
   a. $112 : 32 : 144$
   b. $1.44 : 0.84$
   c. $\frac{1}{4} : \frac{2}{5} : \frac{5}{8}$
   d. $\frac{17}{27} : \frac{3}{4}$
6. Reduce the following ratios to the smallest term of one. Round to two decimals as needed.
   a. $\frac{3}{13} : \frac{3}{7}$
   b. $0.05325 : 0.0615 : 0.0445$
   c. $38 : 95 : 27 : 47$
7. Solve the following proportions for the unknown variables.
   a. $6 : x = 27 : 18$
   b. $10 : 12.5 : 20 = q : r : 50$
8. Prorate the total as indicated by the ratio. Ratio = 9 : 8 : 3, $\text{Total } = \$25,000$

Applications

9. In the year 2000, Canada's population was 31,496,800, and 40.3% of Canadians used the Internet. By 2008, Canada's population was 33,212,696, and 84.3% of Canadians used the Internet. Calculate the annual rate of change, rounded to two decimals, in the number of Canadians using the Internet.
10. Maurice, Jennifer, Natasha, and Sinbad just graduated from their business program at Mohawk College and decided to go into business together. Each individual invested funds in the ratio of 4 : 7 : 3 : 6, respectively. If Natasha invested $24,000, what was the total investment in the business?
11. An Extra Foods store needs to allocate overhead expenses of $336,000 across its three departments based on their sales. If refrigerated foods accounted for $765,000 in sales, non-refrigerated foods accounted for $981,500 in sales, and non-food products accounted for $694,000 in sales, what amount of overhead expenses should be allocated to each department?
12. Boris's neighbourhood just had a neighbourhood garage sale. He decided to make some money by operating a food stand. If he sold 115 hot dogs for $2.50 each, 260 bags of chips for $0.75 each, and 412 soft drinks for $1.25 each, what were his total sales for the day, and how much did he average per transaction (rounded to two decimals)?
13. Julian is trying to understand his investment portfolio better. He currently has $40,000 in Canadian stocks, $7,500 in US stocks, and $2,500 in international stocks. Express the respective ratio of his stocks in their lowest terms.
14. Abdul is at the end of the third year of a five-year investment. His account has a current balance of $1,687.14. Over the past three years, his account has increased 3%, 4%, and 5% respectively. In the next two years, his account is supposed to increase 5.5% and then 6%. How much did Abdul initially invest? How much will he have at the end of the five years?
15. You subscribed to 90 issues of The Hockey News magazine at a total cost of $418.50. After receiving 57 issues, you decide to cancel your subscription. What refund should you receive on your subscription?
16. If one Canadian dollar can purchase $0.7328 euros, 0.6379 pounds, or 83.3557 yen, how much of each currency could you purchase with $925 Canadian?

Challenge, Critical Thinking, & Other Applications

17. Little Caesars Pizza just delivered two large pizzas. Each has been cut up into 12 slices. If Sergei can eat only one-fifth of what Richard can eat, Basilio eats twice as much as Sergei, and Alphonso eats 100% more than Basilio, how many slices of pizza should each receive (assuming they'll eat all of the pizza)? If the total bill including tip came to $15.00, how much does each person owe based on how much he ate?
18. Vincent Manufacturing has two divisions with three different product lines in each. Sales for the current year are as follows:

	Division One		Division Two	
Product	**Unit Price**	**Unit Sales**	**Unit Price**	**Unit Sales**
A	$19.95	3,456	$25.31	3,543
B	$31.56	6,852	$17.41	8,524
C	$22.32	1,894	$29.99	2,005

    a. The company wants to set up a ratio of the weighted average price in Division One compared to the weighted average price in Division Two. Calculate the ratio and reduce the ratio to the smallest term of one. Round all numbers to two decimals throughout.
    b. Division One's weighted average price is forecasted to increase by 6% next year and 3% in the third year. Division Two's weighted average price is forecasted to increase by 4% next year and 6% in the third year. Calculate the ratios in each of the second and third years, and reduce each ratio to the smallest term of one. Round all numbers to two decimals throughout.

19. In aviation, one of the measures of success is known as the load factor. This is a percentage that is calculated by taking the number of air miles that consumers flew and dividing it by the total air miles that were available for consumers to purchase. In 2008, WestJet Airlines had 17.139 billion available miles, of which consumers purchased 13.731 billion miles. In 2009, it had 17.558 billion available miles, of which consumers purchased 13.835 billion miles. In 2010, it had 19.535 billion available miles, of which consumers purchased 15.613 billion miles. Calculate the percent change in the load factors for each year and the average annual rate of change. Round your answers to four decimal places.

20. As you prepare for vacation, you need to take your spending money of $750 Canadian and convert it to various currencies. On your vacation, you will be spending seven days in New Zealand, eleven days in India, and six days in South Africa. You figure that you will spend your money in proportion to the number of days in each country. If one Canadian dollar equals 1.3136 New Zealand dollars, 46.3478 Indian rupees, and 7.2401 South African rand, how much of each currency will you need to leave home with?

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CHAPTER OVERVIEW

4: Human Resources and Economic Applications (It's All about the People)

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4.0: Introduction

Has your employer paid you all that you have earned? While this problem is difficult to track, the Canadian Payroll Association estimates the frequency of payroll errors to be 5.7%.[1] Studies in the United States have found that 33% of employers have made some form of payroll mistake, whether in the calculation of pay or in deductions.[2] This means that you must always check your paycheque to ensure that you receive the correct amount of pay. If you are overpaid, you are not entitled to keep the overage and must reimburse your employer. If you are underpaid, your employer owes you money.

We all work to earn enough income to cover our expenses, such as mortgages, car payments, and recreational activities. There are many ways to earn that income, ranging from straight salaries and hourly wages to commission rates and piecework wages. The amount of money you generate by working is called your gross earnings. If you are an employee, this amount is then subject to automatic deductions, including federal taxes, provincial taxes, employment insurance, Canada Pension Plan, along with any other deductions required by your employer. All of these government deductions are submitted to the government on your behalf. What is left over after all of these deductions is your net pay, which is your take-home amount.

Payroll forms one of the largest and most contentious expenses in most organizations. It is critical to get it right. Workers often go on strike if they believe their take-home pay is unfair. And payroll errors are costly and time consuming to fix, aside from their impact on employee satisfaction.

Payroll decisions are affected by inflation. Ask your parents how much things cost when they were young. In the 1950s, my parents attended movie theaters for 25¢ admission. A chocolate bar was 5¢. Today, attending a first-run movie costs $8.99 or more, and chocolate bars are typically around one dollar. To keep up with these rising prices, incomes have constantly increased. In 1959, family income was approximately $4,500 compared to the 2009 median family income of $68,410.[3] Can you still afford the same things that people could afford back then? Business indexes are regularly used to understand the change over time in various quantities or how a quantity at one time point compares to the corresponding value at a certain reference time point.

For example, the next time you are offered a raise at work, an index can tell you whether you are truly receiving a raise or not! Assume that product prices have risen by 2% year-over-year. If you are offered a raise less than 2%, you have taken a wage cut and have less purchasing power than before. A raise of 2% allows you to at least break even, and anything above 2% increases your purchasing power.

Whether you are pursuing a career in human resources, economics, or some other field, the issues of gross earnings, income taxes, and indexes remain important both personally and professionally. This chapter examines some of the basic mathematical concepts involved in these calculations.

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4.1: Gross Earnings (Off to Work You Go)

You work hard at your job, and you want to be compensated properly for all the hours you put in. Assume you work full time with an hourly rate of pay of $10. Last week you worked eight hours on Sunday and eight hours on Monday, which was a statutory holiday. Then you took Tuesday off, worked eight hours on each of Wednesday and Thursday, took Friday off, and worked 10 hours on Saturday. That's a total of 42 hours of work for the week. What is your gross pay? Give or take a small amount depending on provincial employment standards, it should be about $570. But if you don't understand how to calculate gross earnings, you could be underpaid without ever realizing it.

Here are some notes about the content in this chapter: About 10% of Canadian workers fall under federal employment standards, which are not discussed here. This textbook generalizes the most common provincial employment standards; however, to calculate your earnings accurately requires you to apply your own provincial employment standards legislation. Part-time employment laws are extremely complex, so this textbook assumes in all examples that the employee is full time.

This section addresses the calculation of gross earnings, which is the amount of money earned before any deductions from your paycheque. The four most common methods of employee remuneration include salaries, hourly wages, commissions, and piecework wages.

Salary and Hourly Wages

One ad in the employment classifieds indicates compensation of $1,270 biweekly, while a similar competing ad promotes wages of $1,400 semi-monthly. If both job ads are similar in every other way, which job has the higher annual gross earnings? To make this assessment, you must understand how salaries work. A salary is a fixed compensation paid to a person on a regular basis for services rendered. Most employers pay employees by salary in occupations where the employee's work schedule generally remains constant.

In contrast, an hourly wage is a variable compensation based on the time an employee has worked. In contrast to a salary, this form of compensation generally appears in occupations where the number of hours is unpredictable or continually varies from period to period.

Employment Contract Characteristics[1]

Salaried and hourly full-time employees are similar with regard to their gross earnings. The major earnings issues in an employment contract involve regular earnings structure, overtime, and holidays.

Regular Earnings Structure

An agreement with your employer outlines the terms of your employment, including the time frame and frequency of pay.

1. **Time Frame**. For salaried employees, the time frame that the salary covers must be clearly stated. For example, you could receive a salary of $2,000 monthly or $50,000 annually. Notice that each of these salaries is followed by the specific time frame for the compensation. For hourly employees, the time frame requires identification of the wage earned per hour.
2. **Frequency**. How often the gross earnings are paid out to the employee must be defined.
    - **Monthly**: Earnings are paid once per month. By law, employees must receive compensation from their employer at least once per month, which equals 12 times per year.
    - **Daily**: Earnings are paid at the end of every day. This results in about 260 paydays per year (5 days per week multiplied by 52 weeks per year). In a leap year, there might be one additional payday.
    - **Weekly**: Earnings are paid once every week. This results in 52 paydays in any given year since there are 52 weeks per year.
    - **Biweekly**: Earnings are paid once every two weeks. This results in 26 paydays in any given year since there are $52 \div 2 = 26$ biweekly periods per year.
    - **Semi-monthly**: Earnings are paid twice a month, usually every half month (meaning on the 15th and last day of the month). This results in 24 paydays per year.

Thus, the earnings structure specifies both the time frame and the frequency of earnings. For a salaried employee, this may appear as “$2,000 monthly paid semi-monthly” or “$50,000 annually paid biweekly.” For an hourly employee, this may appear as “$10 per hour paid weekly.” No matter whether you are salaried or hourly, earnings determined by your regular rate of pay are called your regular earnings.

Overtime

Overtime is work time in excess of your regular workday, regular workweek, or both. In most jurisdictions it is paid at 1.5 times your regular hourly rate (called time-and-a-half), though your company may voluntarily pay more or a union may have negotiated a more favorable rate such as two times your regular hourly rate (called double time). A contract with an employer will specify your regular workday and workweek, and some occupations are exempt from overtime. Due to the diversity of occupations, there is no set rule on what constitutes a regular workday or workweek. In most jurisdictions, a regular workweek is eight hours per day and 40 hours per week. Once you exceed these regular hours, you are eligible to receive overtime or premium earnings, which are based on your overtime rate of pay.

Holidays

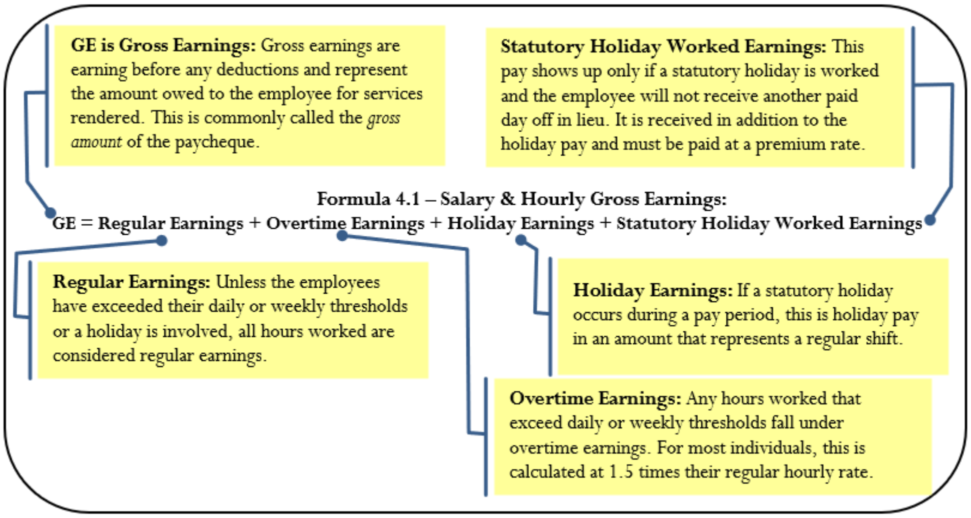
A statutory holiday is a legislated day of rest with pay. Five statutory holidays are recognized throughout Canada, namely, New Year's Day, Good Friday (or Easter Monday in Quebec), Canada Day, Labour Day, and Christmas Day. Each province or territory has an additional four to six public holidays (or general holidays), which may include Family Day (known as Louis Riel Day in Manitoba and Islander Day in PEI) in February, Victoria Day in May, the Civic Holiday in August, Thanksgiving Day in October, Remembrance Day in November, and Boxing Day in December. These public holidays may or may not be treated the same as statutory holidays, depending on provincial laws.

Statutory and public holidays generally require employees to receive the day off with pay. If the holiday falls on a nonworking day, it is usually the next working day that is given off instead. For example, if Christmas Day falls on a Saturday, typically the following Monday is given off with pay. Here's how holidays generally work (though you should always consult legislation for your specific jurisdiction):

- You should be given the day off with pay, called holiday earnings. The holiday earnings are in the amount of a regular day's earnings, and the hours involved count toward your weekly hourly totals for overtime purposes (preventing employers from shifting your work schedule that week).
- If you are required to work, the employer must offer another day off in lieu with pay. Your work on the statutory holiday is then paid at regular earnings and the hours involved contribute toward your weekly hourly totals for overtime purposes. You are paid holiday earnings on your future day off.
- If you are required to work and no day or rest is offered in lieu, this poses the most complex situation. Under these conditions:
    - You are entitled to the holiday earnings you normally would have received for the day off. The hours that make up your holiday earnings contribute toward your weekly hourly totals for overtime purposes (again, consult your local jurisdiction).
    - In addition, for the hours you worked on the statutory holiday you are entitled to overtime earnings known as statutory holiday worked earnings. These hours do not contribute toward your weekly hourly totals for overtime purposes since you are already compensated at a premium rate of pay. For example, assume you work eight hours on Labour Day, your normal day is eight hours, and you won't get another day off in lieu. Your employer owes you the eight hours of holiday earnings you should have received for getting the day off plus the eight hours of statutory holiday worked earnings for working on Labour Day.

The Formula

The four forms of compensation consist of regular earnings, overtime earnings, holiday earnings, and statutory holiday worked earnings. Add these four elements together to determine total gross earnings. Formula 4.1 shows the relationship.



GE is Gross Earnings: Gross earnings are earning before any deductions and represent the amount owed to the employee for services rendered. This is commonly called the *gross amount* of the paycheque.

Statutory Holiday Worked Earnings: This pay shows up only if a statutory holiday is worked and the employee will not receive another paid day off in lieu. It is received in addition to the holiday pay and must be paid at a premium rate.

Formula 4.1 – Salary & Hourly Gross Earnings:

GE = Regular Earnings + Overtime Earnings + Holiday Earnings + Statutory Holiday Worked Earnings

Regular Earnings: Unless the employees have exceeded their daily or weekly thresholds or a holiday is involved, all hours worked are considered regular earnings.

Holiday Earnings: If a statutory holiday occurs during a pay period, this is holiday pay in an amount that represents a regular shift.

Overtime Earnings: Any hours worked that exceed daily or weekly thresholds fall under overtime earnings. For most individuals, this is calculated at 1.5 times their regular hourly rate.

Formula 4.1

How It Works

Salaried Employees

To calculate the total gross earnings for a salaried employee, follow these steps:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. If the employee has only regular hours of pay, skip to step 6.

Step 2: Calculate the employee's equivalent hourly rate of pay. This means converting the salary into an equivalent hourly rate:

$$\text{Equivalent Hourly Rate } = \frac{\text{Annual Salary}}{\text{Annual Hours Worked}}$$

For example, use a $2,000 monthly salary requiring 40 hours of work per week. Express the salary annually by multiplying it by 12 months, yielding $24,000. Express the 40 hours per week annually by multiplying by 52 weeks per year, yielding 2,080 hours worked. The equivalent hourly rate is $\$24,000 \div 2,080 = \$11.538461$.

Step 3: Calculate any holiday earnings. Take the unrounded hourly rate and multiply it by the number of hours in a regular shift, or

$$\text{Holiday Earnings } = \text{ Unrounded Hourly Rate } \times \text{ Hours in a Regular Shift}$$

A salaried employee earning $11.538461 per hour having a daily eight-hour shift receives $\$11.538461 \times 8 = \$92.31$ in holiday earnings.

Step 4: Calculate any overtime earnings.

a. Determine the overtime hourly rate of pay by multiplying the unrounded hourly rate by the minimum standard overtime factor of 1.5 (this could be higher if the company pays a better overtime rate than this):

$$\text{Overtime Hourly Rate } = \text{ Unrounded Hourly Rate } \times 1.5$$

b. Round the final result to two decimals. For the salaried worker, $\$11.538461 \times 1.5 = \$17.31$ per overtime hour.
c. Multiply the overtime hourly rate by the overtime hours worked.

Step 5: Calculate any statutory holiday worked earnings which is similar to calculating overtime earnings:

$$\text{Statutory Holiday Worked Earnings} = \text{Statutory Hourly Rate } \times \text{ Statutory Hours Worked}$$

The statutory hourly rate is at minimum 1.5 times the unrounded hourly rate of pay. The salaried employee working eight hours on a statutory holiday receives $\$17.31 \times 8 = \$138.48$.

Step 6: Calculate the gross earnings paid at the regular rate of pay. Take the amount of the salary and divide it by the number of pay periods involved, then subtract any holiday earnings:

$$\text{Regular Earnings } = \frac{\text{Salary}}{} - \text{ Holiday Earnings}$$

$$\text{Pay Periods}$$

You need to calculate the number of pay periods based on the regular earnings structure. For example, an annual $52,000 salary paid biweekly would have 26 pay periods annually. Therefore, a regular paycheque is $\$52,000 \div 26 = \$2,000$ per paycheque. As another example, a $\$2,000$ monthly salary paid semi-monthly has two pay periods in a single month, resulting in regular earnings of $\$2,000 \div 2 = \$1,000$ per paycheque. If a holiday is involved in the pay period, you must deduct the holiday earnings from these amounts.

Step 7: Calculate the total gross earnings by applying Formula 4.1.

Hourly Employees

To calculate the total gross earnings for an hourly employee, follow steps similar to those for the salaried employee:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. It is usually best to set up a table similar to the one below. This table allows you to visualize the employee's week at a glance along with totals, enabling proper assessment of their hours worked.

This table separates the four types of earnings into different rows. Enter the information into the table about the employee's workweek, placing it in the correct day and on the correct row. If any daily thresholds are exceeded, place the appropriate hours into the overtime row. Once you have completed this, total the regular hours and holiday hours for the week and check to see if they exceed any regular weekly threshold. If so, starting with the last workday of the week and working backwards, convert regular hours into overtime hours until you have reduced the regular hours to the regular weekly total.

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular										
Holiday										
Overtime										
Holiday Worked										
								Total Earnings		

Once you have completed the table, if the employee has only regular hours of pay, skip to step 5. Otherwise, proceed with the next step.

Step 2: Calculate any holiday earnings. Take the hourly rate and multiply it by the number of hours in a regular shift:

$$\text{Holiday Earnings} = \text{Hourly Rate} \times \text{Hours in a Regular Shift}$$

Step 3: Calculate any overtime earnings.

a. Determine the overtime hourly rate of pay rounded to two decimals by multiplying the hourly rate by the minimum standard overtime factor of 1.5 (or higher).

$$\text{Overtime Hourly Rate} = \text{Hourly Rate} \times 1.5$$

b. Multiply the overtime hourly rate by the overtime hours worked.

Step 4: Calculate any statutory holiday worked earnings. This is the same procedure as for a salaried employee.

Step 5: Calculate the gross earnings paid at the regular rate of pay. Take the number of hours worked and multiply it by the hourly rate of pay:

$$\text{Regular Earnings} = \text{Hours Worked} \times \text{Hourly Rate}$$

For example, 20 hours worked at 10 per hour with no holiday earnings is $20 \times \$10 = \$200$.

Step 6: Calculate the total gross earnings by applying Formula 4.1.

Things To Watch Out For

Be careful about the language of the payment frequency. It is very common to confuse semi and bi, and sometimes businesses use the terms incorrectly. The term semi generally means half. Therefore, to be paid semi-monthly means to be paid every half month. The term bi means two. Therefore, to be paid biweekly means to be paid every two weeks. Some companies that pay semi-monthly mistakenly state that they pay bimonthly, which in fact would mean they paid every two months.

Paths To Success

In calculating the pay for a salaried employee, this textbook assumes for simplicity that a year has exactly 52 weeks. In reality, there are 52 weeks plus one day in any given year. In a leap year, there are 52 weeks plus two days. This extra day or two has no impact on semi-monthly or monthly pay, since there are always 24 semi-months and 12 months in every year. However, weekly and biweekly earners are impacted as follows:

- If employees are paid weekly, approximately once every six years there are 53 pay periods in a single year. This would "reduce" the employees' weekly paycheque in that year. For example, assume they earn $52,000 per year paid weekly. Normally, they are paid $\$52,000 \div 52 = \$1,000$ per week. However, since there are 53 pay periods approximately every sixth year, this results in $\$52,000 \div 53 = \$981.13$ per week for that year.
- If employees are paid biweekly, approximately once every 12 years there are 27 pay periods in a single year. This has the same effect as the extra pay period above. For example, if they are paid $\$52,000$ per year biweekly they normally receive $\$52,000 \div 26 = \$2,000$ per biweekly cheque. Approximately every twelfth year, they are paid $\$52,000 \div 27 = \$1,925.93$ per biweekly cheque for that year.

Many employers ignore these technical nuances in pay structure since the extra costs incurred to modify payroll combined with the effort required to calm down employees who don't understand the smaller paycheque are not worth the savings in labour. Therefore, most employers treat every year as if it has 52 weeks (26 biweeks) regardless of the reality. In essence, employees receive a bonus paycheque approximately once every six or twelve years!

? Exercise $4.1.1$: Give It Some Thought

A salaried employee whose normal workweek is 8 hours per day and 40 hours per week works 8 hours each day from Monday to Saturday inclusive, where Monday was a statutory holiday. Which of the following statements is correct (assuming she will not get another day off in lieu of the holiday)?

a. The employee receives only her regular weekly earnings for 40 hours.
b. The employee receives 32 hours of regular earnings, 8 hours of holiday earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
c. The employee receives 40 hours of regular earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
d. The employee receives 40 hours of regular earnings and 8 hours of overtime earnings.

Answer

> The correct answer is b. When working on the statutory holiday and not getting another day off in lieu, the salaried employee is eligible for eight hours of holiday earnings plus eight hours of statutory holiday worked earnings. The holiday earnings count toward the weekly total, but not the statutory holiday worked earnings. Thus the employee from Tuesday to Friday inclusive worked an additional 32 regular hours, bringing her weekly total to 40 hours. The work on Saturday exceeds her 40 hour workweek, and therefore all eight hours are paid as overtime earnings.

✓ Example $4.1.1$: Salary with Overtime and a Holiday

Tristan is compensated with an annual salary of $65,000 paid biweekly. His regular workweek consists of four 10-hour days, and he is eligible for overtime at 1.5 times pay for any work in excess of his regular requirements. Tristan worked regular hours for the first two weeks. Over the next two weeks, Tristan worked his regular hours and became eligible for 11 hours of overtime. During these two weeks, he worked his regular shift on Good Friday but his employer has agreed to give him another day off with pay in the future.

a. Determine Tristan's gross earnings for the first two-week pay period.

b. Determine Tristan's gross earnings for the second two-week pay period.

Solution

You have been asked to calculate Tristan's gross earnings, or GE, for two consecutive pay periods.

What You Already Know

Step 1:

You know Tristan's compensation:

Annual Salary = $65,000

Pay Periods = biweekly = 26 times per year

$\text{Annual Hours } = 10/\text{day} \times 4 \text{ days/week } \times 52 \text{ weeks } = 2,080$

You also know his work schedule:

First Two Weeks = regular pay

Overtime in First Two Weeks = 0

Second Two Weeks = regular pay

Overtime in Second Two Weeks = 11 hours

There is a holiday in the second two weeks, but he will receive another day off in lieu.

How You Will Get There

For each biweekly pay period, apply the following steps:

Step 2: Calculate Tristan’s equivalent hourly rate of pay.

Step 3: Calculate holiday earnings using the equivalent hourly rate.

Step 4: Calculate overtime earnings by taking the overtime hourly pay rate multiplied by hours worked.

Step 5: Calculate statutory holiday worked earnings at the premium rate of pay.

Step 6: Calculate regular earnings:

Step 7: Determine total gross earnings using Formula 4.1.

Perform

Step 2:

First two weeks:

There are only regular earnings; skip to step 6.

$$\text{Equivalent Hourly Rate } = \frac{\$65,000}{2,080} = \$31.25$$

Step 3:

Holiday earnings = $0

Tristan will take his holiday pay another day. Therefore, he is not eligible for this pay, and his work counts as regular hours; holiday earnings = $0

Step 4:

First two weeks:

Overtime earnings = $0

$\text{Overtime hourly rate } = \$31.25 \times 1.5 = \$46.88$

$\text{Overtime Earnings } = \$46.88 \times 11 = \$515.68$

Step 5:

Statutory Worked = $0

Second two weeks:

Since Tristan is receiving another day off in lieu, he is not eligible for this pay; statutory Worked = $0

Step 6:

$$\text{Regular Earnings } = \frac{\$65,000}{26} - \$0 = \$2,500$$

Second two weeks:

$$\text{Regular Earnings } = \frac{\$65,000}{26} - \$0 = \$2,500$$

Step 7:

$GE = \$2,500 + \$0 + \$0 + \$0 = \$2,500$

Second two weeks:

$GE = \$2,500 + \$515.68 + \$0 + \$0 = \$3,015.68$

For the first two-week pay period, Tristan worked only his regular hours and therefore is compensated $2,500 as per his salary. For the second two-week pay period, Tristan is eligible to receive his regular hours plus his overtime, but he receives no additional pay for the worked holiday since he will receive another day off in lieu. His total gross earnings are $3,015.68.

✓ Example 4.1.2: A Week in the Life of an Hourly Wage Earner

Marcia receives an hourly wage of $32.16 working on an automotive production line. Her union has negotiated a regular work day of 7.25 hours for five days totalling 36.25 hours for the week. Overtime is paid at 1.5 times her regular rate for any work that exceeds the daily or weekly limits. If work is required on a statutory holiday, her company does not give a day off in lieu and pays a premium rate of 2.5 times her regular rate. Last week, Marcia worked nine hours on Monday, her regular hours on Tuesday through Friday inclusive, and three hours on Saturday. Friday was a statutory holiday. Calculate Marcia's gross earnings for the week.

Solution

You need to calculate Marcia's gross earnings, or GE, for the week.

What You Already Know

Step 1: You know Marcia's pay structure and workweek:

Regular Hourly Rate = $32.16

Overtime Hourly Pay = 1.5 times

Statutory Holiday Worked Rate = 2.5 times

Exceeding 7.25 hours daily or 36.25 hours weekly is overtime.

Sunday	0
Monday	9
Tuesday	7.25
Wednesday	7.25
Thursday	7.25
Friday (Statutory holiday)	7.25
Saturday	3

How You Will Get There

Step 1 (continued): Take Marcia's hours and place them into the table. Assess whether any daily or weekly totals are considered overtime and make any necessary adjustments.

Step 2: Calculate holiday earnings using the hourly rate.

Step 3: Calculate overtime earnings by determining the overtime hourly rate of pay multiplied by hours worked.

Step 4: Calculate statutory holiday worked earnings at the premium rate of pay.

Step 5: Calculate regular earnings.

Step 6: Determine total gross earnings.

Perform

Step 1:

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular	0	(1)7.25	7.25	7.25	7.25		3	(3)39.25	$32.16	
Holiday						(2)7.25				
Overtime		(1)7.25						1.75		
Holiday Worked						(2)7.25		7.25		
								Total Earnings		

1. She worked nine hours. Therefore, the first 7.25 hours are regular pay and the last 1.75 are overtime pay.
2. Friday was a statutory holiday, and she will not receive another day off in lieu. She must receive statutory holiday worked pay in addition to her hours worked.
3. Note the weekly total of 36.25 has been exceeded by three hours. Move Saturday's hours into overtime.

The following table is the final layout of her workweek:

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular	0	7.25	7.25	7.25	7.25			36.25	$32.16	
Holiday						7.25				
Overtime		1.75					3	4.75		
Holiday Worked						7.25		7.25		
								Total Earnings		

Steps 2–6:

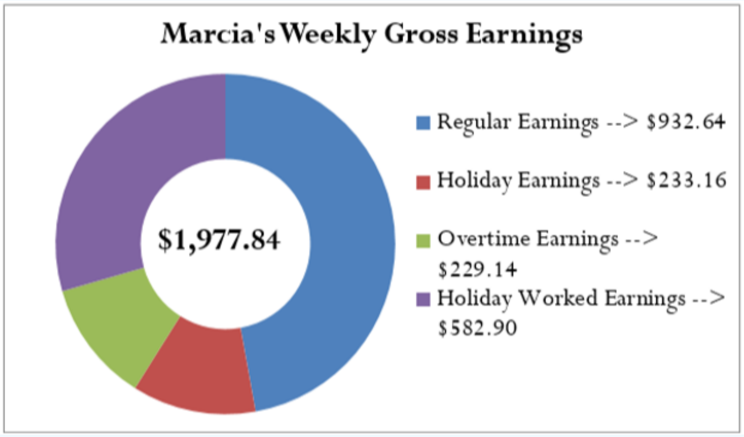
Explanations are noted in the table below.

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular	0	7.25	7.25	7.25	7.25			36.25	$32.16	(4)$932.64
Holiday						7.25				(1)$233.16

Overtime		1.75					3	4.75	(2)$48.24	(2)$229.14
Holiday Worked						7.25		7.25	(3)$80.40	(3)$582.90
								Total Earnings		(5)$1,977.84

1. $\text{Holiday Earnings } = 7.25 \times \$32.16 = \$233.16$
2. $\text{Overtime Hourly Rate } = \$32.16 \times 1.5 = \$48.24;\ \text{Overtime Earnings } = 4.75 \times \$48.24 = \$229.14$
3. $\text{Statutory Holiday Worked Rate } = \$32.16 \times 2.5 = \$80.40;\ \text{Statutory Worked Earnings } = 7.25 \times \$80.40 = \$582.90$
4. $\text{Regular Earnings } = 29 \times \$32.16 = \$932.64$
5. $GE = \$932.64 + \$229.14 + \$233.16 + \$582.90 = \$1,977.84$

Marcia will receive total gross earnings of $1,977.84 for the week.



Commission

Over the last two weeks you sold $50,000 worth of machinery as a sales representative for IKON Office Solutions Canada. IKON's compensation plan involves a straight commission rate of 3.5%. What are your gross earnings? If you sold an additional $12,000 in machinery, how much more would you earn?

Particularly in the fields of marketing and customer service, many workers are paid on a commission basis. A commission is an amount or a fee paid to an employee for performing or completing some form of transaction. The commission typically takes the form of a percentage of the dollar amount of the transaction. Marketing and customer service industries use this form of compensation as an incentive to perform: If the representative doesn't sell anything then the representative does not get paid. Issues to be discussed about commission include what constitutes regular earnings, how to handle holidays and overtime, and the three different types of commission structures.

- **Regular Earnings**. All commissions are considered to be regular earnings. To calculate the gross earnings for an employee, take the total amount of the transactions and multiply it by the commission rate:

  $$\text{Gross Earnings } = \text{ Total Transaction Amount } \times \text{ Commission Rate}$$

  This is not a new formula. It is a specific application of Formula 2.2: Rate, Portion, Base. In this case, the Base is the total amount of the transactions, the Rate is the commission rate, and the Portion is the gross earnings for the employee.
- **Holidays and Overtime**. Commission earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the provincial standards on these matters vary widely and the mathematics involved do not necessarily follow any one procedure or calculation. As such, this textbook leaves these issues to be covered in a payroll administration course.
- **Types of Commission**. Commission earnings typically follow one of the following three structures:

- Straight Commission. If your entire earnings are based on your dollar transactions and calculated strictly as a percentage of the total, you are on straight commission. An application of Formula 2.2 (Rate, Portion, Base) calculates your gross earnings under this structure.
- Graduated Commission. Within a graduated commission structure, you are offered increasing rates of commission for higher levels of performance. The theory behind this method of compensation is that the higher rewards motivate employees to perform better. An example of a graduated commission scale is found in the table below.

Transaction Level	Commission Rate
$0–$100,000.00	3%
$100,000.01–$250,000.00	4.5%
$250,000.01–$500,000.00	6%
Over $500,000.00	7.5%

Recognize that the commission rates are applied against the portion of the sales falling strictly into the particular category, not the entire balance. Thus, if the total sales equal $150,000, then the first $100,000 is paid at 3% while the next $50,000 is paid at 4.5%.

- **Salary Plus Commission**. If your earnings combine a basic salary together with commissions on your dollar transactions, you have a salary plus commission structure. No new mathematics are required for this commission type. You must combine salary calculations, discussed earlier in this section, with either a straight commission or graduated commission, as discussed above. Usually this form of compensation pays the lowest commission rate since a basic salary is already provided.

How It Works

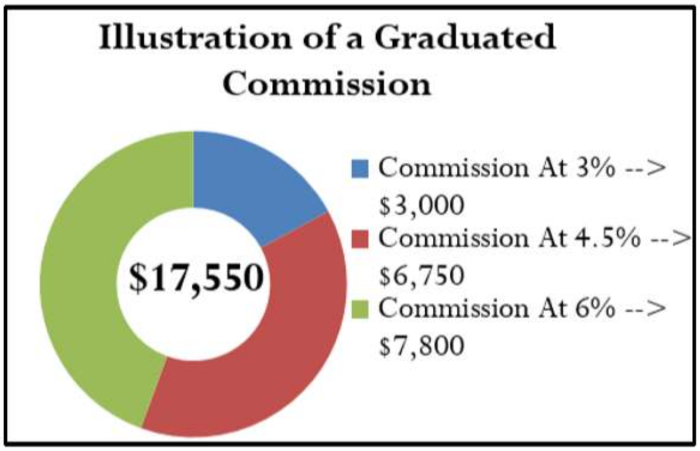
Follow these steps to calculate commission earnings:

Step 1: Determine which commission structure is used to pay the employee. Identify information on commission rates, graduated scales, and any salary.

Step 2: Determine the dollar amounts that are eligible for any particular commission rate and calculate commissions.

Step 3: Sum all earnings from every eligible commission rate plus any salary.

Let’s assume $380,000 of merchandise is sold. Using the previous table as our graduated commission scale, calculate commission earnings.



Step 1: Sales total $380,000 and all commission rates and scales are found in the table.

Step 2: The first $100,000 is compensated at 3%, equaling $3,000. The next $150,000 is compensated at 4.5%, equaling $6,750. The last $130,000 is compensated at 6%, equaling $7,800. There is no compensation at the 7.5% level since sales did not exceed $500,000.

Step 3: The total commission on sales of $380,000 is $\$3,000+\$6,750+\$7,800=\$17,550$.

✓ Example $4.1.3$: Different Types of Commissions

Josephine is a sales representative for Kraft Foods Canada. Over the past two weeks, she closed $325,000 in retail distribution contracts. Calculate the total gross earnings that Josephine earns if

a. She is paid a straight commission of 3.45%.
b. She is paid 2% for sales on the first $100,000, 3% on the next $100,000, and 4% on all remaining sales.
c. She is paid a base salary of $2,000 plus a commission of 3.5% on all sales above $100,000.

Solution

You need to calculate the gross earnings, or GE, for Josephine under various commission structures.

What You Already Know

Step 1:

For all three options, total sales = $325,000

a. This is a straight commission where the commission rate = 3.45%.

b. This is a graduated commission, structured as follows:

Sales Level	Rate
$0–$100,000.00	2%
$100,000.01–$200,000.00	3%
$200,000.01 and above	4%

c. This is a salary plus graduated commission, with salary = $2,000 and a graduated commission structure as follows:

Sales Level	Rate
$0–$100,000.00	0%
$100,000.01 and above	3.5%

How You Will Get There

Step 2:

Determine the sales eligible for each commission and calculate total commissions. Note that all calculations are $\text{Portion} = \text{Rate} \times \text{Base}$.

Step 3:

Sum all commissions plus any salary to calculate total gross earnings.

Perform

Steps 2 & 3:

a. $3.45\% \times \$325,000 = 0.0345 \times \$325,000 = \$11,212.50 = \text{ Total Gross Earnings}$

b.

Sales Level	Rate	Eligible Sales at Each Rate	Commission Earned
$0–$100,000.00	2%	$100,000 – $0 = $100,000	0.02 × $100,000 = $2,000
$100,000.01–$200,000.00	3%	$200,000 – $100,000 = $100,000	0.03 × $100,000 = $3,000
$200,000.01 and above	4%	$325,000 – $200,000 = $125,000	0.04 × $125,000 = $5,000

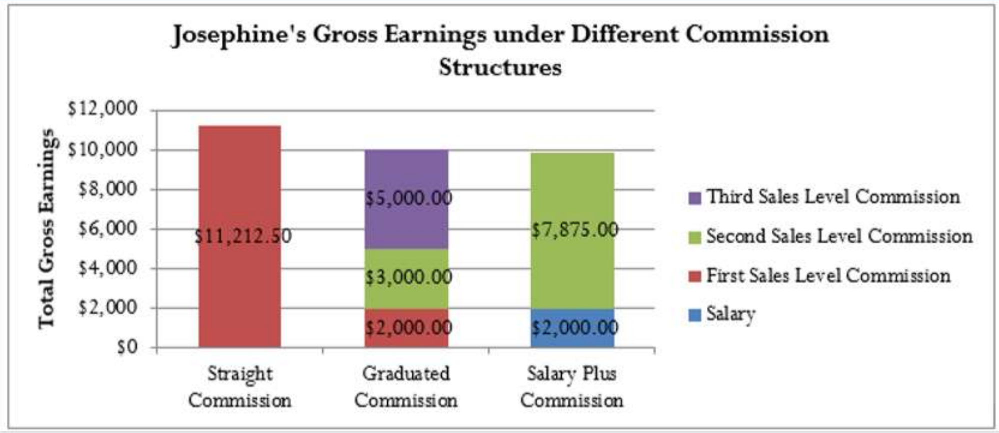
$\text{Total Gross Earnings } = \$2,000 + \$3,000 + \$5,000 = \$10,000$

c. $\text{Recall Base Salary } = \$2,000$

Sales Level	Rate	Eligible Sales at Each Rate	Commission Earned
\$0–\$100,000.00	0%	\$100,000 – \$0 = \$100,000	0 × \$100,000 = \$0
\$100,000.01 and above	3.5%	\$325,000 – \$100,000 = \$225,000	0.035 × \$225,000 = \$7,875

$\text{Total Gross Earnings } = \$0 + \$7,875 + \$2,000 = \$9,875$

If Josephine is paid under straight commission, her total gross earnings will be \$11,212.50. Under the graduated commission, she will receive \$10,000 in total gross earnings. For the salary plus commission, she will receive \$9,875 in total gross earnings.



Piecework

Have you ever heard the phrase "pay-for-performance"? Although this phrase has many interpretations in different industries, for some people this phrase means that they get paid based on the quantity of work that they do. For example, many workers in clothing manufacturing are paid a flat rate for each article of clothing they produce. As another example, employees in fruit orchards may get paid by the number of pieces of fruit that they harvest, or simply by the kilogram. As you can see, these workers are neither salaried nor paid hourly, nor are they on commission. They earn their paycheque for performing a specific task. Therefore, a piecework wage compensates such employees on a per-unit basis.

This section focuses on the regular earnings only for piecework wage earners. Similar to workers on commission, piecework earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the standards vary widely from province to province, and there is not necessarily any one formula to calculate these earnings. As with commissions, this textbook leaves those calculations for a payroll administration course.

The Formula

To calculate the regular gross earnings for a worker paid on a piecework wage, you require the piecework rate and how many units they are to be paid for:

$$\text{Gross Earnings} = \text{Piecework Rate } \times \text{ Eligible Units}$$

This is not a new formula but another application of Formula 2.2 (Rate, Portion, Base). The Piecework Rate is the Rate, the Eligible Units are the Base, and the Gross Earnings are the Portion.

How It Works

To calculate an employee’s gross earnings for piecework, follow these steps:

Step 1: Identify the piecework rate and the level of production or units.

Step 2: Perform any necessary modifications on the production or units to match how the piecework is paid.

Step 3: Calculate the commission gross earnings by multiplying the rate by the eligible units.

Assume that Juanita is a piecework earner at a blue jean manufacturer. She is paid daily and earns $1.25 for every pair of jeans that she sews. On a given day, Juanita sewed 93 pairs of jeans. Her gross earnings are calculated as follows:

Step 1: Her piecework rate = $1.25 per pair with production of 93 units.

Step 2: The rate and production are both expressed by the pair of jeans. No modification is necessary.

Step 3: Her gross piecework earnings are the product of the rate and units produced, or $\$1.25 \times 93 = \$116.25$.

Things To Watch Out For

Pay careful attention to step 2 in the procedure. In some industries, the piecework rate and the units of production do not match. For example, a company could pay a piecework rate per kilogram, but a single unit may not represent a kilogram. This is typical in some canning industries, where workers are paid per kilogram for canning the products, but the cans may only be 200 grams in size. Therefore, if workers produce five cans, they are not paid for five units produced. Rather, they are paid for only one unit produced since five cans $\times 200\text{g} = 1,000\text{g} = 1\text{kg}$. Before calculating piecework earnings, ensure that both the piecework rate and the eligible units are in the same terms, whether it be metric tonnes, kilograms, or otherwise.

✓ Example $4.1.4$: A Telemarketer Earning Piecework Wages

In outbound telemarketing, some telemarketers are paid on the basis of "completed calls." This is not commission since their pay is not based on actually selling anything. Rather, a completed call is defined as simply any phone call for which the agent speaks with the customer and a decision is reached, regardless of whether the decision was to accept, reject, or request further information. If a telemarketer produces five completed calls per hour and works 7½-hour shifts five times per week, what are the total gross earnings she can earn over a biweekly pay period if her piecework wage is $3.25 per completed call?

Solution

We are looking for the total gross earnings, or GE, for the telemarketer over the biweekly pay period.

What You Already Know

Step 1:

The frequency of the telemarketer's pay, along with her hours of work, piecework wage, and unit of production are known:

Piecework Rate = $3.25 per completed call

Hourly Units Produced = 5

Hours of Work = 7½ hours per day, five days per week

Frequency of Pay = biweekly

How You Will Get There

Step 2:

You must determine the telemarketer’s production level. Calculate how many completed calls she achieves per biweekly pay period:

$$\text{Eligible Units } = \text{ Units Produced per Hour } \times \text{ Hours per Day } \times \text{ Days per Week } \times \text{ Weeks}$$

Step 3:

Apply Formula 2.2 (adapted for piecework wages) to get the portion owing.

Perform

Step 2:

$$\text{Eligible Units } = 1.5 \times 7.5 \times 5 \times 2 = 375$$

Step 3:

$$GE = \$3.25 \times 375 = \$1,218.75$$

> Over a biweekly period, the telemarketer completes 375 calls. At her piecework wage, this results in total gross earnings of $1,218.75.

1. Special thanks to Steven Van Alstine (CPM, CAE), Vice-President of Education, the Canadian Payroll Association, for assistance in summarizing Canadian payroll legislation and jurisdictions.

Contributors and Attributions

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4.2: Personal Income Tax (The Taxman Taketh)

You have just filled out your income tax return, where you see that your total payable for the year is more than $5,000. You are glad you do not have to pay this lump sum all at once! Instead, almost all of it has actually been paid already: Your employer has made regular deductions from each paycheque to cover your federal and provincial taxes.

In Canada, businesses are taxed differently than individuals and there are many nuances and complexities to address. For these reasons, this textbook does not cover business taxes, which require a course in corporate taxes.

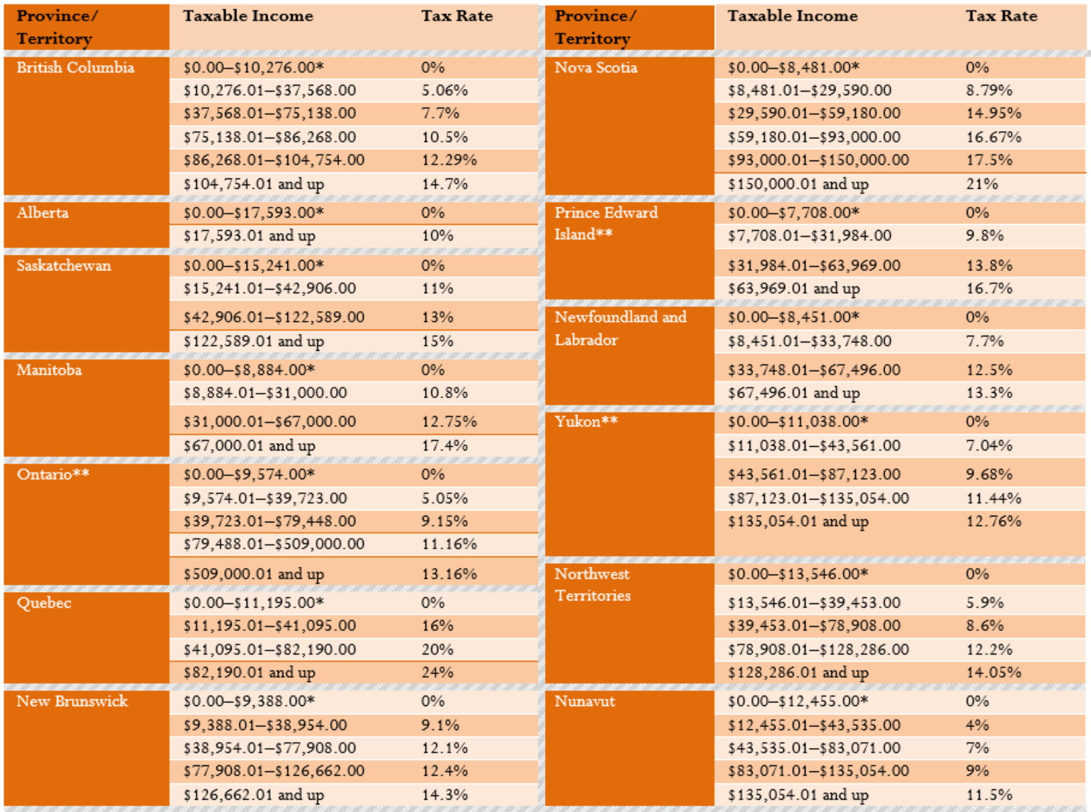
For individuals, taxes are relatively straightforward. This section of the book uses the 2013 tax year federal and provincial/territorial personal income tax brackets to calculate the amount of these taxes that are ultimately deducted from your gross earnings. You can find current-year tax brackets at www.taxtips.ca/marginaltaxrates.htm.

Federal and Provincial/Territorial Tax Brackets and Tax Rates

Personal income tax is a tax on gross earnings levied by both the federal and provincial/territorial governments. All of Canada uses a progressive tax system in which the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above minimum thresholds. Thus, higher income earners pay higher marginal tax rates than lower income earners. The format is similar to a graduated commission structure.

The federal and provincial/territorial governments offer a basic personal amount, which is the amount of income for which the wage earner is granted a tax exemption. In other words, it is tax-free income. This is designed to help low-income earners in Canada. Tax brackets are adjusted annually federally and in most provinces and territories to reflect increases in the cost of living, as measured by the consumer price index (CPI).

2013 Provincial Taxes



Province/ Territory	Taxable Income	Tax Rate
British Columbia	$0.00–$10,276.00*	0%
	$10,276.01–$37,568.00	5.06%
	$37,568.01–$75,138.00	7.7%
	$75,138.01–$86,268.00	10.5%
	$86,268.01–$104,754.00	12.29%
	$104,754.01 and up	14.7%
Alberta	$0.00–$17,593.00*	0%
	$17,593.01 and up	10%
Saskatchewan	$0.00–$15,241.00*	0%
	$15,241.01–$42,906.00	11%
	$42,906.01–$122,589.00	13%
	$122,589.01 and up	15%
Manitoba	$0.00–$8,884.00*	0%
	$8,884.01–$31,000.00	10.8%
	$31,000.01–$67,000.00	12.75%
	$67,000.01 and up	17.4%
Ontario**	$0.00–$9,574.00*	0%
	$9,574.01–$39,723.00	5.05%
	$39,723.01–$79,448.00	9.15%
	$79,488.01–$509,000.00	11.16%
	$509,000.01 and up	13.16%
Quebec	$0.00–$11,195.00*	0%
	$11,195.01–$41,095.00	16%
	$41,095.01–$82,190.00	20%
	$82,190.01 and up	24%
New Brunswick	$0.00–$9,388.00*	0%
	$9,388.01–$38,954.00	9.1%
	$38,954.01–$77,908.00	12.1%
	$77,908.01–$126,662.00	12.4%
	$126,662.01 and up	14.3%

Province/ Territory	Taxable Income	Tax Rate
Nova Scotia	$0.00–$8,481.00*	0%
	$8,481.01–$29,590.00	8.79%
	$29,590.01–$59,180.00	14.95%
	$59,180.01–$93,000.00	16.67%
	$93,000.01–$150,000.00	17.5%
	$150,000.01 and up	21%
Prince Edward Island**	$0.00–$7,708.00*	0%
	$7,708.01–$31,984.00	9.8%
	$31,984.01–$63,969.00	13.8%
	$63,969.01 and up	16.7%
Newfoundland and Labrador	$0.00–$8,451.00*	0%
	$8,451.01–$33,748.00	7.7%
	$33,748.01–$67,496.00	12.5%
	$67,496.01 and up	13.3%
Yukon**	$0.00–$11,038.00*	0%
	$11,038.01–$43,561.00	7.04%
	$43,561.01–$87,123.00	9.68%
	$87,123.01–$135,054.00	11.44%
	$135,054.01 and up	12.76%
Northwest Territories	$0.00–$13,546.00*	0%
	$13,546.01–$39,453.00	5.9%
	$39,453.01–$78,908.00	8.6%
	$78,908.01–$128,286.00	12.2%
	$128,286.01 and up	14.05%
Nunavut	$0.00–$12,455.00*	0%
	$12,455.01–$43,535.00	4%
	$43,535.01–$83,071.00	7%
	$83,071.01–$135,054.00	9%
	$135,054.01 and up	11.5%

*There are additional income surtaxes in these provinces and Yukon.

**These surtaxes are not covered in this textbook. These amounts represent the basic personal amount.

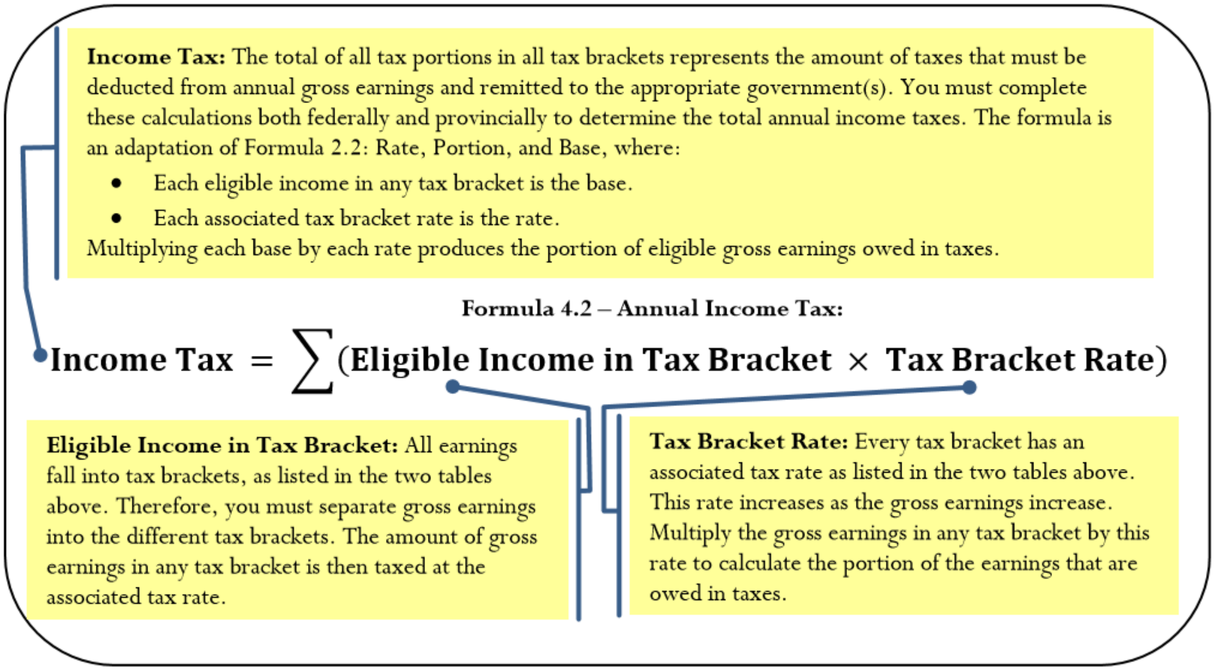
2013 Federal Taxes

Taxable Income	Tax Rate
$0.00–$11,038.00*	0%
$11,038.01–$43,561.00	15%
$43,561.01–$87,123.00	22%
$87,123.01–$135,054.00	26%
$135,054.01 and up	29%

*This is the basic personal amount.

The Formula

To calculate the income taxes for any individual, you must total all taxes from all eligible gross earnings in every tax bracket. Formula 4.2 shows the calculation.



Income Tax: The total of all tax portions in all tax brackets represents the amount of taxes that must be deducted from annual gross earnings and remitted to the appropriate government(s). You must complete these calculations both federally and provincially to determine the total annual income taxes. The formula is an adaptation of Formula 2.2: Rate, Portion, and Base, where:

- Each eligible income in any tax bracket is the base.
- Each associated tax bracket rate is the rate.

Multiplying each base by each rate produces the portion of eligible gross earnings owed in taxes.

Formula 4.2 – Annual Income Tax:

$$\textbf{Income Tax} = \sum(\textbf{Eligible Income in Tax Bracket} \times \textbf{Tax Bracket Rate})$$

Eligible Income in Tax Bracket: All earnings fall into tax brackets, as listed in the two tables above. Therefore, you must separate gross earnings into the different tax brackets. The amount of gross earnings in any tax bracket is then taxed at the associated tax rate.

Tax Bracket Rate: Every tax bracket has an associated tax rate as listed in the two tables above. This rate increases as the gross earnings increase. Multiply the gross earnings in any tax bracket by this rate to calculate the portion of the earnings that are owed in taxes.

Formula 4.2

HOW IT WORKS

To calculate the total annual income tax owing in any tax year, follow these steps:

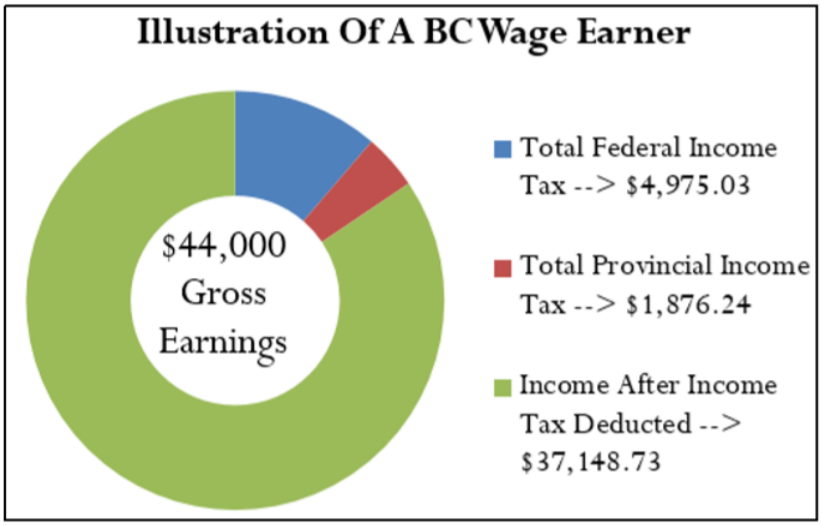
Step 1: Determine the total amount of gross earnings that are taxable for the individual.

Step 2: Calculate the federal income taxes owing. Using the federal tax table, split the gross earnings into each of the listed tax brackets, then calculate the taxes owing in each tax bracket by multiplying the income in the bracket and the tax rate. Round any tax amounts to two decimals.

Step 3: Calculate the provincial/territorial income taxes owing. Repeat the same process as in step 2, but use the appropriate tax brackets from the provincial tax table instead. Round any tax amounts to two decimals.

Step 4: Calculate the total annual income taxes owing by applying Formula 4.2. This means adding all calculated tax amounts from both steps 2 and 3 together.

Assume you live in British Columbia and your taxable gross income is $44,000. Here is how you would calculate your annual total federal and provincial taxes:



Step 1: The gross earnings are $44,000.

Step 2: Federally, your income falls into the first three brackets. Your first $11,038 is taxed at 0%; therefore, there are no income taxes on this amount. The next $32,523 (from $11,038.01 to $43,561.00) is taxed at 15%, thus $\$32,523 \times 15\% = \$4,878.45$. The last $439 (from $43,561.01 to $44,000.00) is taxed at 22%, thus $\$439 \times 22\% = \$96.58$.

Step 3: Provincially, your income falls into the first three brackets for British Columbia. Your first $10,276 is taxed at 0%; therefore, there are no income taxes on this amount. The next $27,292 (from $10,276.01 to $37,568.00) is taxed at 5.06%, thus $\$27,292 \times 5.06\% = \$1,380.98$. The last $6,432 (from $37,568.01 to $44,000) is taxed at 7.7%, thus $\$6,432 \times 7.7\% = \$495.26$.

Step 4: Your total federal income tax is $\$0.00 + \$4,878.45 + \$96.58 = \$4,975.03$. Your total provincial income tax is $\$0.00 + \$1,380.98 + \$495.26 = \$1,876.24$. Applying Formula 4.1, $\text{income tax} = \$4,975.03 + \$1,876.24 = \$6,851.27$, which is the amount that will be deducted from total gross earnings.

Things To Watch Out For

Have you ever heard someone say, "I earn more income and moved to a higher tax bracket, so now I am earning less money and my paycheque is lower"? The progressive tax system used in Canada makes this statement untrue.

Remember that tax rates apply only to the portion of the gross earnings in the tax bracket and are not retroactive to lower levels of income. For example, if your taxable gross income increased from $80,000 to $88,000, your highest tax bracket is now 26% instead of 22%. However, your federal income tax is not calculated at 26% for your entire income. Rather, the first $11,038 is tax free, the next $32,523 is taxed at 15%, the next $43,562 is taxed at 22%, and the final $877 is taxed at 26%.

Paths To Success

It is sometimes beneficial to pre-calculate the total income taxes in any tax bracket under the assumption that the individual's income comprises the entire tax bracket. This technique is particularly useful when repetitive calculations are required, such as determining the federal income taxes for each employee in an entire company. For example, the second federal tax bracket extends from $11,038.01 to $43,561, representing $32,523 of employee income. Since this bracket is taxed at 15%, someone who earns a higher income always owes the full amount of tax in this category, which is $\$32,523 \times 15\% = \$4,878.45$. The next tax bracket covers $43,561.01 to $87,123, representing $43,562 of income. Someone earning a higher income always owes the full amount of tax in this category, which is $\$43,562 \times 22\% = \$9,583.64$.

Assume taxable gross earnings of $88,000. With your pre-calculated income taxes in each bracket, you know the income taxes in the first three tax brackets are $0, $4,878.45, and $9,583.64, or $14,462.09 in total. You only need to calculate the tax on the portion of income in the final tax bracket of $\$877 \times 26\% = \$228.02$. The grand total is $\$14,462.09 + \$228.02 = \$14,690.11$.

Example 4.2.1: Federal and Provincial Income Taxes

A Canadian wage earner has taxable gross income of $97,250. Calculate the federal and provincial annual income taxes individually and then sum the amounts to calculate the total annual income taxes if the wage earner lives in

a. Ontario
b. New Brunswick
c. Alberta

Solution

For each of the provinces, you need to calculate the individual's total federal income tax and total provincial income tax. Then sum these amounts to arrive at the annual income tax.

What You Already Know

Step 1:

The employee's gross taxable earnings are $97,250. From the tax tables, you also know the federal and provincial income tax brackets and corresponding tax rates.

How You Will Get There

Step 2:

Calculate the federal income tax. This amount is the same in all three provinces and needs to be calculated only once. Looking at the federal tax table, you see that this wage earner falls into the first four tax brackets, which requires the sum of the income tax calculations for each bracket.

Step 3:

Calculate the provincial income tax. Using the provincial tax table, the income spans all four of Ontario's tax brackets. In New Brunswick, the individual falls into the first four tax brackets. In Alberta, the individual spans both of Alberta's tax brackets. In all three provinces, the total provincial income is the sum of the income tax calculations for each bracket.

Step 4:

Calculate total annual income taxes using Formula 4.2.

Perform

Step 2:

$$\begin{aligned}
&(\$11,038-\$0)\times 0=\$0.00\\
&(\$43,561-\$11,038)\times 0.15=\$4,878.45\\
&(\$87,123-543,561)\times 0.22=\$9,583.64\\
&(\$97,250-\$87,123)\times 0.26=\$2,633.02\\
&\text{ Total federal income tax}=\$0.00+54,878.45+59,583.64+\$2,633.02=\$17,095.11
\end{aligned}$$

Step 3:

Ontario

$$\begin{aligned}
&(59,574-50)\times 0=\$0.00\\
&(539,723-59,574)\times 0.0505=\$1,522.52\\
&(579,448-\$39,723)\times 0.0915=\$3,634.84\\
&(597,250-579,448)\times 0.1116=\$1,986.70\\
&\text{Total Ontario income tax}=\$0.00+\$1,522.52\$\$+\$3,634.84+\$1,986.70=\$7,144.06
\end{aligned}$$

New Brunswick

$$\begin{aligned}
(59,388-50)\times 0&=\$0.00\\
(\$38,954-\$9,388)\times 0.091&=\$2,690.51\\
(\$77,908-\$38,954)\times 0.121&=\$4,713.43\\
(\$97,250-\$77,908)\times 0.124&=\$2,398.41\\
\text{Total New Brunswick income tax}&=\$0.00+\$2,690.51+54,713.43+52,398.41=\$9,802.35
\end{aligned}$$

Alberta

$$\begin{aligned}(\$17,593-\$0)\times 0&=\$0.00\\(597,250-517,593)\times 0.10&=\$7,965.70\\\text{Total New Alberta income tax}&=\$0.00+\$7,965.70=\$7,965.70\end{aligned}$$

Step 4:

$$\text{Total income tax if living in Ontario }=\$17,095.11+\$7,144.06=\$24,239.17$$

$$\text{Total income tax if living in New Brunswick }=\$17,095.11+\$9,802.35=\$26,897.46$$

$$\text{Total income tax if living in Alberta }=\$17,095.11+\$7,965.70=\$25,060.81$$

For an individual earning $97,250 in taxable gross income, the person will pay $24,239.17 in total income tax if living in Ontario, $26,897.46 total income tax if living in New Brunswick, and $25,060.81 if living in Alberta.

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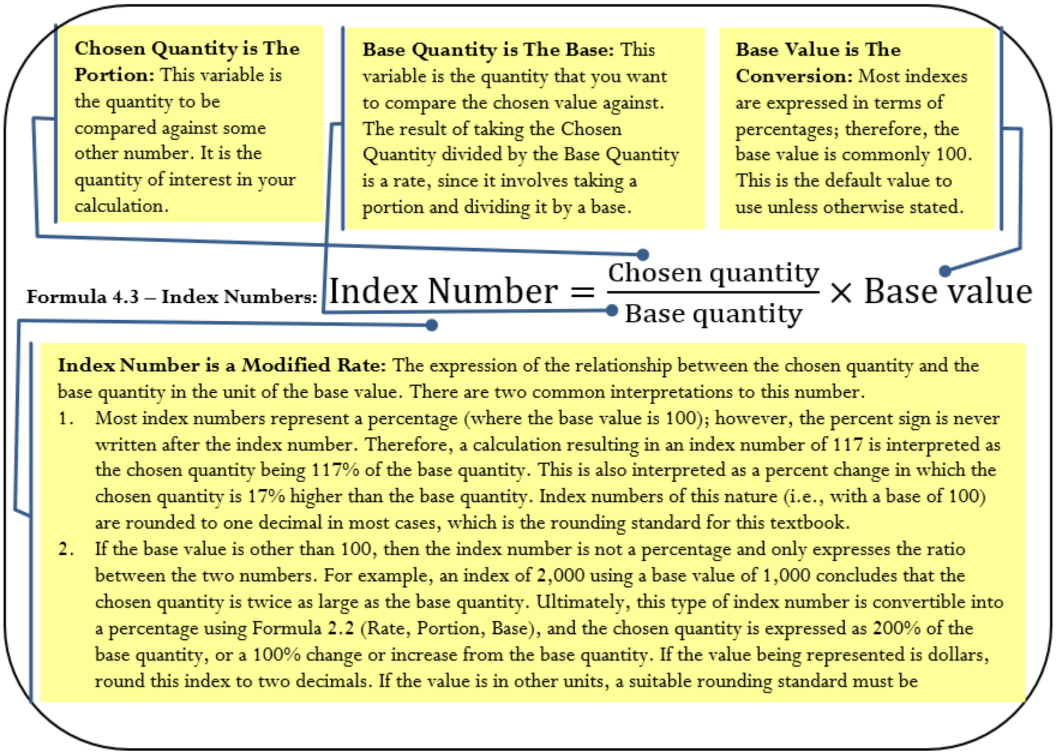
4.3: Indexes (The Times Are Changing)

You read in the Financial Post that the S&P/TSX Composite Index is up 12.74 points while the NASDAQ is down 45.98, and you wonder what this means for your investment portfolio. The consumer price index rose 3.1% since last year, and you wonder if the raise your employer offered you keeps pace. You need to understand indexes to make sense of these numbers.

What Is An Index?

An index is a number used to compare two quantities sharing the same characteristic as measured under different circumstances. Usually, the comparison involves two different points of time. The two quantities are expressed as a rate of one another and are then converted to an index number using a base value (usually 100 or 1,000). The result is called an index number, which expresses the relationship between the two quantities.

The Formula



Chosen Quantity is The Portion: This variable is the quantity to be compared against some other number. It is the quantity of interest in your calculation.

Base Quantity is The Base: This variable is the quantity that you want to compare the chosen value against. The result of taking the Chosen Quantity divided by the Base Quantity is a rate, since it involves taking a portion and dividing it by a base.

Base Value is The Conversion: Most indexes are expressed in terms of percentages; therefore, the base value is commonly 100. This is the default value to use unless otherwise stated.

Formula 4.3 – Index Numbers: $\text{Index Number} = \dfrac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$

Index Number is a Modified Rate: The expression of the relationship between the chosen quantity and the base quantity in the unit of the base value. There are two common interpretations to this number.

1. Most index numbers represent a percentage (where the base value is 100); however, the percent sign is never written after the index number. Therefore, a calculation resulting in an index number of 117 is interpreted as the chosen quantity being 117% of the base quantity. This is also interpreted as a percent change in which the chosen quantity is 17% higher than the base quantity. Index numbers of this nature (i.e., with a base of 100) are rounded to one decimal in most cases, which is the rounding standard for this textbook.
2. If the base value is other than 100, then the index number is not a percentage and only expresses the ratio between the two numbers. For example, an index of 2,000 using a base value of 1,000 concludes that the chosen quantity is twice as large as the base quantity. Ultimately, this type of index number is convertible into a percentage using Formula 2.2 (Rate, Portion, Base), and the chosen quantity is expressed as 200% of the base quantity, or a 100% change or increase from the base quantity. If the value being represented is dollars, round this index to two decimals. If the value is in other units, a suitable rounding standard must be

Formula 4.3

How It Works

Follow these steps to calculate an index number:

Step 1: Identify the base and the base value for the index.

Step 2: Identify the value of the chosen quantity to be compared against the base value.

Step 3: Calculate the index number by applying Formula 4.3.

In the above steps, note that in the event that the index number is known and you are solving for a different unknown, identify the known variables in steps 1 and 2. Assume the price of tuition was $3,535.00 in 2009 and $3,978.67 in 2012. The goal is to express the 2012 tuition indexed against the 2009 tuition.

Step 1: The tuition in 2009 is the amount to be compared against; thus, Base Quantity = $3,535. No base value is specified; thus, use the default of Base Value = 100.

Step 2: The 2012 tuition is the quantity for which you wish to compute the index, resulting in Chosen Quantity = $3,978.67.

Step 3: $\text{The Index Number } = \frac{\$3,978.67}{\$3,535.00} \times 100 = 112.6$. You can conclude that tuition in 2012 is 112.6% higher than 2009, or that tuition has increased by 12.6% over the three years.

✓ Example 4.3.1: Indexing the Price of Gold

In January 2000, the average price of gold per troy ounce was $284.32 (in US dollars). In May 2011, the average price of gold was $1,494.89. Express the gold price in 2011 indexed against the gold price in 2000.

Solution

You need to calculate an index number that compares the 2011 gold price against the 2000 gold price.

What You Already Know

Step 1:

Base Quantity = 2000 price = $284.32; assume the standard Base Value = 100

Step 2:

Chosen Quantity = 2011 price = $1,494.89

How You Will Get There

Step 3:

Apply Formula 4.3.

Perform

Step 3:

$$\text{Index Number } = \frac{\$1,494.89}{\$284.32} \times 100 = 525.8$$

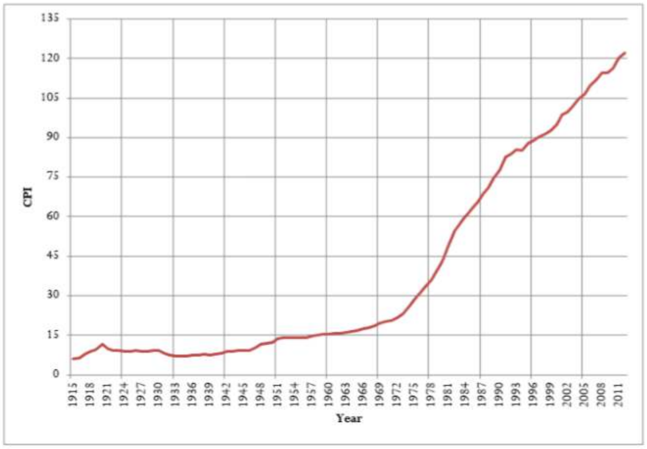
The May 2011 gold price has an index number of 525.8 when compared to the January 2000 price. This means that the price of gold has increased by 425.8% over the approximate 11-year period.

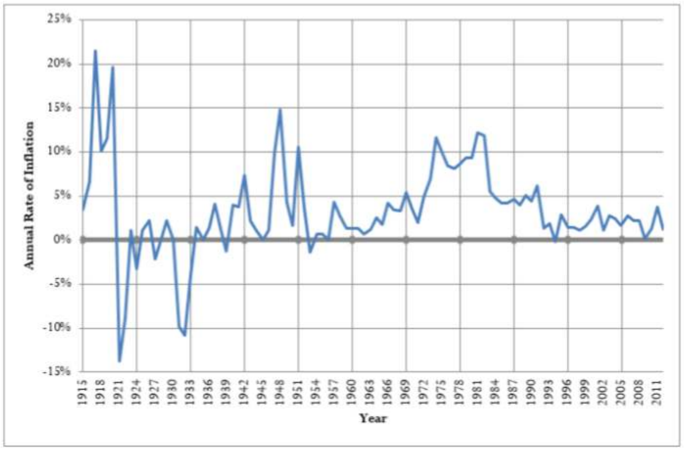
Specialty Indexes

While indexes can be created for any comparison of two quantities, two specialized indexes are widely used across the business and financial sectors of Canada: the consumer price index and the S&P/TSX Composite Index.

Consumer Price Index

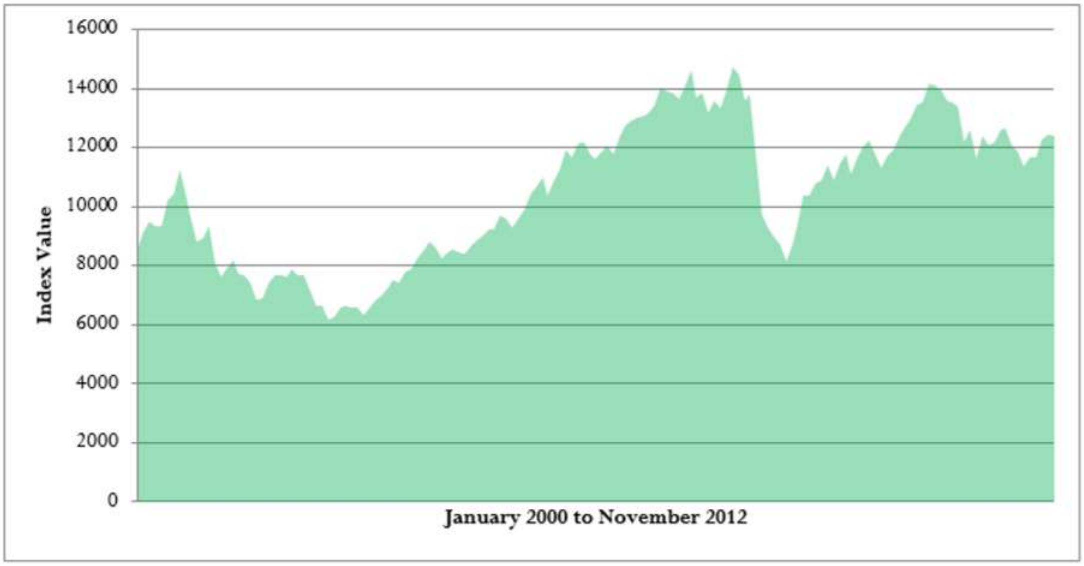
One of the most closely watched price indexes in Canada is the consumer price index, or CPI. Historical values for this index (base year = 2002) are illustrated the graph on the left while the historical rates of inflation (1915-2012) are depicted in the graph on the right.[1]





The CPI, calculated by Statistics Canada on a monthly basis, is used as a measure for estimating the rate of inflation or the cost of living (notice the slope of the CPI line increases or decreases in accordance with the change in the rate of inflation). The CPI measures the average price of the goods and services that a typical Canadian household commonly purchases, which is called the market basket. This basket contains about 600 items in categories such as food, housing, transportation, furniture, recreation, and clothing. The CPI is used for many purposes, such as increases in wages, pensions, salaries, prices, the Canada Pension Plan (CPP), and Old Age Security (OAS).

In the section opener, you need to know the CPI in discussing your raise with your employer. Since the CPI reflects the changing cost of living, you must ensure that any raise your employer offers covers, at the barest minimum, the increased costs of living. For example, if the CPI increased from 114.3 to 116.5 over the year, reflecting a 1.78% rate of inflation, then your raise needs to be at least 1.78% to keep you in the same financial position as last year.



S&P/TSX Composite Index

Another index reported daily and nationally across Canada is the Standard & Poor's/Toronto Stock Exchange Composite Index, or S&P/TSX Composite Index. This index captures the equity prices of approximately 200 of Canada's largest companies on the Toronto Stock Exchange as measured by their market capitalization, representing approximately 70% of the Canadian market capitalization listed on the TSX. The historical values of the index are illustrated on the previous page. The exact number of companies in the index continually changes because of strict requirements on market capitalization, liquidity, and domicile that must be met. The index is a measure of the strength and direction of the economy of Canada. For example, during the 2008 subprime mortgage crisis in the United States, the index retreated from a value of 14,714 to just over 8,123 in a very short time frame (demonstrating the close interconnectedness of Canada’s economy with that of the United States). At this time, the Canadian economy entered its first recession in years.

In the section opener, knowing the S&P/TSX Composite Index is invaluable for the meeting with your financial adviser to evaluate the health of your investment portfolio. For example, from May 2010 to May 2011 the index rose approximately 15.7%. If your investment portfolio rose by 9% over the same time frame (which on its own sounds impressive), you know from comparison to the index that your portfolio in fact had a subpar performance.

The Formula

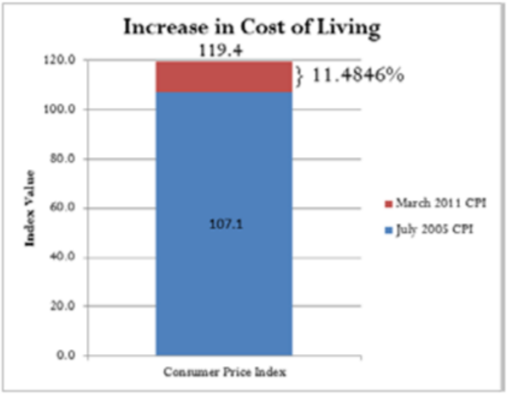
The base year used in CPI calculations is mid-2002, and the base value is 100. The CPI is always rounded to one decimal. The base year used in S&P/TSX calculations is 1975, and the base value is 1,000. The S&P/TSX is always rounded to two decimals. Regardless of which index is being calculated, Formula 4.3 remains unchanged.

Often when you are working with these indexes, you want to compare how quantities have increased or decreased. This requires you to use Formula 3.1 (Percent Change) to calculate percent changes.

How It Works

No new steps or procedures are required when you work with the CPI or the S&P/TSX Composite Index. To calculate the value of an index, apply the index number steps. To calculate the percent change between two index values, apply the percent change steps.

For example, in July 2005 the CPI was 107.1. In March 2011 the CPI was 119.4. How much did the cost of living rise from July 2005 to March 2011?



To solve this problem, apply the percent change steps:

Step 1: New = 119.4 and Old = 107.1. You are looking for the $\Delta\%$.

Step 2: Substituting into Formula 3.1: $\Delta\% = \frac{119.4 - 107.1}{107.1} \times 100 = 11.4846\%$ You can conclude that the cost of living has risen by 11.4846% over the approximate six-year time frame.

Important Notes

From time to time, Statistics Canada updates the base period for the CPI and adjusts all CPI numbers accordingly. Previously, the base period was 1992 before being adjusted to 2002. Another update to the base period can probably be expected soon; however, this occurs retroactively since it takes time to recalculate all indexes and gather all of the latest data.

✓ Example $4.3.2$: How Much Money Do You Have?

If you had invested $15,000 into an S&P/TSX portfolio in 1975, how much money would you have in 2011 when the S&P/TSX Composite Index indicated a value of 13,607.25?

Solution

You need to determine the amount of money in the portfolio at the current time period of 2011. This amount represents the Chosen Quantity.

What You Already Know

Step 1:

Since the money was invested in the base year (1975), the amount of the investment is the base quantity.

Base Quantity = $15,000 Base Value = 1,000

Step 2:

In this case, seek the Chosen Quantity with a known

Index Number = 13,607.25

How You Will Get There

Step 3:

Apply Formula 4.3 and rearrange for Chosen Quantity.

Perform

Step 3:

$$13,607.25 = \frac{\text{Chosen Quantity}}{\$15,000} \times 1,000$$

$$204,108,750 = \text{ Chosen Quantity } \times 1,000$$

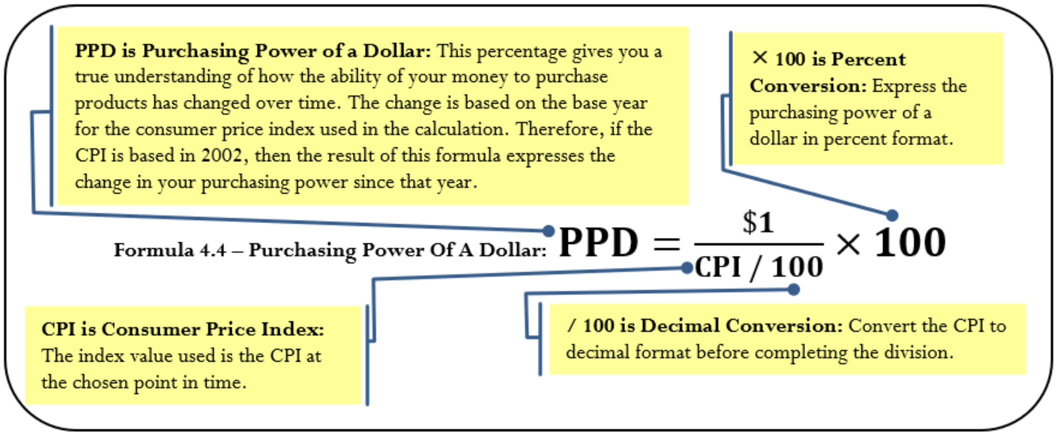
$$\$204,108.75 = \text{ Chosen Quantity}$$

If you had invested $15,000 in the S&P/TSX Composite Index back in 1975, your investment portfolio is now valued at $204,108.75, representing an increase of 1,260.725%.

Purchasing Power of a Dollar

The consumer price index is applied in determining the purchasing power of a dollar, which is the amount of goods and services that can be exchanged for a dollar. The result of the calculation is a percentage showing how much more or less product is received in return for every dollar spent. For example, recall that my parents could attend a movie for 25¢ in the 1950s. If my parents had lost that quarter under their mattress and found it today, the quarter no longer can purchase admission to the movie (nor much else!). Therefore, the purchasing power of that quarter has drastically declined.

The Formula



PPD is Purchasing Power of a Dollar: This percentage gives you a true understanding of how the ability of your money to purchase products has changed over time. The change is based on the base year for the consumer price index used in the calculation. Therefore, if the CPI is based in 2002, then the result of this formula expresses the change in your purchasing power since that year.

× 100 is Percent Conversion: Express the purchasing power of a dollar in percent format.

Formula 4.4 – Purchasing Power Of A Dollar: $\text{PPD} = \frac{\$1}{\text{CPI}/100} \times 100$

CPI is Consumer Price Index: The index value used is the CPI at the chosen point in time.

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.

Formula 4.4

How It Works

Follow these steps to calculate the purchasing power of a dollar:

Step 1: Choose the point in time to perform the calculation. Identify the value of the CPI at that point in time as well as noting the base year upon which the CPI is determined (so you can properly interpret the calculated amount). Note that in the event that the purchasing power is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the purchasing power of a dollar by applying Formula 4.4.

Assume the CPI is 111.1 today. How has your purchasing power changed since 2002 (the base year for the CPI)?

Step 1: The $CPI = 111.1$ and the base year is 2002.

Step 2: Substitute into the formula: $PPD = \frac{\$1}{111.1/100} \times 100 = 90.009\%$. Therefore, your purchasing power is 90.009% what it was in 2002. In 2002, if you could buy 10 loaves of bread with your dollar, today you could buy approximately nine loaves of bread with the same amount of money. In other words, your dollar purchases about 10% less product with the same amount of money.

✓ Example $4.3.3$: How Has Your Purchasing Power Changed?

In March 2006, the CPI was 108.6. By March 2011 it had risen to 119.4. As a percent change, how did your purchasing power change over that time frame?

Solution

You need to calculate the percent change, or $\Delta\%$, between the purchasing power of a dollar (PPD) in 2006 and 2011.

What You Already Know

Step 1:

2006 represents your purchasing power of a dollar in the past, and 2011 represents the present.

Old = PPD in 2006 New = PPD in 2011

CPI 2006 = 108.6 CPI 2011 = 119.4

The current CPI is based in 2002.

How You Will Get There

Step 2:

Apply Formula 4.4 to calculate the PPD for both 2006 and 2011.

Step 3:

Apply Formula 3.1 to calculate the percent change.

Perform

Step 2:

$$PPD\ 2006=\frac{\$1}{108.6/100}\times 100=92.0810\%$$

$$PPD\ 2011=\frac{\$1}{119.4/100}\times 100=83.7521\%$$

Step 3:

$$\Delta\%=\frac{0.837521-0.920810}{0.920810}\times 100=-9.0452\%$$

Calculator Instructions

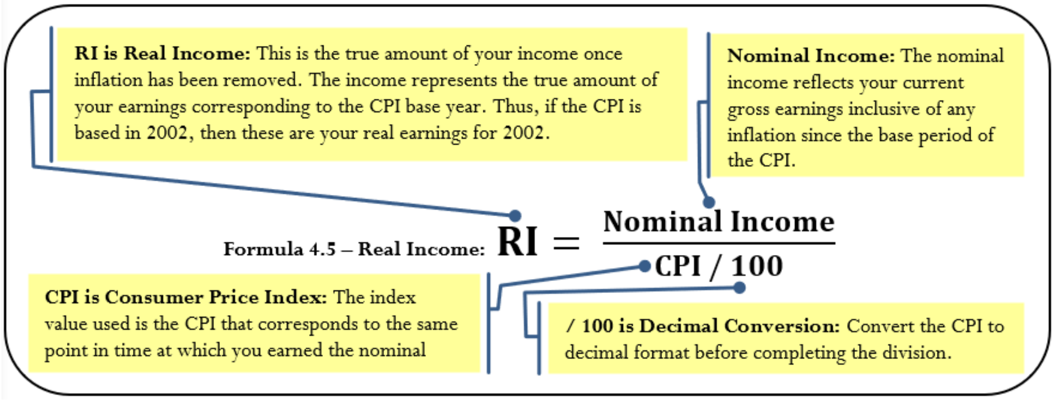
OLD	NEW	%CH	#PD
0.920810	0.837521	Answer: -9.045188	1

From March 2006 to March 2011, your purchasing power declined by 9.0452%. This means that if in March 2006 you were able to buy approximately 92 items with a certain sum of money, in March 2011 you were only able to buy about 83 items with the same amount of money.

Real Income

Another application of the consumer price index allows you to assess your real earnings. With the cost of living (inflation) always changing, it is difficult to assess how much more or less money you are truly making. Real income allows you to remove the effects of inflation from your income, allowing for a fair comparison of earnings at different points in time. For example, if your gross earnings last year were $50,000 and you received a $1,500 raise this year, how much more are you truly earning if the CPI changed from 114.6 to 116.6 over the same time frame?

The Formula



RI is Real Income: This is the true amount of your income once inflation has been removed. The income represents the true amount of your earnings corresponding to the CPI base year. Thus, if the CPI is based in 2002, then these are your real earnings for 2002.

Nominal Income: The nominal income reflects your current gross earnings inclusive of any inflation since the base period of the CPI.

Formula 4.5 – Real Income: $\mathrm{RI} = \dfrac{\text{Nominal Income}}{\text{CPI} / 100}$

CPI is Consumer Price Index: The index value used is the CPI that corresponds to the same point in time at which you earned the nominal

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.

Formula 4.5

How It Works

Follow these steps to calculate your real income:

Step 1: Identify the nominal income. Determine the CPI value that corresponds to the same time frame as the nominal income and note the base year for interpretation purposes. In the event that the real income is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the real income by applying Formula 4.5.

Assume your gross earnings were $40,000 when the CPI was 111.7. Currently your income is $42,000 and the CPI is 113.3. Calculate your real income to assess how your income has changed.

Step 1: In this case, you have two nominal incomes to convert for comparison purposes. In the past, nominal income = $40,000 and CPI = 111.7. Currently, nominal income = $42,000 and CPI = 113.3.

Step 2: In the past, $RI = \dfrac{\$40,000}{111.7/100} = \$35,810.21$. Currently, $RI = \dfrac{\$42,000}{1.133/100} = \$37,069.73$. Comparing the amounts, you see that your real income expressed in 2002 dollars has risen by $37,069.73 − $35,810.21 = $1,259.52. Notice that your nominal income rose by $2,000. This means that of your $2,000 increase in income, you really are only making $1,259.82 more. The other $740.18 represents your increased cost of living.

✓ Example $4.3.4$: Did You Get a Raise?

Now that your boss has conducted your performance review, the time has come to discuss your raise. Currently, you earn $81,250 annually. Your boss has offered you a new salary of $83,000 annually. You know last year that the CPI was 104.1 and this year it sits at 106.6. Assess the salary offer.

Solution

To assess the offer, remove inflation from your nominal incomes, allowing for a fair comparison between the two salary amounts. Calculate the real income (RI) for each salary.

What You Already Know

Step 1:

You know the following information about your old and new salaries:

Old Nominal Income = $81,250 Old CPI = 104.1

New Nominal Income = $83,000 New CPI = 106.6

Both CPI amounts are based on the year 2002.

How You Will Get There

Step 2:

Apply Formula 4.5 to each salary.

Perform

Step 2:

$$\begin{aligned}\text{RI Last Year} &= \frac{\$81,250}{104.1/100} = \$78,049.95\\ \text{RI This Year} &= \frac{\$83,000}{106.6/100} = \$77,861.16\end{aligned}$$

$\$77,861.16 - \$78,049.95 = -\$188.79$

Although it appears as if you are receiving a $1,750 wage increase, when you factor in the increased cost of living you are actually earning $188.79 less than you did last year. Perhaps you should discuss this matter further with your boss. Since the CPI had a percent change of 2.4015%, your income must rise to at least $83,201.25 for you to break even.

References

1. Values determined from Bank of Canada “Inflation Calculator” (all index values measured in May of each year). www.bankofcanada.ca/rates/related/inflation-calculator/?page_moved=1 (accessed November 26, 2012).

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4.S: Summary

Key Concepts Summary

4.1: Gross Earnings (Off to Work You Go)

- Calculating salary and hourly gross earnings
- Discussion of employment contract characteristics for salary and hourly earners
- Calculating commission gross earnings
- Calculating piecework gross earnings

4.2: Personal Income Tax (The Taxman Taketh)

- Provision of 2013 federal and provincial/territorial tax brackets and corresponding tax rates
- Calculation of income taxes

4.3: Indexes (The Times Are Changing)

- What is an index and how is it calculated?
- Specialty indexes, including the consumer price index and the S&P/TSX Composite Index
- Using the CPI to calculate the purchasing power of a dollar
- Using the CPI to calculate real income

The Language of Business Mathematics

basic personal amount

The amount of income for which the wage earner is granted a tax exemption.

commission

An amount or a fee paid to an employee for performing or completing some form of transaction.

consumer price index

A measure of the average price of a typical Canadian market basket, which is used to estimate inflation in Canada.

graduated commission

A form of compensation where an employee is offered increasing rates of commission for higher levels of performance.

gross earnings

The amount of money earned before any deductions are removed from a paycheque.

holiday earnings

Earnings paid to an employee on a statutory holiday for which no work is performed..

hourly wage

A variable compensation based on the time an employee has worked.

index

A number used to compare two quantities sharing the same characteristic as measured under different circumstances.

index number

The expression of a relationship between two quantities; it is a result of an index calculation.

market basket

The average price of the goods and services that a typical Canadian household commonly purchases; used in calculating the consumer price index.

overtime

Work time in excess of an employee’s regular workday or regular workweek.

overtime or premium earnings

Earnings determined by an employee’s overtime rate of pay and that occur when regular hours are exceeded.

personal income tax

A tax on earned income that is levied by both the federal and provincial/territorial governments.

piecework

A form of compensation where an employee is paid on a per-unit basis.

progressive tax system

A personal income tax system where the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above a minimum threshold.

public holiday

A provincially recognized day for which employees may or may not get a day of rest and may or may not receive pay depending on provincial employment standards.

purchasing power of a dollar

The amount of goods and services that can be purchased with a dollar.

real income

Income that has the effects of inflation removed from its amount.

regular earnings

Earnings determined by an employee’s regular rate of pay.

S&P/TSX Composite Index

This index captures the equity prices of approximately 200 of Canada's largest companies on the Toronto Stock Exchange as measured by their market capitalization, representing approximately 70% of the Canadian market capitalization listed on the TSX.

salary

A fixed compensation paid to a person on a regular basis for services rendered.

salary plus commission

A form of compensation in which gross earnings combine a basic salary together with commissions on transactions.

statutory holiday

A legislated day of rest with pay.

statutory holiday worked earnings

Earnings paid to an employee on a statutory holiday at a premium rate for working on the statutory holiday.

straight commission

A form of compensation where the employee's entire earnings are based on dollar transactions and calculated strictly as a percentage of the total transactions.

The Formulas You Need to Know

Symbols Used

CPI = consumer price index

GE = gross earnings

PPD = purchasing power of a dollar

RI = real income

Formulas Introduced

Formula 4.1 Salary and Hourly Gross Earnings:

$$\begin{aligned}\text{GE} &= \text{Regular Earnings } + \text{ Overtime Earnings } + \text{ Holiday Earnings} \\ &\quad + \text{ Statutory Holiday Worked Earnings}\end{aligned}$$

Formula 4.2 Annual Income Tax:

$$\text{Income Tax} = \sum(\text{ Eligible Income in Tax Bracket } \times \text{ Tax Bracket Rate })$$

Formula 4.3 Index Numbers: $\text{Index Number } = \dfrac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{ Base value}$

Formula 4.4: Purchasing Power of a Dollar: $PPD = \dfrac{\$1}{CPI/100}$

Formula 4.5: Real Income: $RI = \dfrac{\text{Nominal Income}}{CPI/100}$

Technology

Calculator

The percent change ($\Delta\%$) function is used in this chapter. See the end of Chapter 3 for a full discussion of this function.

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4.CS: Case Study-Payroll Planning

The Situation

As Lightning Wholesale reviews the payroll for one of its divisions, it needs to produce a report on the overall payroll for last year. The tables below list each employee along with his or her compensation for last year. The report needs to include details about each employee along with overall company figures.

The Data

Last Year's Employee Compensation Information:

Name	Annual Salary	Hourly Wage	Average O/T Hours per Pay Period	Average Sales per Pay Period
Jack	$39,000		2	
Gail				$169,500
Jennifer		$16.98	5	
Evelyn				$134,000
Indiana	$76,500		3	
Hakim		$34.94	4	

Per Pay Period Graduated Commission Scale:

Commission Scale	Commission Rate
$0.00–$50,000.00	1.01%
$50,000.01–$100,000.00	2.98%
$100,000.01 and up	4.44%

Important Information

- A normal workweek for any employee is 7.5 hours per day from Monday through Friday.
- Lightning Wholesale pays overtime at time-and-a-half and is never open on statutory holidays.
- Lightning Wholesale issues payroll every two weeks.
- Lightning Wholesale is situated in Ontario.
- Federal and provincial income tax information can be found in Tables in Section 4.2.
- A 52-week year is assumed.

Your Tasks

1. Create a report on each employee and summarize the information in a table.
    a. Calculate the annual regular earnings.
    b. Calculate the annual overtime earnings.
    c. Calculate the annual gross earnings
    d. Calculate the annual federal income taxes.
    e. Calculate the annual provincial income taxes.
    f. Calculate the employee's take-home pay after deducting taxes.
    g. Calculate the totals of all of the above.

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4.E: Human Resources and Economic Applications (Exercises)

4.1 Exercises

Mechanics

1. Laars earns an annual salary of $60,000. Determine his gross earnings per pay period under each of the following payment frequencies:
   a. Monthly
   b. Semi-monthly
   c. Biweekly
   d. Weekly
2. A worker earning $13.66 per hour works 47 hours in the first week and 42 hours in the second week. What are his total biweekly earnings if his regular workweek is 40 hours and all overtime is paid at 1.5 times his regular hourly rate?
3. Marley is an independent sales agent. He receives a straight commission of 15% on all sales from his suppliers. If Marley averages semi-monthly sales of $16,000, what are his total annual gross earnings?
4. Sheila is a life insurance agent. Her company pays her based on the annual premiums of the customers that purchase life insurance policies. In the last month, Sheila's new customers purchased policies worth $35,550 annually. If she receives 10% commission on the first $10,000 of premiums and 20% on the rest, what are her total gross earnings for the month?
5. Tuan is a telemarketer who earns $9.00 per hour plus 3.25% on any sales above $1,000 in any given week. If Tuan works 35 regular hours and sells $5,715, what are his gross earnings for the week?
6. Adolfo packs fruit in cans on a production line. He is paid a minimum wage of $9.10 per hour and earns $0.09 for every can packed. If Adolfo manages to average 160 cans per hour, what are his total gross earnings daily for an eight-hour shift?

Applications

7. Charles earns an annual salary of $72,100 paid biweekly based on a regular workweek of 36.25 hours. His company generously pays all overtime at twice his regular wage. If Charles worked 85.5 hours over the course of two weeks, what are his gross earnings?
8. Armin is the payroll administrator for his company. In looking over the payroll, he notices the following workweek (from Sunday to Saturday) for one of the company's employees: 0, 6, 8, 10, 9, 8, and 9 hours, respectively. Monday was a statutory holiday, and with business booming the employee will not be given another day off in lieu. Company policy pays all overtime at time-and-a-half, and all hours worked on a statutory holiday are paid at twice the regular rate. A normal workweek consists of five, eight-hour days. If the employee receives $22.20 per hour, what are her total weekly gross earnings?
9. In order to motivate a manufacturer's agent to increase his sales, a manufacturer offers monthly commissions of 1.2% on the first $125,000, 1.6% on the next $150,000, 2.25% on the next $125,000, and 3.75% on anything above. If the agent managed to sell $732,000 in a single month, what commission is he owed?
10. Humphrey and Charlotte are both sales representatives for a pharmaceutical company. In a single month, Humphrey received $5,545 in total gross earnings while Charlotte received $6,388 in total gross earnings. In sales dollars, how much more did Charlotte sell if they both received 5% straight commission on their sales?
11. Mayabel is a cherry picker working in the Okanagan Valley. She can pick 17 kg of cherries every hour. The cherries are placed in pails that can hold 13.6 kg of cherries. If she works 40 hours in a single week, what are her total gross earnings if her piecework rate is $17.00 per pail?
12. Miranda is considering three relatively equal job offers and wants to pick the one with the highest gross earnings. The first job is offering a base salary of $1,200 semi-monthly plus 2% commission on monthly sales. The second job offer consists of a 9.75% straight commission. Her final job offer consists of monthly salary of $1,620 plus 2.25% commission on her first $10,000 in monthly sales and 6% on any monthly sales above that amount. From industry publications, she knows that a typical worker can sell $35,000 per month. Which job offer should she choose, and how much better is it than the other job offers?
13. A Canadian travel agent is paid a flat rate of $37.50 for every vacation booked through a certain airline. If the vacation is in North America, the agent also receives a commission of 2.45%. If the vacation is international, the commission is 4.68%. What are the total monthly gross earnings for the agent if she booked 29 North American vacations worth $53,125 and 17 international vacations worth $61,460?
14. Vladimir's employer has just been purchased by another organization. In the past, he has earned $17.90 per hour and had a normal workweek of 37.5 hours. However, his new company only pays its employees a salary semi-monthly. How much does

Vladimir need to earn each paycheque to be in the same financial position?

Challenge, Critical Thinking, & Other Applications

15. An employee on salary just received his biweekly paycheque in the amount of $1,832.05, which included pay for five hours of overtime at time-and-a-half. If a normal workweek is 40 hours, what is the employee's annual salary?
16. A graduated commission scale pays 1.5% on the first $50,000, 2.5% on the next $75,000, and 3.5% on anything above. What level of sales would it take for an employee to receive total gross earnings of $4,130?
17. A sales organization pays a base commission on the first $75,000 in sales, base + 2% on the next $75,000 in sales, and base + 4% on anything above. What is the base commission if an employee received total gross earnings of $7,500 on $200,000 in sales?
18. A typical sales agent for a company has annual sales of $4,560,000, equally spread throughout the year, and receives a straight commission of 2%. As the new human resource specialist, to improve employee morale you have been assigned the task of developing different pay options of equivalent value to offer to the employees. Your first option is to pay them a base salary of $2,000 per month plus commission. Your second option is to pay a base commission monthly on their first $100,000 in sales, and a base + 2.01% on anything over $200,000 per month. In order to equate all the plans, determine the required commission rates, rounded to two decimals in percent format, in both options.
19. Shaquille earns an annual salary of $28,840.50 paid biweekly. His normal workweek is 36.25 hours and overtime is paid at twice the regular rate. In addition, he is paid a commission of 3% of sales on the first $25,000 and 4% on sales above that amount. What are his total gross earnings during a pay period if he worked 86 hours and had sales of $51,750?
20. Mandy is paid $9.50 per hour and also receives a piecework wage of $0.30 per kilogram, or portion thereof. A regular workday is 7.5 hours and 37.5 hours per week. Overtime is paid at time-and-a-half, and any work on a statutory holiday is paid at twice the regular rate. There is no premium piecework wage. Mandy's work record for a two-week period is listed below. Determine her total gross earnings.

Week		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
1	Hours worked	7.5	7.5	9	8	7.5	3
	250 g items produced	1,100	1,075	1,225	1,150	1,025	450
2	Hours worked	4 Statutory holiday, no day off in lieu	7.5	10	7.5	8	
	250 g items produced	575	1,060	1,415	1,115	1,180	

Section 4.2 Exercises

Use the provincial and federal tax tables introduced earlier in this chapter when you determine income taxes in these exercises.

Mechanics

For questions 1–3, determine only the federal income taxes on the following gross taxable incomes.

1. $22,375
2. $158,914
3. $102,100

For questions 4–6, determine only the provincial income taxes on the following gross taxable incomes.

4. $61,000 in Newfoundland and Labrador
5. $83,345 in Saskatchewan
6. $78,775 in British Columbia

For questions 7 and 8, determine the total federal and provincial/territorial taxes on the following gross taxable incomes.

7. $48,910 in the Northwest Territories

8. $65,525 in Prince Edward Island

Applications

9. Nadia lives in Manitoba and has three part-time jobs. Her gross annual taxable income from each job was $5,300, $21,450, and $25,390. How much federal and provincial income tax does Nadia owe?
10. If Delaney's gross taxable income increases from $71,000 to $79,000, by what dollar amount will her after-tax pay change once federal and provincial income taxes are deducted in the province of Quebec?
11. Helen just moved from Saskatchewan to her new job in Alberta at the same rate of pay of $51,225. By what dollar amount will her provincial income taxes change?
12. Jane has received two job offers that she thinks are relatively equal. The only difference lies in the salary. The first job offer is for a position in Nova Scotia earning gross taxable income of $63,375. The second job offer is for a position in British Columbia earning gross taxable income of $61,990. If Jane will select the job that has the highest after-tax income, which one should she choose? How much better is this option in dollars?
13. Suppose an employee earns $111,300 in gross taxable income. Calculate the federal and provincial income taxes that need to be deducted if he lives in either Saskatchewan, Ontario, or the Northwest Territories. In which province or territory will he earn the most income after income tax deductions? The least? What is the dollar amount difference among the three alternatives?
14. After both federal and provincial income taxes are deducted, who would earn more money, an individual earning $85,000 in New Brunswick or an individual earning $79,000 in Nunavut?

Challenge, Critical Thinking, & Other Applications

15. Rawatha is the human resources manager for her firm located in Yukon. The salespeople for her organization are paid an annual base salary of $30,000 plus 8% straight commission on their annual sales. If the average salesperson sells $210,000 annually, what is the total annual federal and provincial income tax owing on the salesperson's gross taxable income?
16. Esmerelda works on the production line and is paid a piecework rate of $2.25 per unit produced. She is capable of producing an average of five units per hour, and works eight hours every weekday. Assuming a 52-week year, what is the annual federal income tax and provincial income tax owing if she lives in Prince Edward Island?
17. Mary Jane is paid biweekly at a rate of $34.68 per hour and works 36.25 hours every week. Assuming a 52-week year, what are the amounts of the federal income tax deduction and provincial income tax deduction that must be removed from each paycheque if Mary Jane lives in Manitoba? (Hint: income tax would be deducted equally across all paycheques for the year.)
18. Felix was transferred from his head office in Ontario to Alberta. In accepting the transfer, his employer agreed to increase his salary of $88,000 by $7,000. What is the percent change in Felix's after-tax income?
19. If a wage earner paid $2,277.98 in total annual Ontario provincial income taxes, determine his annual gross taxable income.
20. Perform a comparison across all territories and provinces of the income earned after both federal and provincial income taxes are deducted for a Canadian who earns $56,738 in gross taxable income. Which province or territory has the highest income after deductions? Which has the lowest? What is the percent difference between the two?

Section 4.3 Exercises

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. Round index numbers to one decimal and base values to the nearest integer.

	Chosen Quantity ($)	Base Quantity ($)	Base Value	Index Number
1.	$559.99	$539.99	100	?
2.	$315,000.00	?	1,000	12,415.9
3.	$114.30	$112.50	?	101.6
4.	?	$248.75	1,000	1,548.9

For questions 5–9, solve for the unknown variables (identified with a ?) based on the information provided.

	Consumer Price Index (CPI)	Purchasing Power of a Dollar (PPD)

	Consumer Price Index (CPI)	Purchasing Power of a Dollar (PPD)
5.	107.9	?
6.	?	80%

	Nominal Income ($)	Consumer Price Index (CPI)	Real Income (RI)
7.	$45,000.00	120.0	?
8.	?	105.0	$29,523.81
9.	$86,000.00	?	$80,298.71

Applications

10. In Regina, a 4 L bag of 2% milk has an average price of $3.71. Prices in Toronto and Montreal are $4.55 and $5.40, respectively. Calculate an index of these prices using Regina as the base.
11. If Sabrina is currently earning $53,000 and the CPI changes from 105.9 to 107.6, how much money does she need to earn next year just to keep up with inflation?
12. If the CPI rises from 103.4 to 108.8, how much money at the start is required to have the same purchasing power as $100 at the end?
13. George currently earns $28,000 per year. As per his union salary grid, next year he will earn $32,500 per year. If the CPI increases from 104.0 to 106.1 over the same time frame, how much of his raise is real income?
14. Last year the purchasing power of a dollar was 84.3%. This year the purchasing power of a dollar is 81.4%. What is the percent change in the CPI between the two years?

Challenge, Critical Thinking, & Other Applications

15. An investor had $200,000 invested in an S&P/TSX Composite Index portfolio in 2009. By 2011, the index rose to 14,136.50. If the index in 2009 was 8,694.90, how much money would the investor have in her portfolio in 2011, if she was able to match the market performance?
16. Over the years, Hannah's income has changed from $36,000 to $40,000 to $45,000. If the CPI changed from 102.9 to 105.1 to 108.6 over the same time period, determine Hannah's percent change in real income from year to year.
17. The CPI rose from 102.6 in 2003 to 116.5 in 2010. The S&P/TSX Composite Index rose from 6,569.49 to 11,294.42 over the same time period. If an investor had $125,000 invested in 2003, how much of the growth (in dollars) for the portfolio over the seven years represents real growth?
18. An enterprise has 136 employees earning an average income of $42,250 per year. If the CPI rises from 103.7 to 107.2, by what amount do total wages rise if all employee wages are adjusted to match the CPI?
19. Using January 2003 as your base year with a base value = 100, compute a series of indexes for the S&P/TSX Composite Index from select years in 2003 to 2011. Interpret your results.

	2003	2005	2007	2009	2011
January index	6,569.49	9,204.05	13,034.12	8,694.90	13,551.91

20. Below is select information on the CPI from 2004 to 2010:

	2004	2006	2008	2010
July CPI	105.0	109.6	115.8	116.8

Using this information, determine the following:

a. The purchasing power of a dollar for each year. Interpret your results.
b. If you were earning $34,500 in 2002, calculate the nominal income required each year to keep up with the changes in the CPI.

Chapter 4 Review Exercises

For any questions involving income tax calculations, refer to the tables previously introduced in this chapter for federal income taxes and provincial/territorial income taxes.

Mechanics

1. Jim's annual salary is $45,670 paid weekly for 36.25 hours of work each week. Determine his total gross earnings for one pay period in which he worked 8.75 hours of overtime.
2. Tracy is paid 4% commission on any sales above $150,000 plus a base salary of $500. What are her gross earnings for a pay period where her sales were $225,000?
3. Sisko has annual gross taxable earnings of $99,990. Determine his annual federal income tax.
4. Jean-Luc works two part-time jobs in Ontario. If his annual wages from each position are $17,444 and $16,430, determine his annual provincial income tax.
5. If the CPI rises from 102.6 to 110.8, determine the percent change in the purchasing power of a dollar.
6. Since Maybelline graduated from college in 2004, her salary has changed from $23,100 in 2004 to $27,800 in 2009. Over that time, the CPI has increased from 105.0 to 114.9. Express Maybelline's salary in real dollars and determine the percent change in her real earnings.

Applications

7. You have been given two job offers. The first employer is offering $17.50 per hour. The second employer is offering a $33,000 annual salary. Both employers expect that you might be able to earn an additional two hours of overtime per week. Treat any holiday earnings as regular earnings and suppose there are no statutory holiday worked earnings. If both jobs have 7.5 regular hour workdays five times per week for 52 weeks, which employer is offering the higher annual gross earnings? By what amount in dollars?
8. Juanita is an architect. She is paid $41.25 per hour and normally works eight-hour days, Monday through Friday. Her employer has agreed to pay her overtime at double her regular rate of pay and any time worked on a statutory holiday at 2.5 times her regular pay. What are Juanita's gross earnings in a single week from Sunday to Saturday where she worked 4, 9, 8, 11.5, 8, 0, and 6 hours, respectively, where Monday was a statutory holiday and her employer has decided to pay her in lieu of another day off?
9. Lorelei works on the production line for eight hours each day. She earns $9.50 per hour plus $0.13 per kilogram of sardines that she packs in excess of 100 kg per day. What are her daily gross earnings in a single day where she manages to pack 200, 250 g cans of sardines per hour?
10. Determine whether a wage earner in British Columbia or Ontario pays the least provincial income taxes on gross taxable earnings of $58,000. What percentage more does the wage earner with the higher income tax pay?
11. If Henri had $200,000 invested in his S&P/TSX Composite Index portfolio in 1975, how much would the portfolio be worth in March of 2011 when the S&P/TSX Composite Index was 14,116.10?
12. In the province of Saskatchewan, you are paid an hourly wage of $11.25 and you work 7.5-hour days for five days per week. Assuming a 52-week year, determine the amount of provincial income tax deducted from your biweekly paycheque.

Challenge, Critical Thinking, & Other Applications

13. Karen is paid $12.00 per hour for her work as a cashier at an Alberta retail outlet. Her employer pays a standard overtime rate. Her regular workday is eight hours and 40 hours per week.
    a. What are her gross earnings in a single week from Sunday to Saturday where Karen worked 12, 8, 12, 8, 0, 8, and 8 hours, respectively?
    b. If this represents a typical week for Karen, what are the federal and provincial income taxes that should be deducted from her biweekly paycheques (assume 52 weeks in a year)?
14. Living in Nova Scotia, Rufus pays $410.50 per biweekly pay period in total federal and provincial income tax. What are his gross taxable earnings?
15. Gilligan is paid a graduated commission of 1⅞% on his first $50,000 in sales, 3½% on the next $75,000 in sales, 4⅛% on the next $75,000 in sales, and 5⅕% on all sales above that amount. If his total gross earnings were $9,516.25, determine his sales during the pay period.
16. Last year your annual salary was $58,150. This year your salary was increased in accordance with the CPI. Last year, the CPI was 108.9 and this year it is 111.1. Determine the amount of federal income tax that is deducted from your semi-monthly

paycheque throughout the current year.

17. As a sales representative, you average $350,000 per month in sales. Your commission is calculated as 1% of the first $120,000, 2% on the next $120,000, and 3% for anything higher. If you live in Ontario, determine both of the federal and provincial income taxes that are removed from your monthly commission cheques.
18. Rufaro is the payroll administrator for New Wave Industries. There are four employees on the payroll. Buck and Roger receive annual salaries of $41,000 and $85,555, respectively. Luke receives piecework wages of $2.50 per item and produces 45 items per day five days per week. Leah is paid a base salary of $200 per week and receives 6% commission on average weekly sales of $10,000. Assuming a 52-week year, determine the annual total federal income taxes that New Wave Industries must remit to the federal government on behalf of its employees.
19. For every province in Western Canada (British Columbia, Alberta, Saskatchewan, and Manitoba), determine the total annual income tax deducted from an annual gross taxable income of $66,660. Express the total income tax as a percentage of gross income.
20. For a weekly pay period, you know the following information about the payroll at Hopper Industries. Friday is a statutory holiday.

- Christie receives an annual salary of $44,880 based on a 36-hour workweek comprised of four, nine-hour days. She has two hours of overtime at time-and-a-half and did not work the statutory holiday.
- Marie is paid $16.75 per hour based on eight-hour days and a 40-hour workweek. All premium wages are paid at regulatory minimums. From Sunday to Saturday, she worked 8, 0, 8, 8, 8, 3, and 8 hours, respectively. She will not receive another day off in lieu for the statutory holiday.
- Norm receives a graduated commission of 2.25% on his first $100,000 in sales and 4.15% on anything higher. His sales during one week were $123,000. Ignore his holiday or statutory earnings in your calculations.
- Garth is paid a straight piecework wage of $1.12 per unit produced. He worked a total of 30 hours during the week, producing one unit every five minutes. Ignore his holiday or statutory earnings in your calculations.

For all employees combined, determine the total regular earnings, overtime earnings, holiday earnings, and holiday worked earnings that Hopper Industries must report on its weekly company payroll report.

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CHAPTER OVERVIEW

5: Marketing and Accounting Fundamentals (Keeping Your Nose above Water)

This chapter introduces the global model of the relationship between costs and revenues. It shows the overall cost structure of a business and introduces models for developing total costs and total revenues. It explains how you calculate net income and contribution margins. Armed with all of this knowledge, you can then answer the break-even question: How much does the business need to sell to cover its costs?

5.0: Introduction
5.1: Cost-Revenue-Net Income Analysis (Need to Be in the Know)
5.2: Break-Even Analysis (Sink or Swim)
5.S: Marketing and Accounting Fundamentals (Summary)
5.CS: Case Study: Forecasting the Impact of Revenue and Cost Changes
5.E: Marketing and Accounting Fundamentals (Exercises)

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5.0: Introduction

When a new retail store opens in your neighborhood strip mall, you wonder whether it will still be running six months or a year from now. Do its owners know how much merchandise they have to sell to cover their costs? Will any money be left over to help grow the business?

While the numbers vary, approximately 15% of new companies in Canada go out of business in their first year of operations. This number rises to 38% by the third year and to a staggering 49% over the first five years.[1] Why do so many new businesses fail? Economic conditions, the fierceness of competition, changing consumer tastes, or even changes in taxes all contribute to the problem, but one of the most common reasons businesses fail is poor tracking of their basic financial numbers.

So if you want your own business to succeed, you need to understand its financial side. You need to be able to answer questions like the following:

- How does the money leave the business through costs and expenses?
- How does the money come into the business through revenue, which is determined by both the price tags on the merchandise and the volume at which that merchandise sells?
- What is the minimum amount of revenue that is needed to cover the costs?
- What is the resulting profitability of the business?

The next three chapters introduce the concepts behind the key numbers in any business.

- This chapter introduces the global model of the relationship between costs and revenues. It shows the overall cost structure of a business and introduces models for developing total costs and total revenues. It explains how you calculate net income and contribution margins. Armed with all of this knowledge, you can then answer the break-even question: How much does the business need to sell to cover its costs?
- Chapter 6 explores merchandising, including methods of calculating the cost of an individual product along with setting its associated retail price, and the impact of markup and markdown decisions.
- Chapter 7 delves into cost elements that the accounting department typically handles, including sales and property taxes, the mathematics of invoicing, and how to handle currency conversions properly.

References

1. Eileen Fisher and Rebecca Reuber, *The State of Entrepreneurship in Canada: February, 2010,* (Ottawa, ON: Small Business and Tourism Branch, Industry Canada), 9, www.ic.gc.ca/eic/site/061.nsf/vwapj/SEC-EEC_eng.pdf/$file/SEC-EEC_eng.pdf.

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5.1: Cost-Revenue-Net Income Analysis (Need to Be in the Know)

The end of the month is approaching, and bills are coming due. As you sit at your kitchen table trying to figure out your budget for next month, you wonder whether you will be able to afford that concert you had been planning on attending. Some of your costs remain unchanged from month to month, such as your rent, Internet service, gym membership, and insurance. Other costs tend to fluctuate with your usage, such as your utilities, cellphone bill, vehicle fuel, and the amount of money spent on recreational activities. Together, these regular and irregular costs total to next month's costs.

Examining a few recent paycheque stubs, you calculate the average monthly net income you bring home from your hourly cashier position at Sobeys. The exact amount of each paycheque, of course, depends on how many hours you work. Besides your short-term costs, you need to start saving for next year's tuition. Therefore, your budget needs to include regular deposits into your savings account to meet that goal. Once you have put your bills, paycheques, and goals together, you hope that your budget will balance. If there is a shortfall, you will have to miss out on those concert tickets.

Budgeting at work is no different in principle from your home budget. Businesses also need to recognize the different types of costs they incur each month, some of which remain the same and some of which fluctuate. Businesses need to pay for these costs by generating revenues, which correspond to your paycheque. As with your education goals, businesses also require profits to grow. A business needs to understand all of these numbers so it can plan its activities realistically.

This section explores the various types of costs and establishes a model relating total costs to total revenues to determine total profitability levels. You will then apply this model to see how the sale of an individual product contributes to covering costs and how each product individually contributes to overall profitability.

Types of Costs

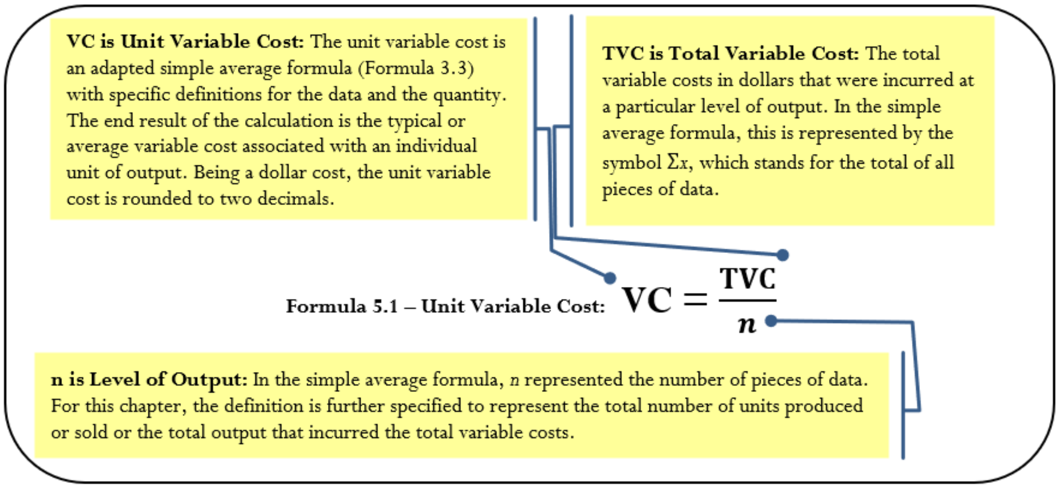
A **cost** is an outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services. Costs can come in three forms:

1. A **fixed cost** is a cost that does not change with the level of production or sales (call this "output" for short). In other words, whether the business outputs nothing or outputs 10,000 units, these costs remain the same. Some examples include rent, insurance, property taxes, salaries unrelated to production (such as management), production equipment, office furniture, and much more. **Total fixed costs** are the sum of all fixed costs that a business incurs.
2. A **variable cost** is a cost that changes with the level of output. In other words, if the business outputs nothing there is no variable cost. However, if the business outputs just one unit (or more) then a cost appears. Some examples include material costs of products, production labour (hourly or piecework wages), sales commissions, repairs, maintenance, and more. **Total variable costs** are the sum of all variable costs that a business incurs at a particular level of output.
3. A **blended cost** is a cost that comprises both fixed cost and variable cost components. In other words, a portion of the total cost remains unchanged while another portion depends on the output. For calculation purposes, you must separate a blended cost into its fixed and variable cost components. A few examples will illustrate the concept of blended costs:

- Residential natural gas bills from Manitoba Hydro include a fixed charge per month of \$14 plus charges for cubic metres of actual consumption based on transportation, distribution, and primary and supplemental gas rates. In this situation, the \$14 is a fixed cost while the actual consumption of natural gas is a variable cost.
- A cell phone bill includes a fixed charge for the phone service plus any additional charges for usage, such as long distance, texting, or data.
- If employees are paid a salary plus commission, then their salaries represent fixed costs while their commissions are a variable cost.

The Formula

In calculating business costs, fixed costs are commonly calculated on a total basis only since the business incurs these costs regardless of any production. However, variable costs are commonly calculated both on a total and per-unit basis to reveal the overall cost along with the cost associated with any particular unit of output. When these variable costs are assigned on an individual basis it is called a **unit variable cost**. The calculation of unit variable cost has a further benefit because it allows managers to explore how the total business costs vary at different levels of output.



VC is Unit Variable Cost: The unit variable cost is an adapted simple average formula (Formula 3.3) with specific definitions for the data and the quantity. The end result of the calculation is the typical or average variable cost associated with an individual unit of output. Being a dollar cost, the unit variable cost is rounded to two decimals.

TVC is Total Variable Cost: The total variable costs in dollars that were incurred at a particular level of output. In the simple average formula, this is represented by the symbol Σx, which stands for the total of all pieces of data.

Formula 5.1 – Unit Variable Cost: $VC = \frac{TVC}{n}$

n is Level of Output: In the simple average formula, n represented the number of pieces of data. For this chapter, the definition is further specified to represent the total number of units produced or sold or the total output that incurred the total variable costs.

Formula 5.1

How It Works

Follow these steps to calculate the unit variable cost:

- **Step 1**: Identify all fixed, variable, and blended costs, along with the level of output. For variable costs, understand any important elements of how the cost is structured. For blended costs, separate the costs into variable and fixed components.
- **Step 2**: Calculate the total variable cost (TVC) by totaling all variable costs at the indicated level of output. This involves taking any known unit variable costs and multiplying each by the level of output.
- **Step 3**: Divide the total variable cost by the total level of output by applying Formula 5.1.

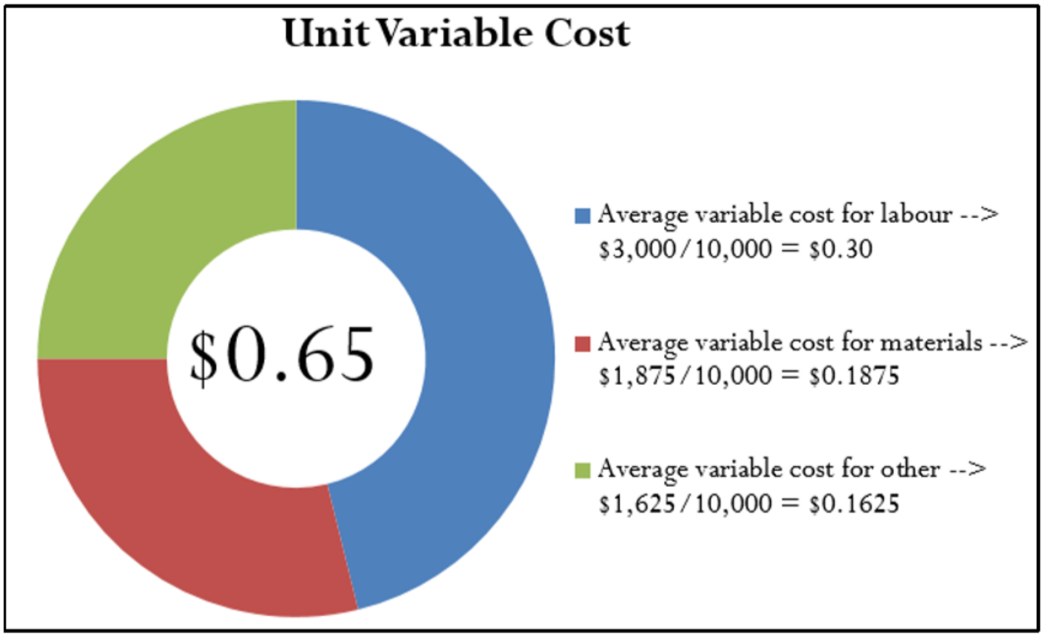
Assume a company produces 10,000 units and wants to know its unit variable cost. It incurs production labor costs of $3,000, material costs of $1,875, and other variable costs totaling $1,625.

Step 1: In this case, all costs are variable costs (production labour and material costs are always variable). The level of output is $n = 10,000$ units.

Step 2: Total all variable costs together to get TVC

$$TVC = \$3,000 + \$1,875 + \$1,625 = \$6,500$$

Step 3: Apply Formula 5.1 to arrive at $VC = \$6,500/10,000 = \$0.65$. This means that, on average, the variable cost associated with one unit of production is $0.65.



Important Notes

The definitions of variable, fixed, and blended costs along with their typical associated examples represent a simplified view of how the real world operates. The complexities involved in real-world business costs complicate the fundamentals of managing a business. Therefore, here is how this textbook addresses the complexities of atypical operations, changing fixed costs, and decreasing unit variable costs:

1. **Atypical Operations**. Although there are "normal" ways that businesses operate, there are also businesses that have atypical operations. What is a fixed cost to one business may be a variable cost to another. For example, rent is usually a fixed cost. However, some rental agreements include a fixed base cost plus a commission on the operational output. These rental agreements form blended costs. This textbook does not venture into any of these atypical costs and instead focuses on common cost categorizations.
2. **Changing Fixed Costs**. In real-world applications, fixed costs do not remain flat at all levels of output. As output increases, fixed costs tend to move upwards in steps. For example, at a low level of output only one manager (on salary) may be needed. As output increases, eventually another manager needs to be hired, perhaps one for every 20,000 units produced. In other words, up to 20,000 units the fixed costs would be constant, but at 20,001 units the fixed costs take a step upwards as another manager is added. The model presented in this textbook does not address these upward steps and treats fixed costs as a constant at all levels of output.
3. **Decreasing Unit Variable Costs**. Production tends to realize efficiencies as the level of output rises, resulting in the unit variable cost dropping. This is commonly known as achieving economies of scale. As a consumer, you often see a similar concept in your retail shopping. If you purchase one can of soup, it may cost $1. However, if you purchase a bulk tray of 12 cans of soup it may cost only $9, which works out to 75¢ per can. This price is lower partly because the retailer incurs lower costs, such as fewer cashiers to sell 12 cans to one person than to sell one can each to 12 different people. Now apply this analogy to production. Producing one can of soup costs 75¢. However, a larger production run of 12 soup cans may incur a cost of only $6 instead of $9 because workers and machines can multitask. This means the unit variable cost would decrease by 25¢ per can. However, the model in this textbook assumes that unit variable costs always remain constant at any given level of output.

✓

You are considering starting your own home-based Internet business. After a lot of research, you have gathered the following financial information:

Dell computer	$214.48 monthly lease payments

Office furniture (desk and chair)	$186.67 monthly rental
Shaw high-speed Internet connection	$166.88 per month
Your wages	$30 per hour
Utilities	$13 per month plus $0.20 per hour usage
Software (and ongoing upgrades)	$20.00 per month
Business licenses and permits	$27.00 per month
Google click-through rate	$10.00 per month + $0.01 per click payable as total clicks per sale

Generating and fulfilling sales of 430 units involves 80 hours of work per month. Based on industry response rates, your research also shows that to achieve your sales you require a traffic volume of 34,890 Google clicks. On a monthly basis, calculate the total fixed cost, total variable cost, and unit variable cost.

Solution

Calculate the monthly total fixed costs (TFC), total variable costs (TVC), and the unit variable cost (VC).

What You Already Know

There are three different types of costs. Other known information includes:

$n = 430$ units Hours = 80

Google clicks = 34,890

How You Will Get There

Step 1:

Sort the costs into fixed and variable. Separate the components for blended costs. You can total the fixed costs to arrive at the total fixed costs, or TFC.

Step 2:

Calculate variable costs, then total them to arrive at the total variable cost, or TVC.

Step 3:

Apply Formula 5.1 to calculate the unit variable cost.

Perform

Step 1:

Fixed Costs

Dell computer	$214.48
Office furniture	$186.67
Shaw high-speed Internet connection	$166.88
Utilities (blended cost)	$13.00 only
Software	$20.00
Business licenses / permits	$27.00
Google click (blended cost)	$10.00
TOTAL FIXED COSTS	TFC = $638.03

Variable Costs

Wages	\$30.00 per hour
Utilities (blended cost)	\$0.20 per hour
Google click (blended cost)	\$0.01 per click

Step 2:

$$\begin{aligned} TVC &= (\$30.00 \times \text{ hours }) + (\$0.20 \times \text{ hours }) + (\$0.01 \times \text{ Google clicks }) \\ &= (\$30.00 \times 80) + (\$0.20 \times 80) + (\$0.01 \times 34,890) = \$2,764.90 \end{aligned}$$

Step 3:

$$VC = \frac{\$2,764.90}{430} = \$6.43$$

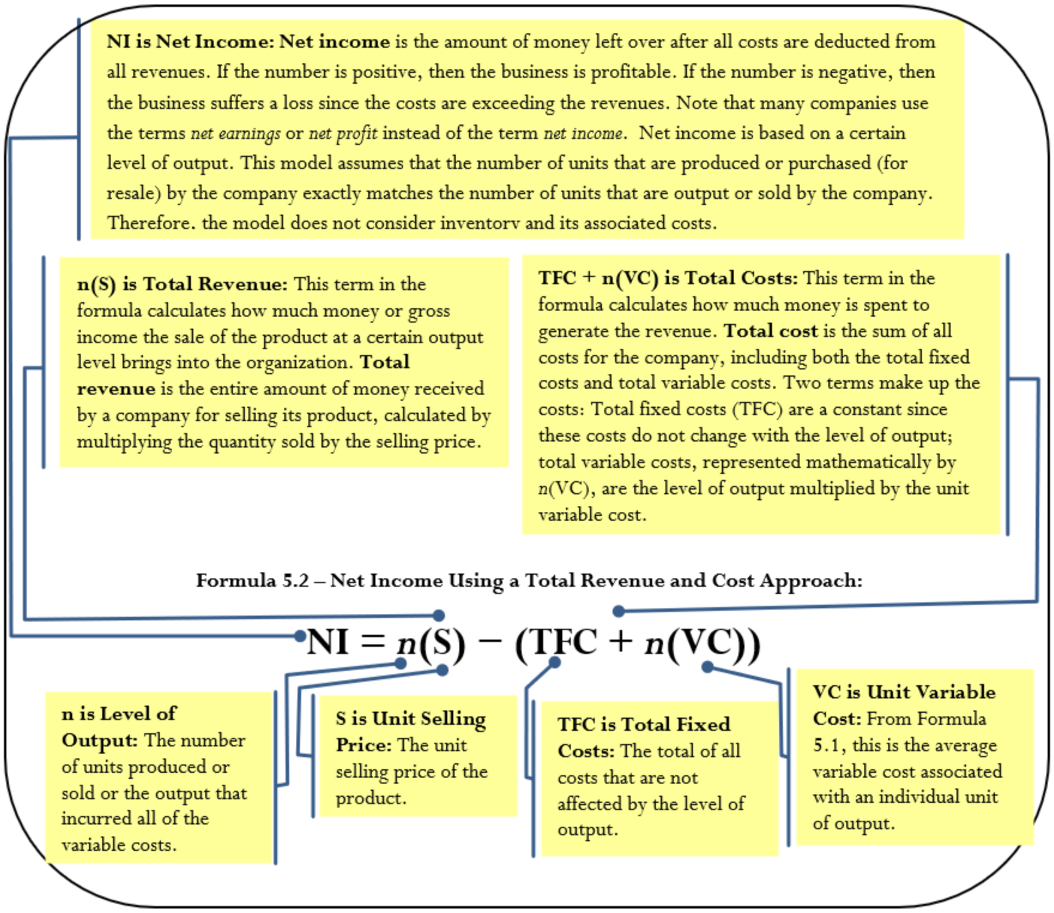
Seven components make up the fixed costs: the computer, furniture, Internet service, fixed utilities, software, licenses/permits, and the fixed Google cost, all totaling \$638.03. Three components make up the variable costs: wages, hourly utilities, and Google clicks, totaling \$2,764.90. The average unit variable cost is \$6.43 per unit sold.

Net Income Using a Total Revenue and Total Cost Approach

Most businesses are "for-profit" businesses, meaning that they operate to make money. In Example $5.1.1$, you figured out the total fixed costs, total variable costs, and the unit variable cost for your Internet business. However, you left unanswered one of the most important questions in business: If you sell the planned 430 units, are you profitable? Is there any money left after you pay for all of those fixed and variable costs? You must also remember that 430 units is just an estimate. What happens if you sell only 350 units? What happens if you sell 525 units? What happens if you decide to pay yourself a higher wage?

There are no guarantees in business, and the future is always uncertain. Successful business managers plan for the future and perform many “what-if” scenarios to answer questions such as those above. This section develops a model for calculating total net income based on total revenues and total costs. The model allows managers to analyze various scenarios and determine the impact on profitability.

The Formula



NI is Net Income: Net income is the amount of money left over after all costs are deducted from all revenues. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss since the costs are exceeding the revenues. Note that many companies use the terms *net earnings* or *net profit* instead of the term *net income*. Net income is based on a certain level of output. This model assumes that the number of units that are produced or purchased (for resale) by the company exactly matches the number of units that are output or sold by the company. Therefore, the model does not consider inventory and its associated costs.

n(S) is Total Revenue: This term in the formula calculates how much money or gross income the sale of the product at a certain output level brings into the organization. **Total revenue** is the entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.

TFC + n(VC) is Total Costs: This term in the formula calculates how much money is spent to generate the revenue. **Total cost** is the sum of all costs for the company, including both the total fixed costs and total variable costs. Two terms make up the costs: Total fixed costs (TFC) are a constant since these costs do not change with the level of output; total variable costs, represented mathematically by *n*(VC), are the level of output multiplied by the unit variable cost.

Formula 5.2 – Net Income Using a Total Revenue and Cost Approach:

$$NI = n(S) - (TFC + n(VC))$$

n is Level of Output: The number of units produced or sold or the output that incurred all of the variable costs.

S is Unit Selling Price: The unit selling price of the product.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

VC is Unit Variable Cost: From Formula 5.1, this is the average variable cost associated with an individual unit of output.

Formula 5.2

How It Works

Follow these steps to calculate net income using a total revenue and total income approach:

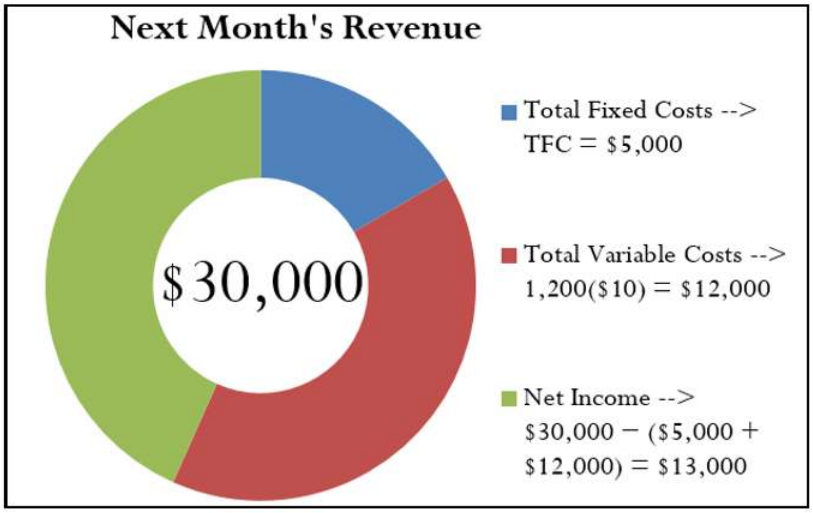
- **Step 1**: Calculate the total revenue. This requires identifying the unit selling price of the product and multiplying it by the level of sales.
- **Step 2**: Calculate your total costs. This requires identifying and separating costs into fixed and variable components. Fixed costs are totaled to arrive at the total fixed cost. Total variable costs are either known or can be calculated through multiplying the unit variable cost by the level of output.
- **Step 3**: Calculate the net income by applying Formula 5.2.

For example, assume that last month a company incurred total variable costs of $10,000 in the course of producing 1,000 units. For next month it forecasts total fixed costs of $5,000 and all variable costs remaining unchanged. Projected production for next month is 1,200 units selling for $25 each. You want to estimate next month's net income.

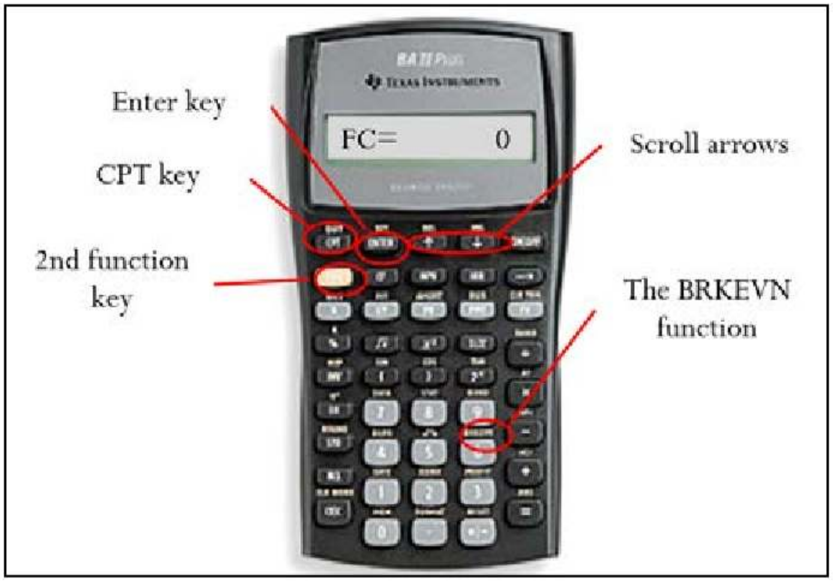
Step 1: Using Formula 5.2, you calculate total revenue from $n(S)$, or the total level of output multiplied by the price of the product. If you project sales of 1,200 units (n) at $25 each (S), then the total forecasted revenue is $1,200(\$25) = \$30,000$.

Step 2: Total fixed costs, or TFC, are $5,000. To get the total variable costs, you must resolve $n(VC)$. You calculate the unit variable cost, or VC, with Formula 5.1. Using the current month figures, you see $VC = \$10,000 \div 1,000 = \$10$. If the projected level of output is 1,200 units, then the total variable costs are $1,200(\$10) = \$12,000$.

Step 3: Applying Formula 5.2 you have $NI = \text{ Total Revenue } - \text{ Total costs } = \$30,000 - (\$5,000 + \$12,000) = \$13,000$. Based on the numbers, you forecast net income of $13,000 for next month.



Important Notes



The Texas Instruments BAII Plus calculator is programmed with a version of Formula 5.2. The function is called Brkevn, and you access it by pressing 2nd and then the number six key. The relationship between the formula symbols and the calculator symbols is displayed in the table below.

Variable	Formula 5.2 Notation	Calculator Notation
Total fixed cost	TFC	FC
Unit variable cost	VC	VC
Price per unit	S	P (for price)
Net income	NI	PFT (for profit)
Level of output	n	Q (for quantity)

To solve Formula 5.2 for the PFT or any other variable, enter data into all of the above variables except one. Keying in a variable requires inputting the value and pressing Enter. Use ↑ and ↓ to scroll through the display. When you are ready, scroll to the unknown variable and press CPT.

Paths To Success

An easy way to remember Formula 5.2 is to understand what the formula represents. As explained, the calculation of $n(S)$ multiplies quantity by price to produce the total revenue. The $(TFC + n(VC)$ takes the total fixed costs and adds the total variable costs (which is a function of quantity multiplied by unit variable cost) to arrive at the total cost. Therefore, Formula 5.2 expressed more simply is:

$$\text{Net Income} = \text{Total Revenue} - \text{Total Cost}$$

Exercise $5.1.1$: Give It Some Thought

In the following situations, explain what would happen to net income and why.

a. The selling price is raised.
b. The hourly wages of production workers are increased to match the increase in the consumer price index.
c. The level of output decreases.
d. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Answer

a. Net income increases because the total revenues increase with no similar increase in costs.
b. Net income decreases because the hourly wages of production workers are variable costs. If the costs go up, then less money is left over.
c. Net income decreases because both total revenue and total variable costs drop; however, the fixed costs remain the same, so there is less revenue to cover proportionally higher costs.
d. Net income increases because the insurance premium is a fixed cost that has become smaller. With lower costs, more money is left over.

✓

Recall from Example $5.1.1$ that you are considering starting your own home-based Internet business. The following information is known:

$$FC = \$638.03,\ VC = \$6.43, \text{ forecasted } n = 430$$

Based on these numbers, calculate:

a. The forecasted net income if your price per unit is $10.
b. The dollar change in net income if you decide to pay yourself a higher wage of $35.38 per hour instead of $30.00 per hour while still working 80 hours. Note the total variable costs excluding wages were $364.90.
c. The dollar change in net income if sales are 30% lower than your initial forecast, ignoring part (b) calculations.

Solution

a. You need to determine net income (NI) using the revenue and cost numbers.
b. Recalculate the total variable cost (TVC) to determine the impact of the higher wage. Then calculate the change in net income (NI).
c. Modify the level of output (n) and calculate the change in net income (NI).

What You Already Know

The costs, level of output, and price are known:

FC = $638.03, VC = $6.43, forecasted n = 430, S = $10

For part (b), the total variable costs excluding wages are known:

TVC (excluding wages) = $364.90

New wage = $35.38

Hours = 80

For part (c), the change in the level of output is known:

$\Delta\% = -30\%$

How You Will Get There

a.

Step 1:

Calculate total revenue at the forecasted output.

Step 2:

Calculate total costs at the forecasted output.

Step 3:

Apply Formula 5.2.

b.

Step 1:

Total revenue is unchanged since the output has not changed.

Step 2:

Fixed costs are unchanged. Recalculate the total variable costs by adding the new wages to the total variable costs excluding wages. By rearranging Formula 5.1, you can see $TVC = n(VC)$. Therefore, substitute TVC in place of $n(VC)$ in Formula 5.2.

Step 3:

Apply Formula 5.2 to recalculate the new net income. Compare to net income in part (a) and determine the change.

c.

Step 1:

The revised level of output is a percent change calculation where $\Delta\% = -30\%$ and $OLD = 430$. Solve Formula 3.1 for New. Once New is known, recalculate total revenue.

Step 2:

Fixed costs are unchanged. Recalculate total variables costs using the revised level of output.

Step 3:

Apply Formula 5.2 to recalculate the new net income. Compare to net income in part (a) and determine the change.

Perform

a.

Step 1:

$$\text{Total revenue } = n(S) = 430(\$10) = \$4,300$$

Step 2:

$$TFC = \$638.03; TVC = n(VC) = 430(\$6.43) = \$2,764.90$$

Step 3:

$$NI = s4,300 - (\$638.03 + \$2,764.90) = \$897.07$$

b.

Step 1:

$$\text{Total revenue } = \$4,300$$

Step 2:

$$TFC = \$638.03; TVC = \$364.90 + (\$35.38 \times 80 \text{ hours }) = \$3,195.30$$

Step 3:

$$NI = \$4,300 - (\$638.03 + \$3,195.30) = \$466.67$$
$$\text{Change in } NI = \$466.67 - \$897.07 = -\$430.40$$

c.

Step 1:

$$\begin{aligned} -30\% = & \frac{\text{New} - 430}{430} \times 100 \\ -129 = & \text{ New } - 430 \\ 301 = & \text{ New} \end{aligned}$$

$$\text{Total revenue } = 301(\$10) = \$3,010$$

Step 2:

$$TFC = \$638.03; TVC = 301(\$6.43) = \$1,935.43$$

Step 3:

$$NI = \$3,010 - (\$638.03 + \$1,935.43) = \$436.54$$
$$\text{Change in } NI = \$436.54 - \$897.07 = -\$460.53$$

Calculator Instructions

	FC	VC	P	PFT	Q
a.	638.03	6.43	10	Answer: 897.07	430
b.	√	$3195.3 \div 430 =$	√	Answer: 466.67	√
c.	√	6.43	√	Answer:436.54	301*

*First use Δ% function to calculate this number:

	OLD	NEW	%CH	#PD
c.	430	Answer: 301	-30	1

Under the initial scenario, a net income of \$897.07 on sales of 430 units is forecast. If you pay yourself a higher wage of \$35.38 per hour, your net income decreases by \$430.40 to \$466.67. If your forecast is inaccurate and is lower by 30%, then your net income is reduced by \$460.53, resulting in a lower net income of \$436.54.

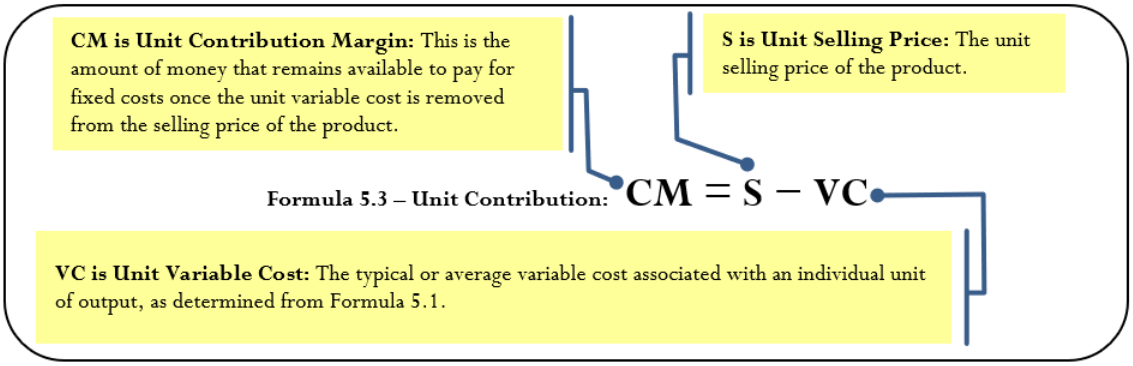
Net Income Using a Total Contribution Margin Approach

In Example 5.1.2, you learned that if you sell the projected 430 units of product for your Internet business, the total net income is \$897.07. What if you sold 431 units of the product? How would your net income change? Clearly it should rise, but by how much? One approach to answering this question is to rerun the numbers through Formula 5.2, revising the total revenues and total variable costs to calculate a new net income. This new net income can then be compared against the existing net income to determine how it changed. This is a multi-step approach and involves a lot of work. An alternative approach explored in this section involves using a unit contribution margin to calculate the net income. The benefit of this approach is that with minimal calculations you can easily assess the impact of any change in the level of output.

In accounting and marketing, the contribution margin is the amount that each unit sold adds to the net income of the business. This approach allows you to understand the impact on net income of each unit sold, and Section 5.2 will explain its further benefit of allowing for a straightforward calculation of a break-even point. The contribution margin determines on a per-unit basis how much money is left over after unit variable costs are removed from the price of the product. This leftover money is then available to pay for the fixed costs. Ultimately, when all fixed costs have been paid for, the leftover money becomes the profits of the business. If not enough money is left over to pay for the fixed costs, then the business has a negative net income and loses money.

The Formula

The difference between the increase in the total revenue and the increase in the total variable costs is how much the sale of an individual product contributes toward the change in your net income. Formula 5.3 expresses this relationship.



CM is Unit Contribution Margin: This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.

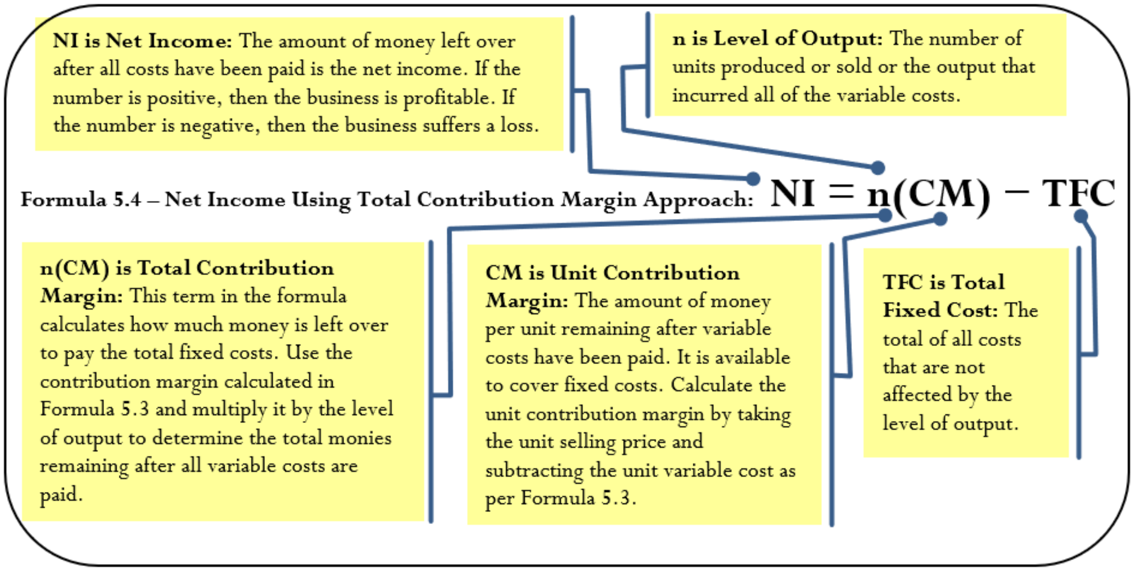
S is Unit Selling Price: The unit selling price of the product.

Formula 5.3 – Unit Contribution: $CM = S - VC$

VC is Unit Variable Cost: The typical or average variable cost associated with an individual unit of output, as determined from Formula 5.1.

Formula 5.3

If you have no units sold, your net income is negative and equal to the total fixed costs associated with your business, since there is no offsetting revenue to pay for those costs. With each unit sold, the contribution margin of each product is available to pay off the fixed costs. Formula 5.4 expresses this relationship when calculating net income.



NI is Net Income: The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.

n is Level of Output: The number of units produced or sold or the output that incurred all of the variable costs.

Formula 5.4 – Net Income Using Total Contribution Margin Approach: $NI = n(CM) - TFC$

n(CM) is Total Contribution Margin: This term in the formula calculates how much money is left over to pay the total fixed costs. Use the contribution margin calculated in Formula 5.3 and multiply it by the level of output to determine the total monies remaining after all variable costs are paid.

CM is Unit Contribution Margin: The amount of money per unit remaining after variable costs have been paid. It is available to cover fixed costs. Calculate the unit contribution margin by taking the unit selling price and subtracting the unit variable cost as per Formula 5.3.

TFC is Total Fixed Cost: The total of all costs that are not affected by the level of output.

Formula 5.4

How It Works

Follow these steps to calculate the net income using a contribution margin approach:

Step 1: If unit information is known, apply Formula 5.3 and calculate the unit contribution margin by subtracting the unit variable cost from the selling price. This may or may not require you to use Formula 5.1 to calculate the unit variable cost.

Step 2: Calculate the total contribution margin by multiplying the contribution margin with the associated level of output.

Step 3: Calculate the total fixed costs by adding all fixed costs.

Step 4: Based on the level of output, calculate the net income by applying Formula 5.4.

For example, using the contribution margin approach, calculate the net income for a product that sells for \$75, has unit variable costs of \$31, total fixed costs of \$23,000, and total sales of 800 units.

Step 1: The unit contribution margin is calculated from Formula 5.3. If the product sells for \$75 and has unit variable costs of \$31, then $CM = \$75 - \$31 = \$44$. This means that every unit sold has \$44 left over to contribute toward fixed costs.

Step 2: Now convert that into a total contribution margin. The first part of Formula 5.4 calculates total contribution margin through $n(CM)$. With sales of 800 units, the total contribution margin is $800(\$44) = \$35,200$.

Step 3: Total fixed costs are known: $TFC = \$23,000$.

Step 4: Apply Formula 5.4, which translates to $\text{Net Income } = \text{ Total contribution margin - Total fixed costs}$, or $\$35,200-\$23,000=\$12,200$.

Paths To Success

When you work with Formula 5.4, sometimes unit information may not be known. Instead, you might just have a single aggregate number representing the total contribution margin for which somebody has already taken the total revenue and subtracted the total variable costs. In this case, skip step 1 and take the provided total contribution margin as the answer for step 2 with no calculations necessary.

✓ Example $5.1.3$: The Contribution Margin for Your Internet Business

Continuing with Examples $5.1.1$ and $5.1.2$, calculate the unit contribution margin and net income using the contribution margin approach. From the previous examples, recall the unit variable cost is $6.43, unit selling price is $10, total fixed costs are $638.03, and the projected sales are 430 units.

Solution

Calculate the unit contribution margin (CM) followed by the net income (NI) using the contribution margin approach.

What You Already Know

The cost, price, and sales information are known:

$VC=\$6.43, S=\$10$

$TFC=\$638.03, n=430$

How You Will Get There

Step 1:

Calculate the unit contribution margin using Formula 5.3.

Step 2:

Calculate the total contribution margin by multiplying the unit contribution margin times the level of output.

Step 3:

Determine the total fixed costs.

Step 4:

To calculate net income, apply Formula 5.4.

Perform

Step 1:

$$CM=\$10.00-\$6.43=\$3.57$$

Step 2:

$$n(CM)=430(\$3.57)=\$1,535.10$$

Step 3:

$$TFC=\$638.03$$

Step 4:

$$NI=\$1,535.10-\$638.03=\$897.07$$

Calculator Instructions

FC	VC	P	PFT	Q
638.03	6.43	10	Answer: 897.07	430

> When a product sells for \$10, \$6.43 goes toward paying for the variable costs of your business, leaving \$3.57 as your unit contribution margin. This means that for every unit increase in sales, your net income rises by \$3.57. Thus, if you sell 430 units you have a total contribution margin of \$1,535.10, which results in a net income of \$897.07 after removing fixed costs of \$638.03.

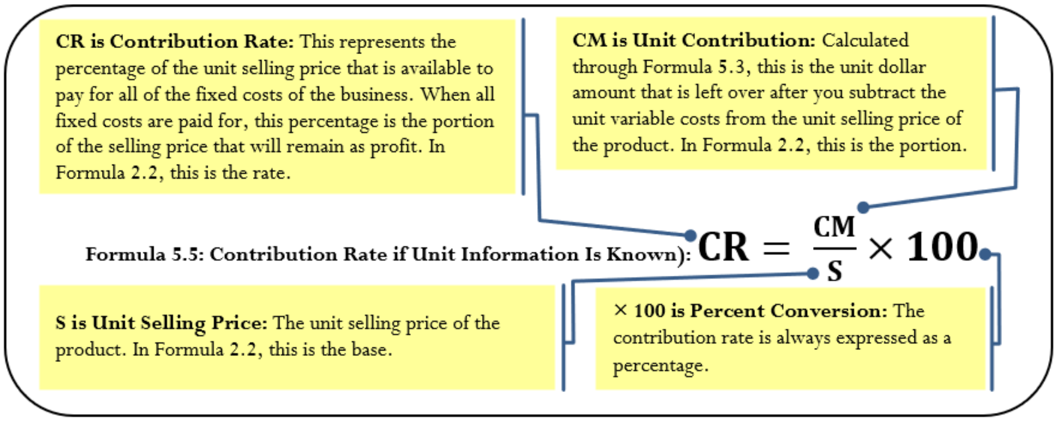
Contribution Rates

It is difficult to compare different products and their respective dollar amount contribution margins if their selling prices and costs vary widely. For example, how do you compare a unit contribution margin of \$1,390 (selling price of \$2,599.99) on a big screen television to a unit contribution margin of \$0.33 on a chocolate bar (selling price of \$0.67)? On a per-unit basis, which contributes relatively more to fixed costs? To facilitate these comparisons, the products must be placed on equal terms, requiring you to convert all dollar amount contribution margins into percentages. A contribution rate is a contribution margin expressed as a percentage of the selling price.

The Formula

Your choice between two formulas for calculating a contribution rate depends on whether you have unit information or only aggregate information. Both formulas are adaptations of Formula 2.2 (Rate, Portion, Base).

If unit information is known, including the unit variable cost and unit selling price, then calculate the contribution rate using unit information as expressed in Formula 5.5.



CR is Contribution Rate: This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit. In Formula 2.2, this is the rate.

CM is Unit Contribution: Calculated through Formula 5.3, this is the unit dollar amount that is left over after you subtract the unit variable costs from the unit selling price of the product. In Formula 2.2, this is the portion.

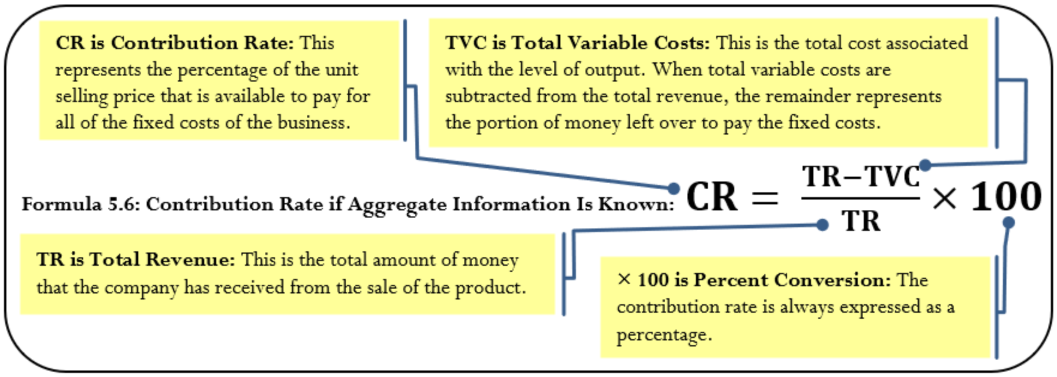
Formula 5.5: Contribution Rate if Unit Information Is Known): $CR = \frac{CM}{S} \times 100$

S is Unit Selling Price: The unit selling price of the product. In Formula 2.2, this is the base.

× 100 is Percent Conversion: The contribution rate is always expressed as a percentage.

Formula 5.5

If any unit information, including the unit variable cost or unit selling price, is unknown or unavailable, then you cannot apply Formula 5.5. Sometimes only aggregate information is known. When total revenue and total variable costs for any product are known or can at least be calculated, then you must calculate the contribution rate from the aggregate information as expressed in Formula 5.6.



CR is Contribution Rate: This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business.

TVC is Total Variable Costs: This is the total cost associated with the level of output. When total variable costs are subtracted from the total revenue, the remainder represents the portion of money left over to pay the fixed costs.

Formula 5.6: Contribution Rate if Aggregate Information Is Known: $CR = \frac{TR - TVC}{TR} \times 100$

TR is Total Revenue: This is the total amount of money that the company has received from the sale of the product.

× 100 is Percent Conversion: The contribution rate is always expressed as a percentage.

Formula 5.6

How It Works

Follow these steps to calculate a contribution rate:

Step 1: Identify the required variables and calculate the margin, if needed.

- If unit information is known, this requires you to calculate the unit contribution margin. Otherwise, calculate the unit contribution margin by applying Formula 5.3. You must identify the unit selling price.
- If aggregate information is known, you need to identify total revenue and total variable costs.

Step 2: Calculate the contribution rate.

- If unit information is known, apply Formula 5.5.
- If only aggregate information is known, apply Formula 5.6.

As an example of these steps, recall earlier that you wanted to compare the relative contributions of the big screen television and the chocolate bar. The television sells for $2,599.99 and has a unit contribution margin of $1,390. The chocolate bar sells for $0.67 and has a unit contribution margin of $0.33. Notice that the information being provided is on a per-unit basis. Calculate the contribution rate of each product.

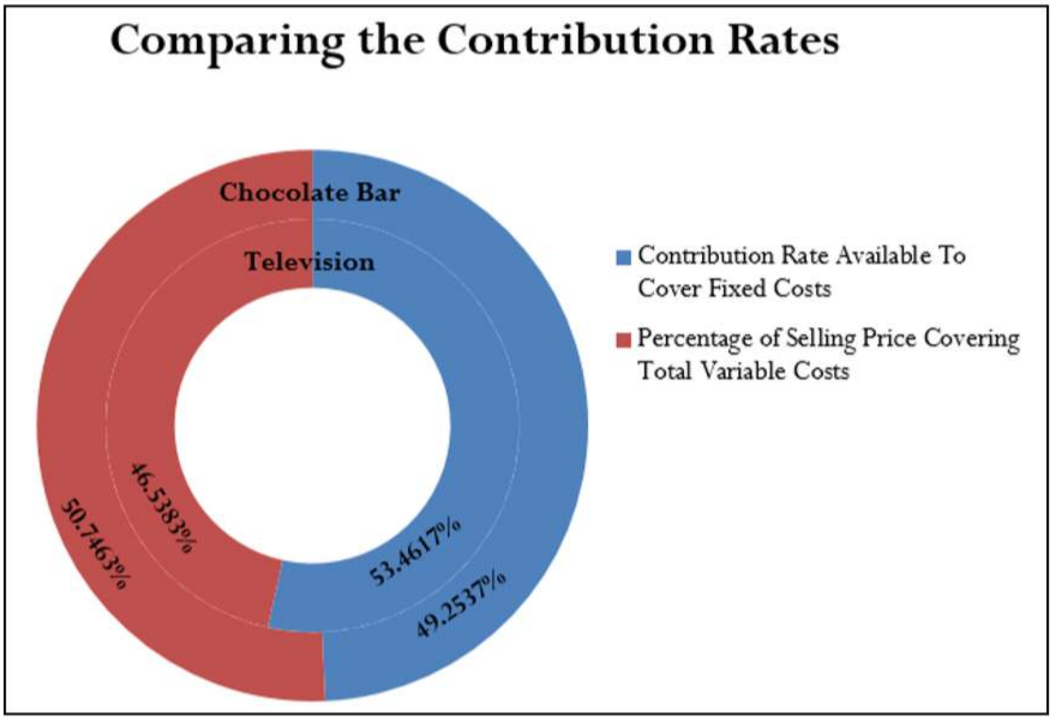
Step 1: The contribution margins are known. For the television, CM = $1,390, and for the chocolate bar, CM = $0.33.

Step 2: Applying Formula 5.5, the television $CR = \$1,390.00 \div \$2,599.99 \times 100 = 53.4617\%$, while the chocolate bar $CR = \$0.33 \div \$0.67 \times 100 = 49.2537\%$. It is now evident from the contribution rate that 4.208% more of the television's selling price is available to pay for fixed costs as compared to the chocolate bar's price.

Now change the facts. This time, assume there is no unit information. Instead, the television's total revenue is $129,999.50 and associated total variable costs are $60,499.50. The chocolate bar has total revenue of $3,886.00 with total variable costs of $1,972.00. Based on this information, you are to determine the product with the higher contribution rate.

Step 1: The aggregate numbers are known for both products. For the television, $TR = \$129,999.50$ and $TVC = \$60,499.50$. For the chocolate bar, $TR = \$3,886.00$ and $TVC = \$1,972.00$.

Step 2: Applying Formula 5.6, the television $CR = \frac{\$129,999.50 - \$60,499.50}{\$129,999.50} \times 100 = 53.4617\%$ while the chocolate bar $CR = \frac{\$3,886 - \$1,972}{\$3,886} \times 100 = 49.2537\%$ You have reached the same conclusion as above.



Exercise $5.1.2$: Give It Some Thought

In each of the following situations, what would happen to the contribution rate and why?

a. The selling price is raised.
b. The hourly wages of production workers are increased to match the increase in the consumer price index.
c. The level of output decreases.
d. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Answer

a. The contribution rate increases, since raising the price increases the unit contribution margin.
b. The contribution rate decreases, since total variable costs rise, eroding some of the unit contribution margin.
c. There is no effect on the contribution rate. The level of output is not a factor in calculating the contribution rate.
d. There is no effect on the contribution rate since the insurance premiums are a fixed cost. Fixed costs are not a factor in calculating contribution rates

Example $5.1.4$: Understanding the Contribution Rate of Your Internet Business

Continuing with your ongoing Internet business from the three previous examples, calculate the contribution rate using both the unit and aggregate methods and show how they arrive at the same number. Remember that you are selling products for $10 each, your unit contribution margin is $3.57, total revenue is projected at $4,300, and total variable costs are $2,764.90.

Solution

Your goal is to calculate the contribution rate (CR) using both the unit and aggregate methods.

What You Already Know

Step 1:

Prices, margins, revenue, and costs are known:

$S = \$10, CM = \$3.57$

$TR = \$4,300, TVC = \$2,764.90$

How You Will Get There

Step 2a:

Using the unit method, apply Formula 5.5.

Step 2b:

Using the aggregate method, apply Formula 5.6.

Perform

Step 2a:

$$CR = \frac{\$3.57}{\$10.00} \times 100 = 35.7\%$$

Step 2b:

$$CR = \frac{54,300.00 - \$2,764.90}{\$4,300.00} \times 100 = 35.7\%$$

Under both the unit and aggregate method, your contribution rate equals 35.7%. This means that every time you sell a $10 product, 35.7% of the revenue remains after recovering the cost of the product.

Putting It All Together

You have studied costs, volume, and net income in this section. So far, you have considered each of these concepts separately while you worked through the various applications. It is time to put the types of costs, unit variable cost, net income, sales, contribution margin, and contribution rate together so that you can see all of these concepts in a single scenario. Look at the following example.

✓ Example 5.1.5: Running a Pizza Delivery Business

In the commercial section of the newspaper you come across an ad for a pizza delivery business for sale. Upon inquiry, you discover that the owner, who wants to sell the business and then retire, has four salaried employees and owns two delivery vehicles. He invites you to look through his books, where you acquire the following information:

Employees: Owner's salary Employee salaries and premiums	$5,000 per month $2,000 per month each	**Business:** Phone Business insurance Building lease	$85 per month $3,600 per year $2,000 per month
Delivery Vehicles: Car insurance Fuel Oil changes Vehicle maintenance and repairs	$1,200 per year per vehicle $1,125 per month per vehicle $225 per month per vehicle $562.50 per month per vehicle	**Operations:** Pizza ingredients, materials, and packaging Selling price of pizza Average number of pizzas sold	$19,125 per month $9 per pizza 4,500 per month

Assume that this is all of the key information. You need to understand this business and therefore want to determine:

a. A unit variable cost
b. A typical monthly net income for the business
c. The contribution margin per pizza
d. The contribution rate

Solution

All calculations require the sorting of costs into fixed and variable components, which will be your first step. You then calculate a unit variable cost (VC) using the historical operations information. Once you obtain this, you can calculate the net income (NI), contribution margin (CM), and contribution rate (CR).

What You Already Know

You have unit information on costs, volume, and revenue, as listed above.

How You Will Get There

a. Unit variable cost:

Step 1:

Sort the costs into fixed and variable categories. Total the monthly fixed costs.

Step 2:

Total the monthly variable costs.

Step 3:

Calculate the unit variable cost by applying Formula 5.1.

b. Net income:

Step 1:

Calculate total revenue at the level of output.

Step 2:

Calculate total costs at the level of output.

Step 3:

Calculate monthly net income by applying Formula 5.2.

c. and d. Contribution margin and rate:

Step 1:

Calculate the contribution margin by applying Formula 5.3.

Step 2:

Calculate the contribution rate by applying Formula 5.5.

Perform

a. Unit variable cost:

Steps 1 and 2:

Fixed Costs per Month

Owner’s salary	\$5,000
Employee salaries and premiums	\$2,000 × 4 = \$8,000
Car insurance	\$1,200 × 2 cars ÷ 12 months = \$200
Phone	\$85
Business insurance	\$3,600 ÷ 12 months = \$300
Building lease	\$2,000
TOTAL FIXED COSTS	TFC = \$15,585 per month

Variable Costs per Month

Fuel	\$1,125 × 2 vehicles = \$2,250
Oil changes	\$225 × 2 vehicles =\$450
Vehicle maintenance and repairs	\$562.50 × 2 vehicles = \$1,125
Pizza ingredients, materials, and packaging	\$19,125
TOTAL VARIABLE COSTS	TVC = \$22,950 per month

Step 3:

$$VC=\frac{\$2,250+\$450+\$1,125+\$19,125}{4,500 \text{ pizzas}}=\$5.10 \text{ per pizza}$$

b. Net income:

Step 1:

$$\text{Total Revenue }=4,500(\$9)=\$40,500$$

Step 2:

$$TFC=\$15,585 ; TVC=4,500(\$5.10)=\$22,950$$

Step 3:

$$NI=\$40,500-(\$15,585+\$22,950)=\$40,500-\$38,535=\$1,965$$

c. and d. Contribution margin and rate:

Step 1:

$$CM=\$9.00-\$5.10=\$3.90$$

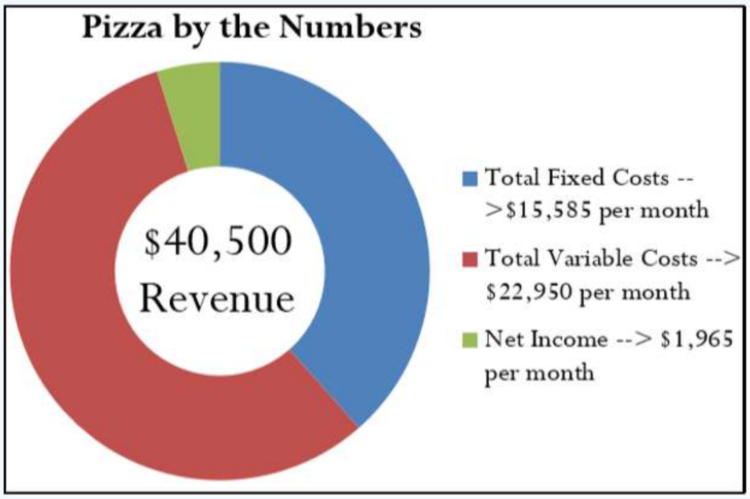
Step 2:

$$CR=\frac{53.90}{59.00} \times 100=43.\overline{3}\%$$

Calculator Instructions

FC	VC	P	PFT	Q
15585	5.1	9	Answer: 1,965	4500

This is a profitable business. For an average month, the business incurs $15,585 in fixed costs along with $22,950 in variable costs, which works out to a unit variable cost of $5.10 for each of the 4,500 pizzas sold. With total revenue of $40,500, after removing both fixed and variable costs, net income is $1,965 per month. Every pizza, after paying for the unit variable costs, has $3.90, or $43.\overline{3}\%$, left over.



Contributors and Attributions

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5.2: Break-Even Analysis (Sink or Swim)

Should you start up the Internet business described in the last section? Right now, all you have are some projected costs and a forecasted level of sales. You imagine you are going to sell 400 units. Is that possible? Is it reasonable to forecast this many sales?

Now you may say to yourself, "400 units a month . . . that's about 13 per day. What's the big deal?" But let's gather some more information. What if you looked up your industry in Statistics Canada data and learned that the product in question sells just 1,000 units per month in total? Statistics Canada also indicates that there are eight existing companies selling these products. How does that volume of 400 units per month sound now? Unless you are revolutionizing your industry, it is unlikely you will receive a 40% market share in your first month of operations. With so few unit sales in the industry and too many competitors, you might be lucky to sell 100 units. If this is the case, are you still profitable?

Simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

What Is Break-Even Analysis?

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through break-even analysis, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This break-even point is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero. To determine the break-even point, you can calculate a break-even analysis in two different ways, involving either the number of units sold or the total revenue in dollars. Each of these two methods is discussed in this section.

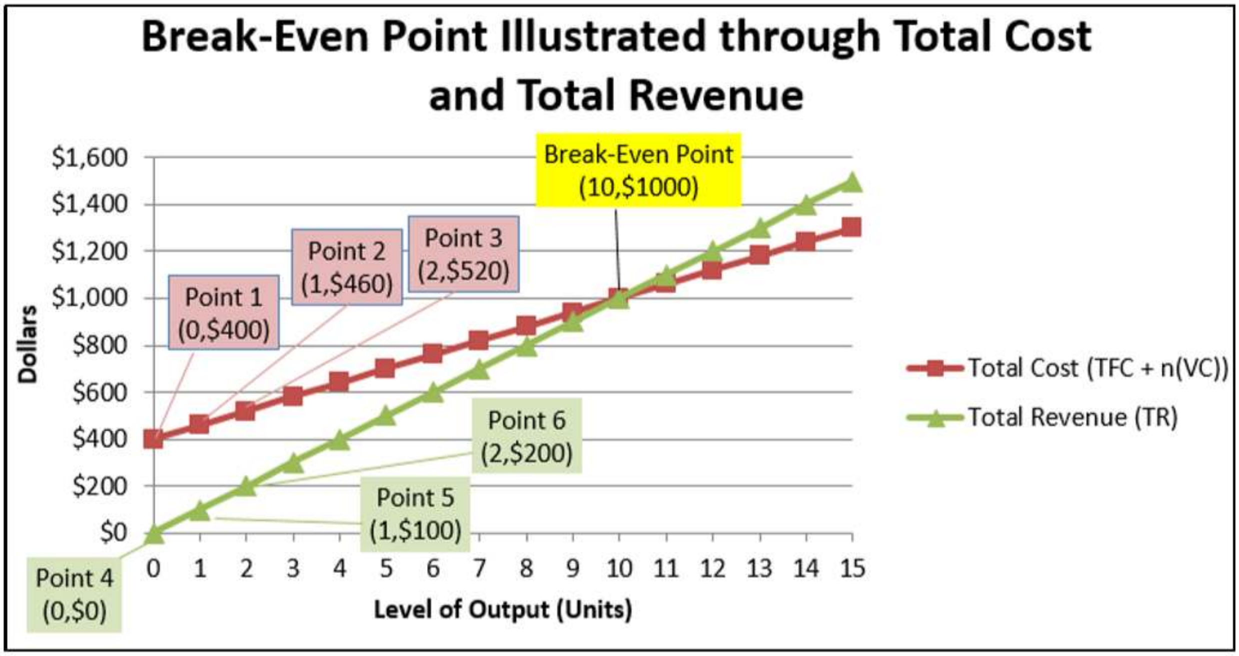
Method 1: Break-Even Analysis in Units

In this method, your goal is to determine the level of output that produces a net income equal to zero. This method requires unit information, including the unit selling price and unit variable cost.

It is helpful to see the relationship of total cost and total revenue on a graph. Assume that a company has the following information:

$$TFC = \$400 \quad S = \$100 \quad VC = \$60$$

The graph shows dollar information on the y-axis and the level of output on the x-axis.



Here is how you construct such a graph:

1. Plot the total costs:
   a. At zero output you incur the total fixed costs of $400. Denote this as Point 1 (0, $400).
   b. As you add one level of output, the total cost rises in the amount of the unit variable cost. Therefore, total cost is $TFC + n(VC) = \$400 + 1(\$60) = \$460$. Denote this as Point 2 (1, $460).
   c. As you add another level of output (2 units total), the total cost rises once again in the amount of the unit variable cost, producing $\$400 + 2(\$60) = \$520$. Denote this as Point 3 (2, $520).
   d. Repeat this process for each subsequent level of output and plot it onto the figure. The red line plots these total costs at all levels of output.
2. Plot the total revenue:
   a. At zero output, there is no revenue. Denote this as Point 4 (0, $0).
   b. As you add one level of output, total revenue rises by the selling price of the product. Therefore, total revenue is $n(S) = 1(\$100) = \$100$. Denote this as Point 5 (1, $100).
   c. As you add another level of output (2 units total), the total revenue rises once again in the amount of the selling price, producing $2(\$100) = \$200$. Denote this as Point 6 (2, $200).
   d. Repeat this process for each subsequent level of output and plot it onto the figure. The green line plots the total revenue at all levels of output.

The purpose of break-even analysis is to determine the point at which total cost equals total revenue. The graph illustrates that the break-even point occurs at an output of 10 units. At this point, the total cost is $\$400 + 10(\$60) = \$1,000$, and the total revenue is $10(\$100) = \$1,000$. Therefore, the net income is $\$1,000 - \$1,000 = \$0$; no money is lost or gained at this point.

The Formula

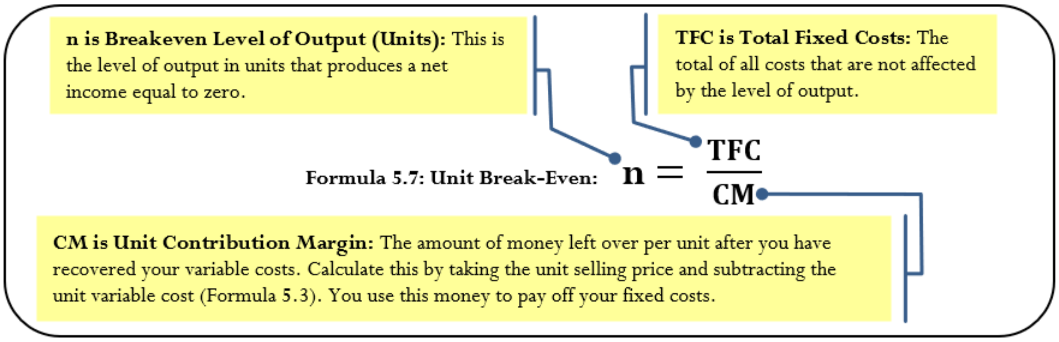
Recall that Formula 5.2 states that the net income equals total revenue minus total costs. In break-even analysis, net income is set to zero, resulting in

$$0 = n(S) - (TFC + n(VC))$$

Rearranging and solving this formula for n gives the following:

$$\begin{aligned} 0 &= n(S) - TFC - n(VC) \\ TFC &= n(S) - n(VC) \\ TFC &= n(S - VC) \\ \frac{TFC}{(S - VC)} &= n \end{aligned}$$

Formula 5.3 states that $CM = S - VC$; therefore, the denominator becomes just CM. The calculation of the break-even point using this method is thus summarized in Formula 5.7.



n is Breakeven Level of Output (Units): This is the level of output in units that produces a net income equal to zero.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

Formula 5.7: Unit Break-Even: $n = \frac{TFC}{CM}$

CM is Unit Contribution Margin: The amount of money left over per unit after you have recovered your variable costs. Calculate this by taking the unit selling price and subtracting the unit variable cost (Formula 5.3). You use this money to pay off your fixed costs.

Formula 5.7

How It Works

Follow these steps to calculate the break-even point in units:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the unit contribution margin (CM) by applying any needed techniques or formulas.

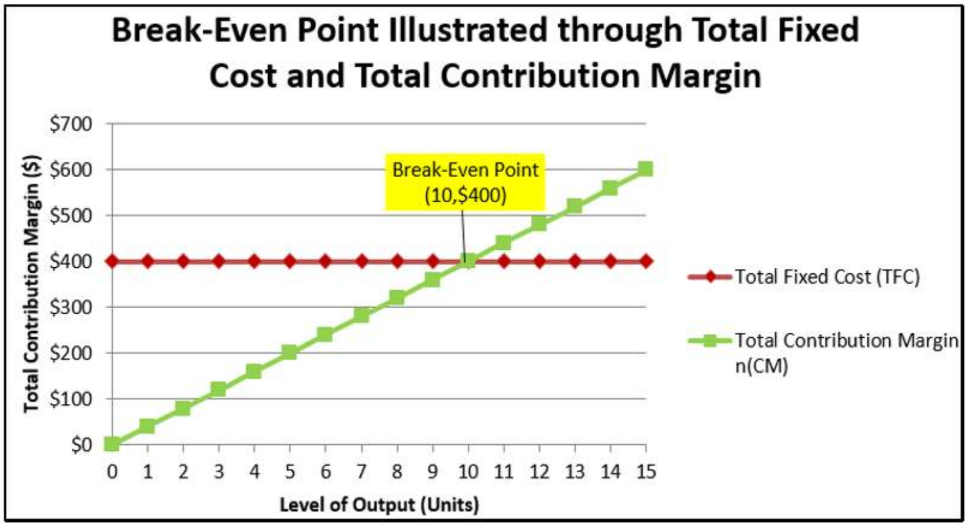
Step 3: Apply Formula 5.7.

Continuing with the example that created the graph on the previous page:

Step 1: Total fixed costs are known, $TFC = \$400$.

Step 2: The unit contribution margin is $\$100 - \$60 = \$40$. For each unit sold this is the amount left over that can be applied against total fixed costs.

Step 3: Applying Formula 5.7 results in $n = \$400 \div \$40 = 10$ units.



Important Notes

When you calculate the break-even units, the formula may produce a number with decimals. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round the level of output down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

✓ Example $5.2.1$: The Break-Even Units for Your Planned Internet Business

Recall from Section 5.1 the Internet business explored throughout Examples 5.1.1 to 5.1.4. Now let’s determine the break-even point in units. As previously calculated, the total fixed costs are $638.03 and the unit contribution margin is $3.57.

Solution

Calculate the break-even point in units sold, or n at break-even.

What You Already Know

Step 1:

The total fixed costs are known: $TFC = \$638.03$.

Step 2:

The contribution rate is known: $CM = \$3.57$.

How You Will Get There

Step 3:

Apply Formula 5.7.

Perform

Step 3:

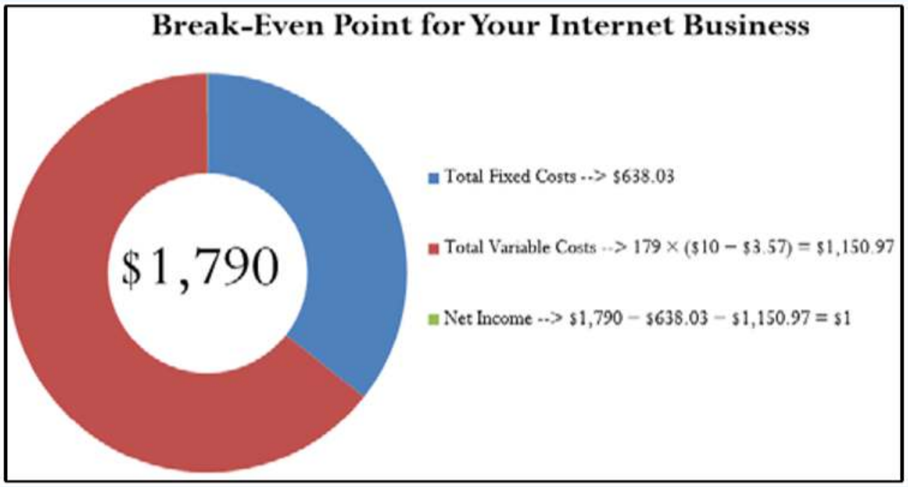
$$n = \frac{\$638.03}{\$3.57} = 178.719888$$

Round this up to 179.

Calculator Instructions

FC	VC	P	PFT	Q
638.03	6.43	10	0	Answer: 178.719888

In order for your Internet business to break even, you must sell 179 units. At a price of \$10 per unit, that requires a total revenue of \$1,790. At this level of output your business realizes a net income of \$1 because of the rounding.



Method 2: Break-Even Analysis in Dollars

The income statement of a company does not display unit information. All information is aggregate, including total revenue, total fixed costs, and total variable costs. Typically, no information is listed about unit selling price, unit variable costs, or the level of output. Without this unit information, it is impossible to apply Formula 5.7.

The second method for calculating the break-even point relies strictly on aggregate information. As a result, you cannot calculate the break-even point in units. Instead, you calculate the break-even point in terms of aggregate dollars expressed as total revenue.

The Formula

To derive the break-even point in dollars, once again start with Formula 5.2, where total revenue at break-even less total fixed costs and total variable costs must equal a net income of zero:

$$NI = TR - (TFC + TVC)$$

$$0 = TR - (TFC + TVC)$$

Rearranging this formula for total revenue gives:

$$0 = TR - TFC - TVC$$

$$TR = TFC + TVC$$

Thus, at the break-even point the total revenue must equal the total cost. Substituting this value into the numerator of Formula 5.6 gives you:

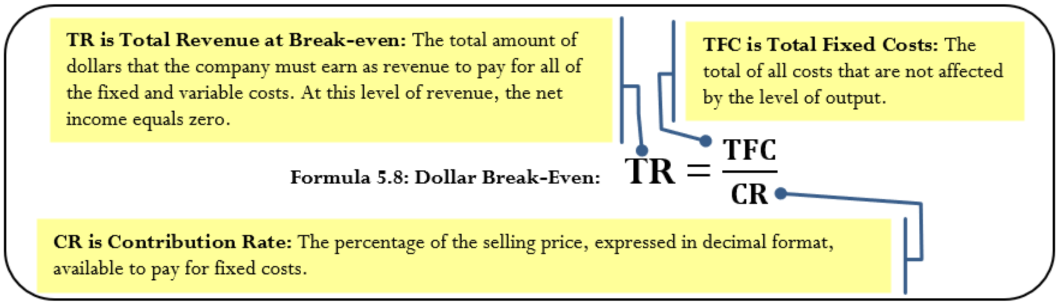
$$CR = (TR - TVC)/TR \times 100$$

$$CR = ((TFC + TVC) - TVC)/TR \times 100$$

$$CR = ((TFC + TVC) - TVC)/TR \times 100$$

$$CR = TFC/TR \times 100$$

A final rearrangement results in Formula 5.8, which expresses the break-even point in terms of total revenue dollars.



TR is Total Revenue at Break-even: The total amount of dollars that the company must earn as revenue to pay for all of the fixed and variable costs. At this level of revenue, the net income equals zero.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

Formula 5.8: Dollar Break-Even: $TR = \frac{TFC}{CR}$

CR is Contribution Rate: The percentage of the selling price, expressed in decimal format, available to pay for fixed costs.

Formula 5.8

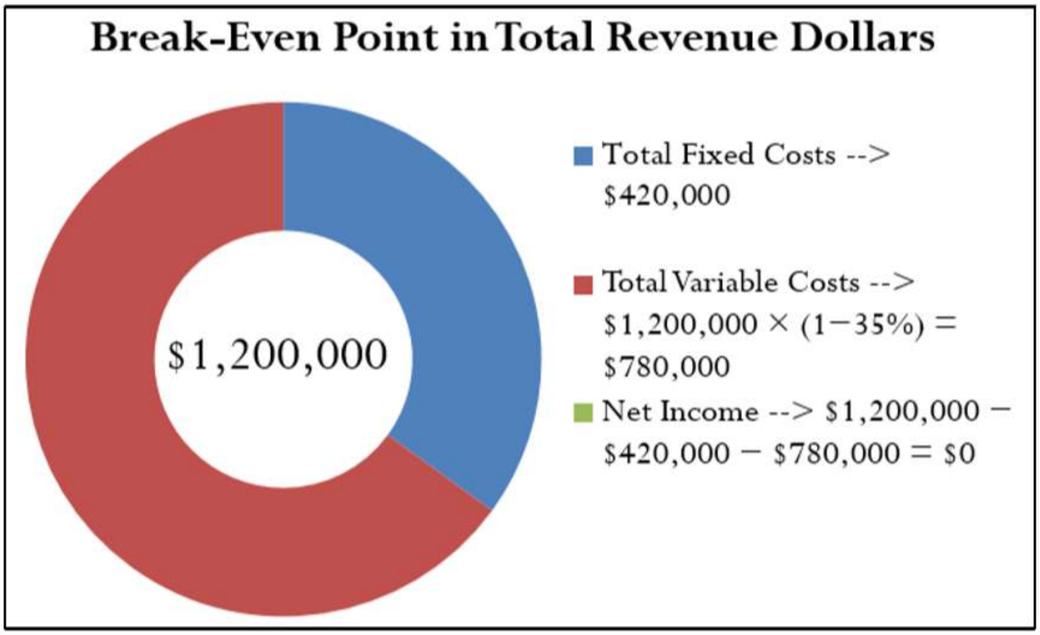
How It Works

Follow these steps to calculate the break-even point in total revenue dollars:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the contribution rate (CR), by applying any needed techniques or formulas. If not provided, typically the CR is calculated using Formula 5.6, which requires aggregate information only.

Step 3: Apply Formula 5.8 to calculate the break-even point in dollars.



Assume that you are looking at starting your own business. The fixed costs are generally easier to calculate than the variable costs. After running through the numbers, you determine that your total fixed costs are $420,000, or $TFC = \$420,000$. You are not sure of your variable costs but need to gauge your break-even point. Many of your competitors are publicly traded companies, so you go online and pull up their annual financial reports. After analyzing their financial statements, you see that your competitors have a contribution rate of 35%, or $CR = 0.35$, on average. What is your estimate of your break-even point in dollars?

Step 1: Total fixed costs are $TFC = \$420,000$.

Step 2: The estimated contribution rate is $CR = 0.35$.

Step 3: Applying Formula 5.8 results in

$TR = \$420,000 \div 0.35 = \$1,200,000$. If you average a similar contribution rate, you require total revenue of $1,200,000 to cover all costs, which is your break-even point in dollars.

Important Notes

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared. The break-even number by itself, whether in units or dollars, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include but is not limited to the following:

- Industry sales (in units or dollars)
- Number of competitors fighting for market share in your industry
- Production capacity of your business

For example, in your Internet business the break-even point is 179 units per month. Is that good? In the section opener, you explored a possibility where your industry had total monthly sales of 1,000 units and you faced eight competitors. A basic analysis shows that if you enter the industry and if everyone split the market evenly, you would have sales of 1,000 divided by nine companies, equal to 111 units each. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share. This doesn't seem very likely, as these other companies are already established and probably have satisfied customers of their own that would not switch to your business.

Break-Even Points Are Not Green Lights. A break-even point alone cannot tell you to do something, but it can tell you not to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even of 179 units put up a whole lot of red lights since it does not seem feasible to obtain. However, what if your industry sold 10,000 units instead of 1,000 units? Your break-even would now be a 1.79% market share (179 units out of 10,000 units), which certainly seems realistic and attainable. This does not mean “Go for it,” however. It just means that from a strictly financial point of view breaking even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number. For instance, if your Google ad is written poorly you might not be able to generate that many sales. The break-even analysis cannot factor in this non-quantitative variable, and for that reason it cannot offer a “go ahead.”

? Exercise $5.2.1$: Give It Some Thought

What would happen to the break-even point in each of the following situations? Would it increase, decrease, or remain the same?

a. The unit contribution margin increases.
b. The total fixed costs increase.
c. The contribution rate decreases.

Answer

a. Decrease. In Formula 5.7, the denominator is larger, producing a lower break-even.
b. Increase. In both Formula 5.7 and Formula 5.8, the numerator is larger, producing a higher break-even.
c. Increase. In Formula 5.8, the denominator is smaller, producing a higher break-even.

✓ Example $5.2.2$: Determining the Break-Even Dollars

In the annual report to shareholders, Borland Manufacturing reported total gross sales of $7,200,000, total variable costs of $4,320,000, and total fixed costs of $2,500,000. Determine Borland's break-even point in dollars.

Solution

Calculate the dollar break-even point, which is the total revenue (TR) at break-even.

What You Already Know

Step 1:

The total fixed costs are known: $TFC = \$2,500,000$.

Other known information includes the following:

$TR = \$7,200,000, TVC = \$4,320,000$

How You Will Get There

Step 2:

Calculate the contribution rate by applying Formula 5.6.

Step 3:

Apply Formula 5.8.

Perform

Step 2:

$$CR = \frac{\$7,200,000 - \$4,320,000}{\$7,200,000} \times 100 = \frac{\$2,880,000}{\$7,200,000} \times 100 = 40\%$$

Step 3:

$$TR = \frac{\$2,500,000}{40\%} = \frac{52,500,000}{0.4} = \$6,250,000$$

Borland Manufacturing achieves its break-even point at $6,250,000 in total revenue. At this point, total fixed costs are $2,500,000 and total variable costs are $3,750,000, producing a net income of zero.

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5.S: Marketing and Accounting Fundamentals (Summary)

Key Concepts Summary

5.1: Cost-Revenue-Net Income Analysis (Need to Be in the Know)

- The three different types of costs
- How to determine the unit variable cost
- Calculating net income when you know total revenue and total cost
- Calculating net income when you know the contribution margin
- Comparing varying contribution margins by calculating a contribution rate
- Integrating all of the concepts together

5.2: Break-Even Analysis (Sink or Swim)

- Explanation of break-even analysis
- How to calculate the break-even point expressed in the level of output
- How to calculate the break-even point expressed in total revenue dollars

The Language of Business Mathematics

blended cost

A cost that comprises both fixed cost and variable cost components.

break-even analysis

An analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost.

break-even point

A quantity that represents the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero.

contribution margin

The amount each unit sold adds to the net income of the business.

contribution rate

A contribution margin expressed as a percentage of the selling price.

cost

An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

economies of scale

The principle that, as production levels rise, per-unit variable costs tend to become lower as efficiencies are achieved.

fixed cost

A cost that does not change with the level of production or sales.

net income

The amount of money left over after all costs are deducted from all revenues.

total fixed cost

The sum of all fixed costs that a business incurs.

total cost

The sum of all costs for the company, including both the total fixed costs and total variable costs.

total revenue

The entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.

total variable cost

The sum of all variable costs that a business incurs at a particular level of output.

unit variable cost

The assignment of total variable costs on a per-unit basis.

variable cost

A cost that changes with the level of production or sales.

The Formulas You Need to Know

Symbols Used

CM = unit contribution margin

CR = contribution rate

n = number of pieces of data, which in this chapter is the level of output

NI = net income

S = unit selling price of the product

TFC = total fixed cost

TR = total revenue

TVC = total variable cost

VC = unit variable cost

Formulas Introduced

Formula 5.1 Unit Variable Cost: $VC=\dfrac{TVC}{n}$

Formula 5.2 Net Income Using a Total Revenue and Cost Approach: $NI=n(S)-(TFC+n(VC))$

Formula 5.3 Unit Contribution Margin: $CM=S-VC$

Formula 5.4 Net Income Using Total Contribution Margin Approach: $NI=n(CM)-TFC$

Formula 5.5 Contribution Rate if Unit Information Is Known: $CR=\dfrac{CM}{S}\times 100$

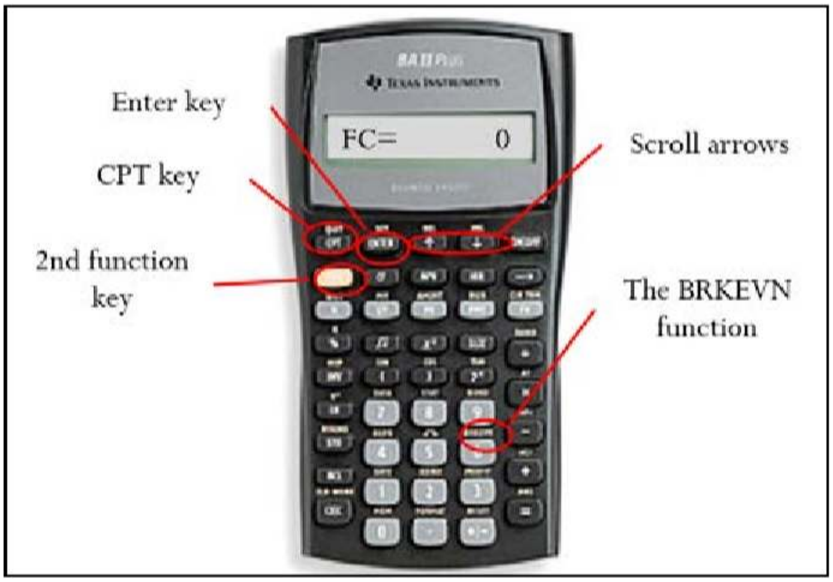
Formula 5.6 Contribution Rate if Aggregate Information Is Known: $CR=\dfrac{TR-TVC}{TR}\times 100$

Formula 5.7 Unit Break-Even: $n=\dfrac{TFC}{CM}$

Formula 5.8 Dollar Break-Even: $TR=\dfrac{TFC}{CR}$

Technology

Calculator


Enter key
CPT key
2nd function key
Scroll arrows
The BRKEVN function
BA II Plus
Texas Instruments
FC= 0

Net Income or Break-Even

- 2nd Brkevn to open the function
- To clear the memory before entering any data, press 2nd CLR Work after opening the function. Use and to scroll through the function.
- Enter four of the following variables and press Enter after each to save:

  FC = Total fixed costs

  VC = Unit variable cost

  P = Selling price per unit

  PFT = Net income

  Q= Level of output

- Press CPT on the unknown variable (when it is on the screen display) to compute.

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5.CS: Case Study: Forecasting the Impact of Revenue and Cost Changes

The Situation

Lightning Wholesale has just wrapped up the 2013 fiscal year of operations. As with all businesses, revenues, costs, and expenses are forecasted to change in 2014. Top management needs to know the impact of these changes on its 2014 operations. This is best completed by performing a year-over-year analysis of 2013 to 2014.

The Data

This table shows the financial figures for three of the company's operating divisions.

	Three Operating Divisions		
	Sporting Goods	Outdoor Clothing	Indoor Clothing
Sales Revenue	\$11,555	\$10,111	\$5,200
Cost of Goods Sold	\$9,585	\$6,154	\$3,531
Operating Expenses			
Sales & Marketing	\$582	\$485	\$356
General & Administrative	\$49	\$35	\$42
Payroll	\$269	\$326	\$184

*All figures are in thousands of dollars.

Important Information

- All variable costs are accounted for in the cost of goods sold.
- All operating expenses are treated as fixed costs.
- The table below shows the percent changes forecast for 2014.

	Three Operating Divisions		
	Sporting Goods	Outdoor Clothing	Indoor Clothing
Sales Revenue	+8%	+4%	−2.3%
Cost of Goods Sold	+4%	+6%	+2.1%
Operating Expenses			
Sales & Marketing	+6%	0%	+3%
General & Administrative	−2%	+1.5%	+2%
Payroll	+2.1%	+3.5%	+2.5%

Your Tasks

1. Top management needs to understand the current 2013 year of operations. Perform the following activities:

   a. For each division, calculate the net income, total contribution margin, and contribution rate.

   b. For each division, calculate the break-even point in dollars.

   c. For all three divisions combined, calculate the total revenue dollars required to break even.

2. Forecast the changes for the 2014 operating year to help management understand how the business is changing.

   a. For each division, calculate the forecasted net income, total contribution margin, and contribution rate. Express these figures as a percentage of the 2013 numbers.

b. For each division, forecast the new break-even point in dollars. Express the forecasted break-even points as a percentage of the 2013 break-even points.

c. For all three divisions combined, calculate the total forecasted revenue dollars required to break even and express the total break-even point as a percentage of the 2013 total break-even point.

3. Write a report to management about your findings for each department and the company as a whole for the 2014 year.

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5.E: Marketing and Accounting Fundamentals (Exercises)

5.1 Exercises

In each of the following questions, round all of the money and percentages to two decimals.

Mechanics

1. Classify each of the following costs as fixed costs, variable costs, or blended costs. If a cost is blended, separate it into its fixed and variable components.
    a. Natural gas bill for $15 per month plus $\$0.33/m^2$.
    b. A chief executive officer salary of $240,000 per year.
    c. An author earning a royalty of 5% of sales.
    d. Placing a commercial on television for $300,000.
    e. A cell phone bill for $40 per month plus $0.25/minute for long distance.
    f. Hourly production worker wages of $18/hr.
    g. Sales staff who are compensated at a salary of $1,000 per month plus 15% of sales.

For questions 2–7, solve for the unknown variables (identified with a ?) based on the information provided.

	Total Fixed Costs	Total Variable Costs	Unit Variable Cost	Selling Price	Total Revenue	Level of Output	Net Income	Contribution Rate	Unit Contribution Margin
2.	$5,000	$6,600	?	$13	?	?	$4,000	?	$7.50
3.	$2,000	?	$5	$10	$10,000	?	?	?	?
4.	?	?	?	$75	$60,000 ?	?	$14,500	35%	?
5.	$18,000	$45,000	?	?	$84,600	1,800	?	?	?
6.	?	?	?	?	$78,000	3,000	$1,8000	?	$13
7.	?	$94,050	$75.24	?	?	?	−$19,500	38%	?

Applications

8. In the current period, Blue Mountain Packers in Salmon Arm, British Columbia, had fixed costs of $228,000 and a total cost of $900,000 while maintaining a level of output of 6,720 units. Next period sales are projected to rise by 20%. What total cost should Blue Mountain Packers project?
9. Fred runs a designer candle-making business out of his basement. He sells the candles for $15 each, and every candle costs him $6 to manufacture. If his fixed costs are $2,300 per month, what is his projected net income or loss next month, for which he forecasts sales of 225 units?
10. A college print shop leases an industrial Xerox photo copier for $1,500 per month plus 1.5¢ for every page. Additional printing costs are estimated at 2¢ per page, which covers toner, paper, labour, and all other incurred costs. If copies are made for students at 10¢ each, determine the following:
    a. How does net income change with every 100 copies sold?
    b. What is the monthly net income if, on average, the shop makes 25,000 copies for students each month?
11. Gayle is thinking of starting her own business. Total fixed costs are $19,000 per month and unit variable costs are estimated at $37.50. From some preliminary studies that she completed, she forecasts sales of 1,400 units at $50 each, 1,850 units at $48 each, 2,500 units at $46 each, and 2,750 units at $44 each. What price would you recommend Gayle set for her products?
12. Last year, A Child's Place franchise had total sales of $743,000. If its total fixed costs were $322,000 and net income was $81,000, what was its contribution rate?
13. What level of output would generate a net income of $15,000 if a product sells for $24.99, has unit variable costs of $9.99, and total fixed costs of $55,005?
14. In the current year, a small Holiday Inn franchise had sales of $1,800,000, fixed costs of $550,000, and total variable costs of $750,000. Next year, sales are forecast to increase by 25% but costs will remain the same. How much will net income change

(in dollars)?

Challenge, Critical Thinking, & Other Applications

15. Monsanto Canada reported the following on its income statement for one of its divisions:

Sales	$6,000,000
Total Fixed Costs	$2,000,000
Total Variable Costs	$3,200,000
Total Costs	55,200,000
Net Income	$800,000

Calculate the total contribution margin in dollars and the contribution rate for this division.

16. Procter and Gamble is budgeting for next year. For one of its brands, P&G projects it will operate at 80% production capacity next year and forecasts the following:

Sales	$80,000,000
Total Fixed Costs	$20,000,000
Total Variable Costs	$50,000,000
Total Costs	570,000,000
Net Income	$10,000,000

Determine the net income if sales are higher than expected and P&G realizes 90% production capacity.

17. Through market research, a marketing manager determines that consumers are willing to pay 5% more for the company's product. However, some of their customers would not like this price increase, so the level of output would drop by 5%. Should the marketing manager leave things as they are or increase the price by 5%? Justify your solution.
18. Francesca is a departmental manager in the women's wear department for The Bay. She believes that if she places her line of $100 dresses on sale at 20% off, she would see her sales rise by 75%. The contribution rate on her dresses is 50%. On a strictly financial basis, should she place the dresses on sale?

Use the following information for questions 19 and 20:

$$\begin{array}{ll} TFC = \$3,200,000.00 & S = \$99.97 \\ TVC = \$5,009,440.00 & n = 131,000 \end{array}$$

19. Calculate the following information: unit variable cost, total revenue, net income, unit contribution margin, total contribution margin, and contribution rate.
20. Determine a new value for net income if the following situations occur:
    a. Fixed costs rise by 10%.
    b. The selling price is lowered by 25% during a sale, resulting in 50% more volume.
    c. Fixed costs are lowered by 5%, total variable costs rise by 3%, the price is lowered by 5%, and the level of output rises 10%.

5.2 Exercises

In each of the following questions, round all of the money and percentages to two decimals unless otherwise specified.

Mechanics

1. Franklin has started an ink-jet print cartridge refill business. He has invested $2,500 in equipment and machinery. The cost of refilling a cartridge including labour, ink, and all other materials is $4. He charges $14.95 for his services. How many cartridges does he need to refill to break even?
2. Hasbro manufactures a line of children's pet toys. If it sells the toy to distributors for $2.30 each while variable costs are 75¢ per toy, how many toys does it need to sell to recover the fixed cost investment in these toys of $510,000? What total revenue would this represent?
3. You are thinking of starting your own business and want to get some measure of feasibility. You have determined that your total fixed costs would be $79,300. From annual business reports and competitive studies, you estimate your contribution rate to be 65%. What is your break-even in dollars?
4. If a business has total revenue of $100,000, total variable costs of $60,000, and total fixed costs of $20,000, determine its break-even point in dollars.

5. If the break-even point is 15,000 units, the selling price is $95, and the unit variable cost is $75, what are the company's total fixed costs?
6. Louisa runs a secretarial business part time in the evenings. She takes dictation or handwritten minutes and converts them into printed word-processed documents. She charges $5 per page for her services. Including labour, paper, toner, and all other supplies, her unit variable cost is $2.50 per page. She invested $3,000 worth of software and equipment to start her business. How many pages will she need to output to break even?
7. If your organization has a contribution rate of 45% and knows the break-even point is $202,500, what are your organization's total fixed costs?
8. What is the unit contribution margin on a product line that has fixed costs of $1,800,000 with a break-even point of 360,000 units?

Applications

9. Ashley rebuilds old laptops as a home hobby business. Her variable costs are $125 per laptop and she sells them for $200. She has determined that her break-even point is 50 units per month. Determine her net income for a month in which she sells 60 units.
10. Burton Snowboards reported the following figures last week for its Custom V Rocker Snowboard:

Sales	$70,000
Total Fixed Costs	$19,000
Total Variable costs	$35,000
Total costs	554,000
Net Income	$16,000

If the above numbers represent 70% operational capacity, express the weekly break-even point in dollars as a percentage of maximum capacity.

11. Shardae is starting a deluxe candy apple business. The cost of producing one candy apple is $4.50. She has total fixed costs of $5,000. She is thinking of selling her deluxe apples for $9.95 each.
    a. Determine her unit break-even point at her selling price of $9.95.
    b. Shardae thinks her price might be set too high and lowers her price to $8.95. Determine her new break-even point.
    c. An advertising agency approaches Shardae and says people would be willing to pay the $9.95 if she ran some "upscale" local ads. They would charge her $1,000. Determine her break-even point.
    d. If she wanted to maintain the same break-even units as determined in (a), what would the price have to be to pay for the advertising?
12. Robert is planning a wedding social for one of his close friends. Costs involve $865 for the hall rental, $135 for a liquor licence, $500 for the band, and refreshments and food from the caterer cost $10 per person. If he needs to raise $3,000 to help his friend with the costs of his wedding, what price should he charge per ticket if he thinks he can fill the social hall to its capacity of 300 people?
13. Boston Beer Company, the brewer of Samuel Adams, reported the following financial information to its shareholders:

Total Revenue	$388,600,000
Total Variable costs	$203,080,000
Total Fixed costs	$182,372,000
Total Costs	5385,452,000
Net Income	$3,148,000

If this represented sales of 2,341,000 barrels of beer, determine its break-even point in units and dollars.

14. In the beverage industry, PepsiCo and The Coca-Cola Company are the two big players. The following financial information, in millions of dollars, was reported to its shareholders:

	PepsiCo	The Coca-Cola Company
Total Revenue	$14.296	$21.807
Total Variable Costs	$7.683	$12.663
Total Fixed Costs	$6.218	$8.838

	PepsiCo	The Coca-Cola Company
Total Costs	$13.901	$21.501
Net Income	$0.395	$0.306

Compare the break-even points in total dollars between the two companies based on these reports.

Challenge, Critical Thinking, & Other Applications

15. Calculate the following:
    a. By what percentage does the unrounded unit break-even point change if the unit contribution margin increases by 1% while all other numbers remain the same?
    b. By what percentage does the unrounded unit break-even point change if total fixed costs are reduced by 1% while all other numbers remain the same?
    c. What do the solutions to the above questions illustrate?
16. École Van Belleghem is trying to raise funds to replace its old playground equipment with a modern, child-safe structure. The Blue Imp playground equipment company has quoted the school a cost of $49,833 for its 20 m × 15 m megastructure. To raise the funds, the school wants to sell Show 'n' Save books. These books retail for $15.00 each and cost $8.50 to purchase. How many books must the school sell to raise funds for the new playground?
17. The Puzzle Company had total revenue of $4,750,000, total fixed costs of $1,500,000, and total variable costs of $2,750,000. If the company desires to earn a net income of $1,000,000, what total sales volume is needed to achieve the goal?
18. Whirlpool Corporation had annual sales of $18.907 billion with a net income of $0.549 billion. Total variable costs amounted to $16.383 billion.
    a. Determine Whirlpool Corporation's break-even in dollars.
    b. If Whirlpool Corporation managed to increase revenues by 5% the following year while implementing cost-cutting measures that trimmed variable costs by 2%, determine the percent change in the break-even dollars.

Use the following information for questions 19 and 20:

$$S=\$100 \quad VC=\$60 \quad TFC=\$250,000$$

19. Calculate the current break-even point in both units and dollars.
20. A production manager is trying to control costs but is faced with the following tradeoffs under three different situations:
    a. Total fixed costs are reduced by 15%, but unit variable costs will rise by 5%.
    b. Unit variable costs are reduced by 10%, but fixed costs will rise by 5%
    c. Total fixed costs are reduced by 20%, but unit variable costs will rise by 10%.
    d. Unit variable costs are reduced by 15%, but fixed costs will rise by 15%.

    Based strictly on break-even calculations, which course of action would you recommend she pursue?

Review Exercises

In each of the following questions, round all of the money and percentages to two decimals unless otherwise specified.

Mechanics

1. Logitech manufactures a surround sound multimedia speaker system for personal computers. The average selling price is $120 per unit, with unit variable costs totaling $64. Fixed costs associated with this product total $980,000. Calculate the net income or loss if 25,000 units are sold.
2. Microsoft sells its wireless laser desktop mouse and keyboard for $70. Unit variable costs are $45.60 and fixed costs associated with this product total $277,200.
    a. What is the break-even point in units?
    b. What is the net income or loss if 40,000 units are sold?
3. Frances has a home-based Avon business. She purchases makeup from Avon for an average price of $12 and sells it to her customers on average for $20. She assigns $100 of her monthly computer leasing costs to this product line and spends $260 per month advertising this product line. In hosting Avon parties, she estimates her labour cost at $2 per unit sold.

a. What is her net income or loss if 100 units are sold?
b. What is her net income or loss if 35 units are sold?
c. What is the break-even point in both units and dollars?
d. What is the new unit break-even point if Frances increases her advertising costs by 50% per month?

Applications

4. Brynne is developing a business plan. From market research she has learned that her customers are willing to pay $49.95 for her product. Unit variable costs are estimated at $23.75, and her monthly total fixed costs are $12,000. To earn a net income of at least $5,000, what minimum level of monthly sales (in units) does she need?
5. Advanced Microcomputer Systems had annual sales of $31,979,000 with total variable costs of $20,934,000 and total fixed costs of $6,211,000.
    a. What is the company's contribution rate?
    b. What annual revenue is required to break even?
    c. In the following year, a competitor dealt a severe blow to sales and total revenues dropped to $15,000,000. Calculate the net income.
6. In the North American automotive markets, Toyota and Honda are two of the big players. The following financial information (all numbers in millions of dollars) for each was reported to its shareholders:

	Toyota	Honda
Total Revenue	$247,734	$101,916
Total Variable Costs	$204,135	$75,532
Total Fixed Costs	$20,938	$24,453
Total Costs	$225,073	$99,985
Net Income	$22,661	$1,931

Compare the break-even points in total dollars between the two companies based on these reports.

7. Lagimodiere Industry’s unit variable costs are 45% of the unit selling price. Annual fixed costs are $1.5 million.
    a. What total revenue results in an income of $250,000?
    b. Calculate the break-even dollars for Lagimodiere Industry.
    c. If revenues exceed the break-even dollars by 20%, determine the net income.

Challenge, Critical Thinking, & Other Applications

8. Svetlana and Yoric are each thinking of starting the same type of business. Svetlana is young and into technology and mechanization. She has a large fixed cost of $15,000, but her advanced production machinery minimizes her labour, resulting in a low unit variable cost of $25. Yoric is a little older and believes the best quality items are handmade. He has a low fixed cost of $8,000 but incurs more labour costs, resulting in a high unit variable cost of $60. Regardless of who makes the product, market research shows that consumers would pay $100 for the product.
    a. Calculate the unit break-even for both Svetlana and Yoric.
    b. Calculate the net income or loss for both if unit sales are 50% higher than the break-even point.
    c. Calculate the net income or loss for both if unit sales are 50% lower than the break-even point.
    d. Comment on your findings from the above solutions.
9. A surfboard manufacturer lost $500,000 last year during a recession. Total revenue was $5,000,000 and total variable costs were 40% of sales. The production facility ran at 50% capacity. The production manager wants to know the following:
    a. What is the percent capacity required to break even?
    b. When the economy recovers this year, if the plant runs at 100% capacity what net income could the company realize?
    c. There is a possibility that sales could be so strong this year that the plant may be required to run at 120% capacity by offering a lot of overtime to its production workers. This would result in total variable costs rising by 35%. On a strictly financial basis, should the production manager plan to exceed capacity or should he advise top management to freeze production at 100% capacity? Justify your answer.
10. The following annual information is known about a company:

Material inputs to manufacturing	$17,400,000	Marketing and advertising	$3,500,000

Management salaries	$2,750,000	Production worker wages	$9,990,000
Supplies needed for manufacturing	$8,345,000	Utilities used in production activities	$6,235,000
Utilities used in non-production activities	$2,222,000	Property taxes	$2,100,000
# of units sold	75,132	Unit price	$749

a. Calculate the unit variable cost, total fixed costs, unit contribution margin, contribution rate, net income, and break-even in both units and dollars.
b. The marketing research manager shows that if the company lowers its price by 5%, unit sales will rise by 5%. Should the company lower its price? Justify your decision.
c. Assume that the price remained the same. Due to economic pressures, fixed costs are forecasted to rise by 6% and variable costs by 4% next year. At the same level of output, by what percentage must the selling price rise to maintain the same level of net income achieved in (a)?

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CHAPTER OVERVIEW

6: Marketing Applications (What Is It Going to Cost Me?)

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of "deals." You can easily explain why the same product sells for two different prices at two different stores.

6.0: Introduction to Marketing Applications
6.1: Figuring Out the Cost - Discounts (How Much?)
6.2: Markup- Setting the Regular Price (Need to Stay in Business)
6.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)
6.4: Merchandising (How Does It All Come Together?)
6.S: Summary
6.CS: Case Study (Making a Profitable Product Selection)
6.E: Marketing Applications (Exercises)

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6.0: Introduction to Marketing Applications

When you buy an iPod, it is very important that the right price is set. The price should

- be seen as fair by you the buyer,
- pay for the costs (plastics, battery, buttons, circuit boards, headset) and expenses (employees, factory, electricity, distribution) of making the iPod, and
- allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales while lowering the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of "deals." You can easily explain why the same product sells for two different prices at two different stores.

In this chapter, you must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

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6.1: Figuring Out the Cost - Discounts (How Much?)

You mutter in exasperation, “Why can’t they just set one price and stick with it?” Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at $49.99. Toys R’ Us has an outlet in the parking lot where the regular price for the same toy is $59.99, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at $64.99 but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price. You go to Dairy Queen for a Blizzard to soothe your headache while you figure things out.

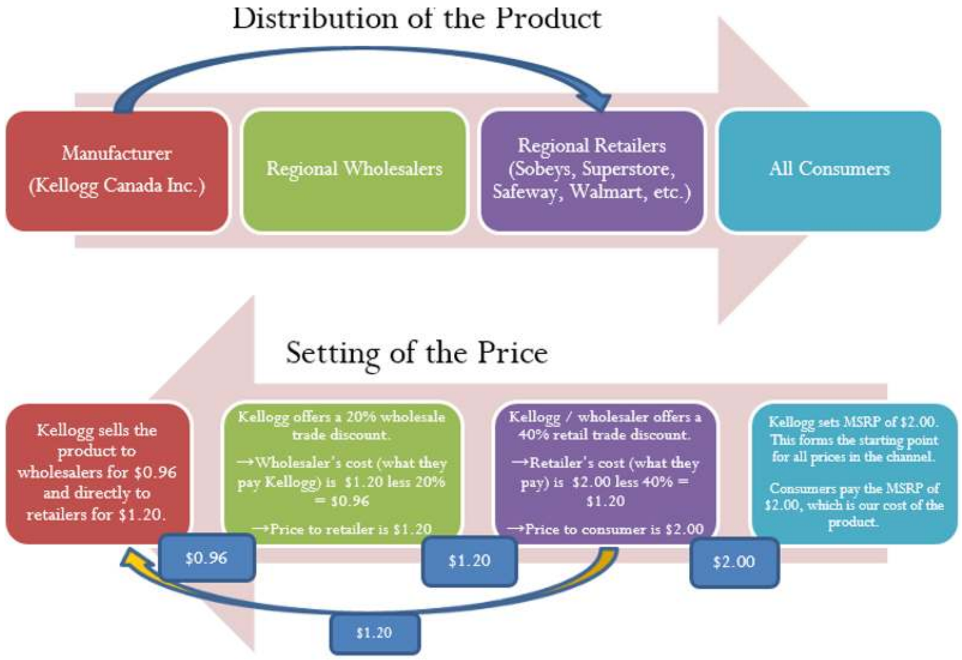
The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a *middleman* or *intermediary*), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labor, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including *sales* or *clearance*. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Since the revenue must cover all costs and expenses associated with the product, the lower price means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

How Distribution and Pricing Work

Start with distribution in the top half of the figure and work left to right. As an example, let’s look at a manufacturer such as Kellogg Canada Inc. (which makes such products as Pop-Tarts, Eggo Waffles, and Rice Krispies). Kellogg’s Canadian production plant is located in London, Ontario. To distribute its products to the rest of Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area; however, some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.


Distribution of the Product
Manufacturer (Kellogg Canada Inc.)
Regional Wholesalers
Regional Retailers (Sobeys, Superstore, Safeway, Walmart, etc.)
All Consumers
Setting of the Price
$0.96
$1.20
$2.00
$1.20

Kellogg sells the product to wholesalers for $0.96 and directly to retailers for $1.20.

Kellogg offers a 20% wholesale trade discount.
→Wholesaler's cost (what they pay Kellogg) is $1.20 less 20% = $0.96
→Price to retailer is $1.20

Kellogg / wholesaler offers a 40% retail trade discount.
→Retailer's cost (what they pay) is $2.00 less 40% = $1.20
→Price to consumer is $2.00

Kellogg sets MSRP of $2.00. This forms the starting point for all prices in the channel.
Consumers pay the MSRP of $2.00, which is our cost of the product.

The relationship of distribution to pricing is illustrated in the bottom half of the figure, working right to left. For now, focus on understanding how pricing works; the mathematics used in the figure will be explained later in this chapter. Kellogg Canada sets a **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Since grocery retailers commonly carry thousands or tens of thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a $2.00 MSRP, which is the price consumers will pay for the product.

The retailer must pay something less than $2.00 to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, it offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is $2.00 less 40%, or $1.20. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is $1.20. Again, Kellogg Canada, knowing that the wholesaler must pay something less than $1.20 to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid by the wholesaler to acquire the product from Kellogg Canada is $1.20 less 20%, or $0.96. This $0.96 forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

1. Companies higher up in the distribution channel pay lower prices than those farther down the channel. Companies receive discounts off the MSRP based on their level in the distribution system. This may result in multiple discounts, such as a wholesaler receiving both the retailer’s discount and an additional discount for being a wholesaler.
2. One organization’s price becomes the next organization’s cost (assuming the typical distribution channel structure):

$$\text{Manufacturer's Price} = \text{Wholesaler's cost}$$

$$\text{Wholesaler's Price} = \text{Retailer's Cost}$$

$$\text{Retailer's Price} = \text{Consumer's Cost}$$

Types of Discounts

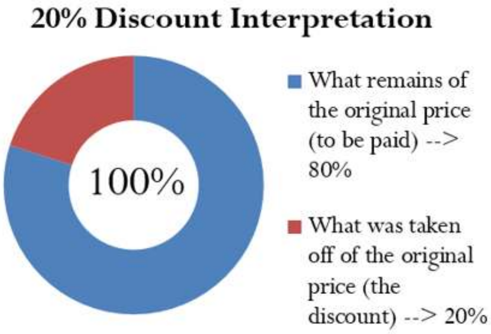
You will perform discount calculations more effectively if you understand how and why single pricing discounts and multiple pricing discounts occur. Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

1. ***Trade Discounts***. A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution system (e.g., as a retailer, wholesaler, or any other member of the distribution system that resells the product). Consumers are ineligible for trade discounts. In the discussion of the figure, two trade discounts are offered. The first is a 40% retail trade discount, and the second is a 20% wholesale trade discount. Typically, a business that is higher up in the distribution system receives a combination of these trade discounts. For example, the wholesaler receives both the 40% retail trade discount and the 20% wholesale trade discount from the MSRP. The wholesaler's cost is calculated as an MSRP of $2.00 less 40% less 20% = $0.96.
2. ***Quantity Discounts***. A **quantity discount** (also called a *volume discount*) is a discount for purchasing larger quantities of a certain product. If you have ever walked down an aisle in a Real Canadian Superstore, you probably noticed many shelf tags that indicate quantity discounts, such as “Buy one product for $2” or “Take two products for $3.” Many Shell gas stations offer a Thirst Buster program in which customers who purchase four Thirst Busters within a three-month period get the fifth one free. If the Thirst Busters are $2.00 each, this is equivalent to buying five drinks for $10.00 less a $2.00 quantity discount.
3. ***Loyalty Discounts***. A **loyalty discount** is a discount that a seller gives to a purchaser for repeat business. Usually no time frame is specified; that is, the offer is continually available. As a consumer, you see this regularly in marketing programs such as Air Miles or with credit cards that offer cash back programs. For example, Co-op gas stations in Manitoba track consumer gasoline purchases through a loyalty program and mails an annual loyalty discount cheque to its customers, recently amounting to 12.5¢ per litre purchased. In business-to-business circles, sellers typically reward loyal customers by deducting a loyalty discount percentage, commonly ranging from 1% to 5%, from the selling price.
4. ***Sale Discounts***. A **sale discount** is a temporary lowering of the price from a product’s regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers. You see such promotional events all the time: LED monitors are on sale at Best Buy; Blu-Ray discs are half off at Walmart; The Brick is having a door crasher event Saturday morning.
5. ***Seasonal Discounts***. A **seasonal discount** is a discount offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts encouraging retailers, wholesalers, or distributors

to purchase products before they are in season. Bombardier Inc. manufactures Ski-Doos, which are sold in Canada from approximately November through March—a time of year when most of the country has snow and consumers would want to buy one. To keep production running smoothly from April through October, Bombardier could offer seasonal discounts to its wholesalers and retailers for the coming winter season. At a retail level, the examples are plentiful. On November 1 most retailers place their Halloween merchandise on seasonal discount to clear out excess inventory, and many retailers use Boxing Day (or Boxing Week) to clear their out-of-season merchandise.

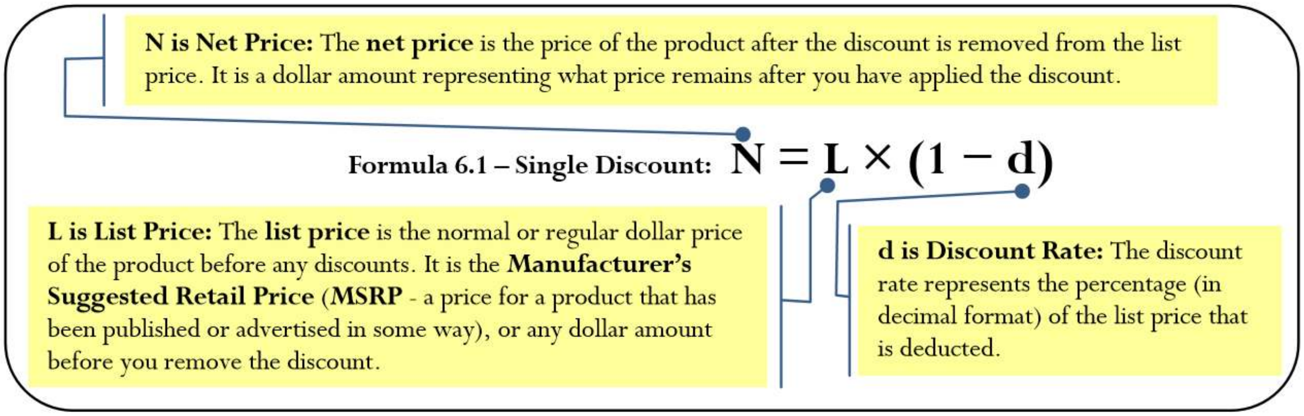
Single Discounts

Let’s start by calculating the cost when only one discount is offered. Later in this section you will learn how to calculate a cost involving multiple discounts.



The Formula

Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed. Note in Formula 6.1 below that you take 1 and subtract the discount rate to determine the rate owing. If you are eligible for a 20% discount, then you must pay 80% of the list price, as illustrated in the figure to the right.



N is Net Price: The **net price** is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

Formula 6.1 – Single Discount: $N = L \times (1 - d)$

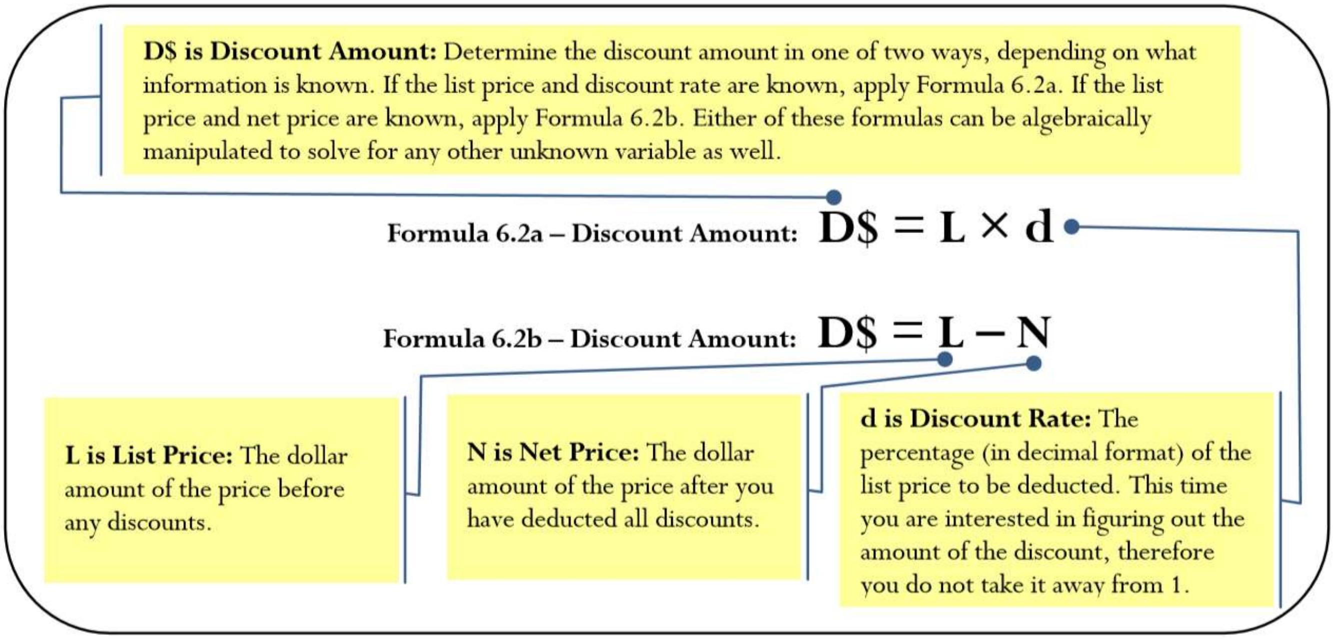
L is List Price: The **list price** is the normal or regular dollar price of the product before any discounts. It is the **Manufacturer’s Suggested Retail Price** (**MSRP** - a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

d is Discount Rate: The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Formula 6.1

Formula 6.1 once again applies Formula 2.2 on rate, portion, and base, where the list price is the base, the (1 - d) is the rate, and the net price represents the portion of the price to be paid.

Notice that Formula 6.1 requires the discount to be in a percentage (decimal) format; sometimes a discount is expressed as a dollar amount, though, such as "Save $5 today." Formulas 6.2a and 6.2b relate the discount dollar amount to the list price, discount percent, and net price. Choose one formula or the other depending on which variables are known.



D$ is Discount Amount: Determine the discount amount in one of two ways, depending on what information is known. If the list price and discount rate are known, apply Formula 6.2a. If the list price and net price are known, apply Formula 6.2b. Either of these formulas can be algebraically manipulated to solve for any other unknown variable as well.

Formula 6.2a – Discount Amount: $D\$ = L \times d$

Formula 6.2b – Discount Amount: $D\$ = L - N$

L is List Price: The dollar amount of the price before any discounts.

N is Net Price: The dollar amount of the price after you have deducted all discounts.

d is Discount Rate: The percentage (in decimal format) of the list price to be deducted. This time you are interested in figuring out the amount of the discount, therefore you do not take it away from 1.

Formula 6.2a and Formula 6.2b

How It Works

Follow these steps to calculate the net price involving a single discount. These steps are adaptable if the net price is a known variable and one of the other variables is unknown.

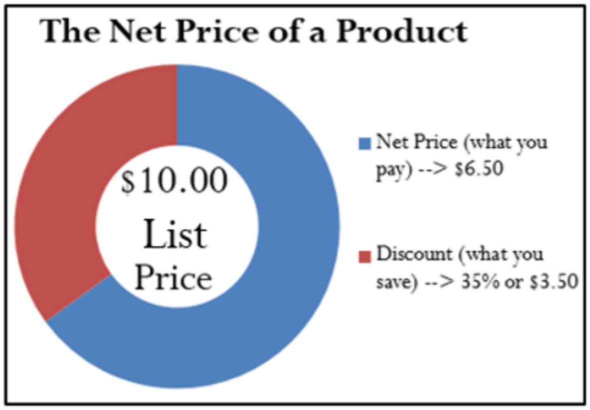
Step 1: Identify any known variables, including list price, discount rate, or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price using an appropriate formula.

Step 3: Calculate the net price.

- If the list price and discount amount are known, apply Formula 6.2b and rearrange for N.

Assume a product sells for $10 and is on sale at 35% off the regular price. Calculate the net price for the product.



Step 1: The list price of the product is L = $10. It is on sale with a discount rate of d = 0.35.

Step 2: List price is known, so this step is not needed.

Step 3: Applying Formula 6.1 results in a new price of $N = \$10 \times (1 - 0.35) = \$6.50$. Note that if you are interested in learning the discount amount, you apply Formula 6.2b to calculate $D\$ = \$10 - \$6.50 = \$3.50$.

Important Notes

You can combine Formula 6.1 and either version of Formula 6.2 in a variety of ways to solve any single discount situation for any of the three variables. As you deal with increasingly complicated pricing formulas, your algebraic skills in solving linear equations and substitution become very important.

Many of the pricing problems take multiple steps that combine various formulas, so you need to apply the PUPP model systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

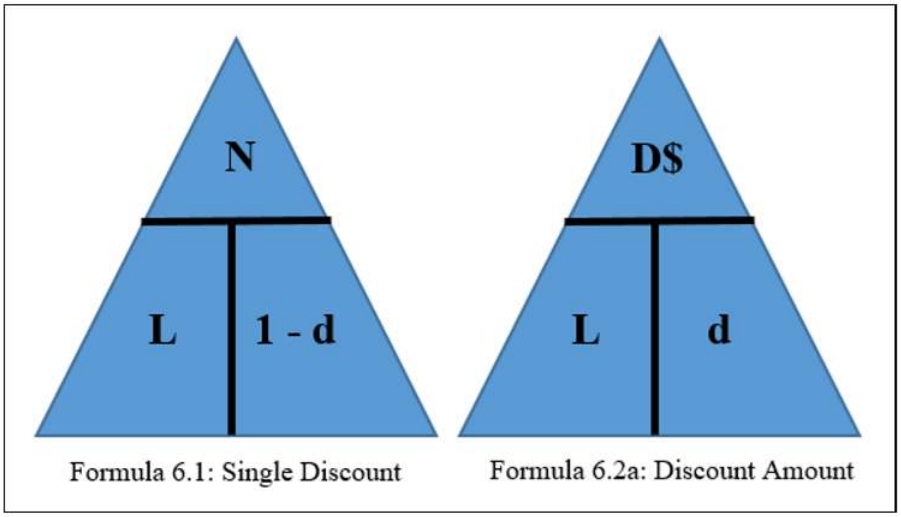
If you find you cannot produce a formula with only one unknown variable, can you find two formulas with the same two unknowns? If so, recall from Section 2.5 that you can use your algebraic skills to find the roots of the two equations simultaneously. Alternatively, you can solve one formula for a variable then substitute it into the other formula, allowing you to isolate the remaining variable. Throughout the examples in this chapter you will see many applications of these algebraic skills.

Things To Watch Out For

Remember to apply the rounding rules discussed in Chapter 2:

1. Until you arrive at the final solution, avoid rounding any interim numbers unless you have some special reason to do so.
2. Round all dollar amounts to the nearest cent. If the dollar amount has no cents, you may write it either without the cents or with the ".00" at the end.
3. Round all percentages to four decimals when in percent format.

Paths To Success



When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. The triangle technique discussed in Section 2.3 can remind you how to rearrange the formulas for each variable, as illustrated in the figure to the right.

? Exercise 6.1.1: Give It Some Thought

1. Will you pay more than, less than, or exactly $10.00 for a product if you are told that you are paying:
   a. a net price of $10.00 when there is a discount of 25%?
   b. a list price of $10.00 when there is a discount of 25%?
2. If an item is subject to a 40% discount, will the net price be more than or less than half of the list price of the product?

Answer

1. a. Exactly $10. The net price is the price after the discount.

b. Less than $10. The discount needs to be removed from the list price.

2. More than half. A 40% discount means that you will pay 60% of the list price.

✓ Example $6.1.1$: Determining the Retailer's Net Price for a Pair of Jeans

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at the MSRP of $59.99. What price should retailers pay for the jeans?

Solution

Calculate how much a retailer should pay for the jeans after the regular price has been discounted to accommodate the trade discount. This is called the net price for the product, or N.

What You Already Know

Step 1: The list price and the discount rate are known:

L = $59.99

d = 0.25

How You Will Get There

Step 2:

List price is known, so skip this step.

Step 3:

Apply Formula 6.1.

Perform

Step 3:

$N = \$59.99 \times (1 - 0.25) = \$59.99 \times 0.75 = \$44.99$

The manufacturer should sell the jeans to the retailers for $44.99.

✓ Example $6.1.2$: Determining the List Price of a Jacket

Winners pays a net price of $27.50 for a winter jacket after receiving a retail trade discount of 45%. What was the MSRP of the jacket?

Solution

Calculate the MSRP for the jacket before Winners received the retail trade discount to arrive at the net price. This is called the list price for the product, or L.

What You Already Know

Step 1:

The net price and the discount rate are known:

N = $27.50

d = 0.45

How You Will Get There

Step 2:

List price is the unknown variable; skip this step.

Step 3:

Apply Formula 6.1, rearranging for L.

Perform

Step 3:

$$\begin{aligned}\$27.50 &= L(1-0.45)\\ L &= \$27.50 \div (1-0.45)\\ L &= \$27.50 \div 0.55\\ &= \$50.00\end{aligned}$$

The MSRP, or list price, of the winter jacket is $50.00.

✓ Example $6.1.3$: Determining the Discount Percent and Discount Amount

You are shopping at Mountain Equipment Co-op for a new environmentally friendly water bottle. The price tag reads $14.75, which is $10.24 off the regular price. Determine the discount rate applied.

Solution

You need to find out how the sale price translates into the discount rate, or d.

What You Already Know

Step 1:

The discount amount and net price are known:

$D\$ = \$10.24 \quad N = \$14.75$

Perform

Step 2:

List price: $\$10.24 = L - \$14.75$, which results in $L = \$24.99$

Step 3:

Discount rate: $d = D\$ \div L$

$d = \dfrac{\$10.24}{\$24.99} = 0.409764 \text{ or } 40.9764\%$

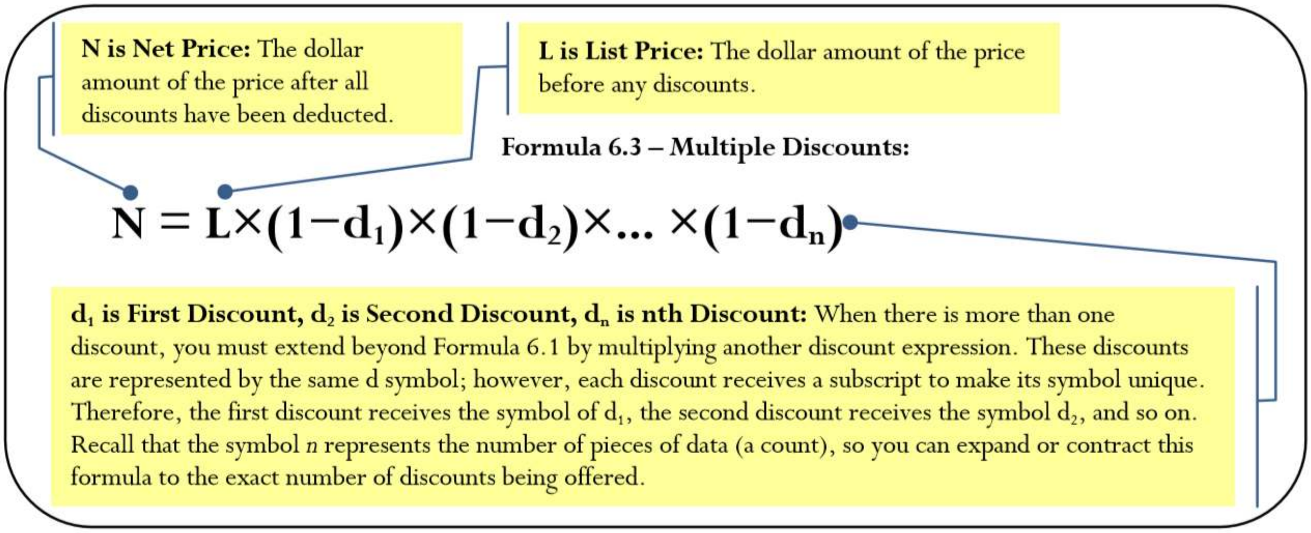
The water bottle today has been reduced in price by the amount of $10.24. This represents a sale discount of 40.9674%.

Multiple Discounts

You are driving down the street when you see a large sign at Old Navy that says, “Big sale, take an additional 25% off already reduced prices!” In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount). Because Formula 6.1 handles only a single discount, you must use an extended formula in this case.

The Formula

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is not a 30% + 10% = 40% discount. The second discount is always applied to the net price after the first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases. Formula 6.3 expresses how to calculate the net price when multiple discounts apply.



N is Net Price: The dollar amount of the price after all discounts have been deducted.

L is List Price: The dollar amount of the price before any discounts.

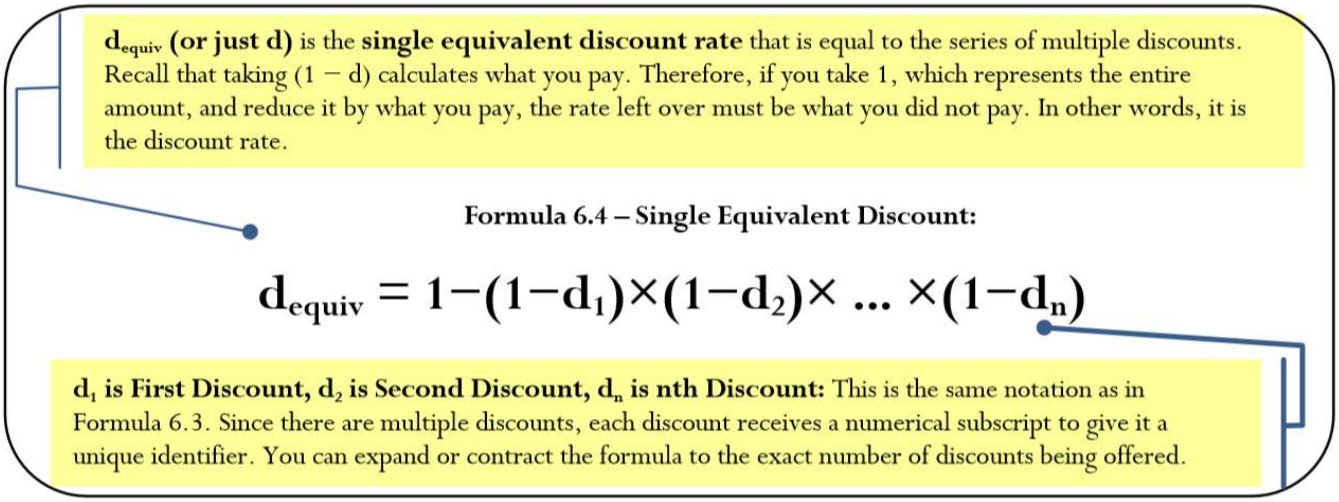
Formula 6.3 – Multiple Discounts:

$$N = L\times(1-d_1)\times(1-d_2)\times\ldots\times(1-d_n)$$

d_1 is First Discount, d_2 is Second Discount, d_n is nth Discount: When there is more than one discount, you must extend beyond Formula 6.1 by multiplying another discount expression. These discounts are represented by the same d symbol; however, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of d_1, the second discount receives the symbol d_2, and so on. Recall that the symbol *n* represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

Formula 6.3

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price. The conversion of multiple discount percentages into a single equivalent discount percent is illustrated in Formula 6.4.



d_{equiv} (or just d) is the **single equivalent discount rate** that is equal to the series of multiple discounts. Recall that taking $(1 - d)$ calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the discount rate.

Formula 6.4 – Single Equivalent Discount:

$$d_{equiv} = 1-(1-d_1)\times(1-d_2)\times\ldots\times(1-d_n)$$

d_1 is First Discount, d_2 is Second Discount, d_n is nth Discount: This is the same notation as in Formula 6.3. Since there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being offered.

Formula 6.4

How It Works

Refer back to the steps in calculating net price. The procedure for calculating a net price involving a single discount extends to a more generic procedure involving multiple discounts. As with the single discount procedures, you can adapt the model if the net price is known and one of the other variables is unknown. Follow these steps to calculate the net price involving any number of discounts:

Step 1: Identify any known variables, including list price, discount rate(s), or discount amount.

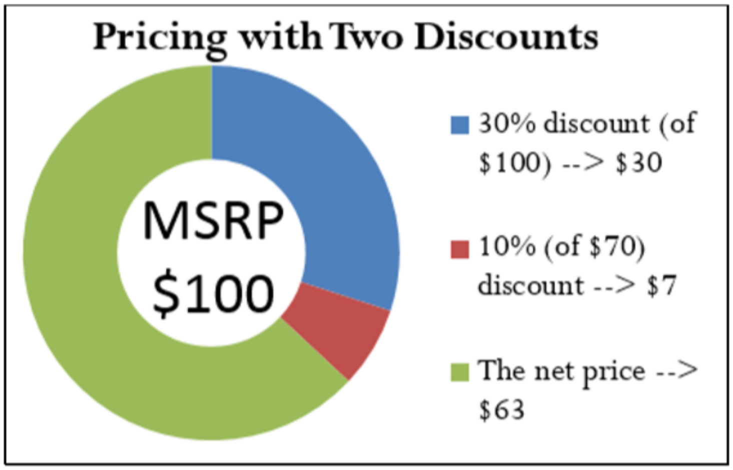
Step 2: If the list price is known, skip this step. Otherwise, solve for list price.

- If more than one discount is involved, the discount amount represents the total discount amount received from all of the discounts combined. This requires you first to convert the multiple discount rates into an equivalent single discount rate using Formula 6.4 and then to apply Formula 6.2a.

Step 3: Calculate the net price.

- If the list price and multiple discount rates are known and involved, apply Formula 6.3.
- If the list price and the total discount amount are known, apply Formula 6.2b and rearrange for N.

Assume a product with an MSRP of $100 receives a trade discount of 30% and a volume discount of 10%. Calculate the net price.



Step 1: The list price and discounts are $L=\$100$, $d_1=0.30$ and $d_2=0.10$.

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 6.3 to calculate the net price:

$$N=\$100\times(1-0.30)\times(1-0.10)=\$63$$

The net price is $63, which is illustrated to the right.

If you are solely interested in converting multiple discounts into a single equivalent discount, you need only substitute into Formula 6.4. In the above example, the product received a trade discount of 30% and a volume discount of 10%. To calculate the single equivalent discount, apply Formula 6.4:

$$\begin{aligned}d_{equiv}&=1-(1-0.30)\times(1-0.10)\\d_{equiv}&=1-(0.70)(0.90)\\d_{equiv}&=1-0.63=0.37\end{aligned}$$

Therefore, whenever discounts of 30% and 10% are offered together, the single equivalent discount is 37%. Whether it is the multiple discounts or just the single equivalent discount that you apply to the list price, the net price calculated is always the same.

Important Notes

Order of Discounts

The order of the discounts does not matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. Therefore, in the above example you could have arrived at the $63 net price through the following calculation:

$$\$100\times(1-0.10)\times(1-0.30)=\$63$$

The order of the discounts does matter if trying to interpret the value of any single discount. If the trade discount is applied before the quantity discount and you are wanting to know the quantity discount amount, then the quantity discount needs to be second. Thus,

$$\$100\times(1-0.30)=\$70$$

$$\$70\times 0.10=\$7$$

which is the amount of the quantity discount.

Price Does Not Affect Single Equivalent Discount

Notice in Formula 6.4 that the list price and the net price are not involved in the calculation of the single equivalent discount. When working with percentages, whether you have a net price of \$6.30 and a list price of \$10, or a net price of \$63 and a list price of \$100, the equivalent percentage always remains constant at 37%.

Things To Watch Out For

A common mistake when working with multiple discounts is to add the discounts together to calculate the single equivalent discount. This mistaken single discount is then substituted into Formula 6.1 to arrive at the wrong net price. Remember that if two discounts of 30% and 10% apply, you cannot sum these discounts. The second discount of 10% is applied on a smaller price tag, not the original price tag. To calculate the net price you must apply Formula 6.3.

Paths To Success

If you happen to know any two of the net price (N), list price (L), or the total discount amount ($D\$$), then you could also use Formula 6.2 to solve for the single equivalent discount, dequiv. For example, if you know the net price is \$63 and the total discount amount for all discounts is \$37, you could use Formula 6.2b to figure out that the list price is \$100, then convert the discount amount into a percentage using Formula 6.2a. This method will also produce a single equivalent discount of 37%.

Another method of calculating the single equivalent discount is to recognize Formula 6.2a as an application of Formula 3.1 involving percent change. The variable d is a discount rate, which you interpret as a negative percent change. The discount amount, $D\$$, is the difference between the list price (representing the Old price) and the net price (representing the New price after the discount). Therefore, Formula 6.2a can be rewritten as follows:

$$D\$ = L \times d$$

$$\text{becomes}$$

$$\text{New } - \text{ Old } = \text{ Old } \times \Delta\% \text{ or}$$

$$\frac{\text{New } - \text{ Old}}{\text{Old}} = \Delta\%$$

Therefore, any question about a single equivalent discount where net price and list price are known can be solved as a percent change. Using our ongoing net price example, you have:

$$\frac{\$63 - \$100}{\$100} = -0.37 \text{ or } -37\%; \text{ this is a discount of } 37$$

? Exercise $6.1.2$: Give It Some Thought

If you are offered discounts in the amount of 25%, 15%, 10%, and 5%, will your total discount percent be 55%, less than 55%, or more than 55%?

Answer

Less than 55%. Each percent discount is calculated from a smaller base.

✓ Example $6.1.4$: Retailer Purchasing Ski-Doos with Multiple Discounts

A retail dealership purchases some Expedition TUV Yeti II Ski-Doos to stock in its stores. Examining the merchandising terms of the manufacturer, Bombardier, the dealership notices that it would be eligible to receive a 35% trade discount, 15% volume discount, and 3% loyalty discount. Because it is June and Ski-Doos are out of season, Bombardier offers a seasonal discount of 12% for purchases made before June 30. If the MSRP for the Ski-Doo is \$12,399.00 and the dealership purchases this item on June 15, what price would it pay?

Solution

You are looking for the net price that the retail dealership will pay for the Ski-Doo, or N.

What You Already Know

Step 1:

The retail dealership is eligible for all four discounts (it qualifies for the seasonal discount since it is purchasing before June 30). Therefore,

$L=\$12,399.00, d_1=0.35, d_2=0.15, d_3=0.03, d_4=0.12$

How You Will Get There

Step 2:

You know the list price, so skip this step.

Step 3:

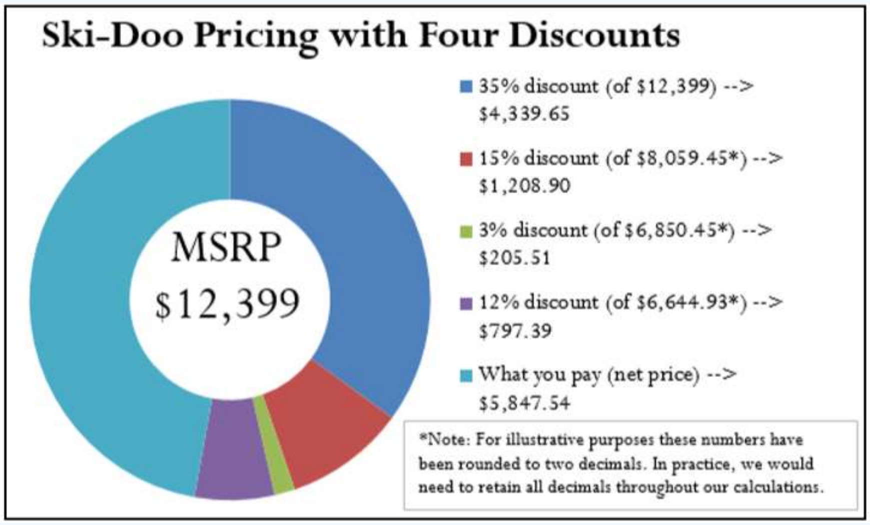
Apply Formula 6.3.

Perform

Step 3:

$$N=\$12,399\times(1-0.35)\times(1-0.15)\times(1-0.03)\times(1-0.12)$$
$$N=\$12,399\times 0.65\times 0.85\times 0.97\times 0.88=\$5,847.54$$

After all four discounts, the retail dealership could purchase the Ski-Doo for $5,847.54.



Ski-Doo Pricing with Four Discounts

*Note: For illustrative purposes these numbers have been rounded to two decimals. In practice, we would need to retain all decimals throughout our calculations.

✓ Example 6.1.5: Reducing Multiple Discounts to a Single Equivalent Discount

The retail dealership in Example 6.1.4 purchases more products subject to the same discounts. It needs to simplify its calculations. Using the information from Example 6.1.4, what single equivalent discount is equal to the four specified discounts?

Solution

You are looking for a single equivalent discount that is equal to the four discount percentages, or d_{equiv} (or just d).

What You Already Know

You know the discount rates:

$d_1=0.35, d_2=0.15, d_3=0.03, \text{ and } d_4=0.12$

How You Will Get There

Apply Formula 6.4.

Perform

$$d_{equiv} = 1 - (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$$
$$d_{equiv} = 1 - (0.65)(0.85)(0.97)(0.88) = 1 - 0.471614 = 0.528386 \text{ or } 52.8386\%$$

The retail dealership can apply a 52.8386% discount to all the products it purchases.

✓ Example 6.1.6: Making a Smart Consumer Purchase

You are shopping on Boxing Day for an 80" HDTV. You have just one credit card in your wallet, a cashback Visa card, which allows for a 1% cash rebate on all purchases. While scanning flyers for the best deal, you notice that Visions is selling the TV for $5,599.99 including taxes, while Best Buy is selling it for $5,571.99 including taxes. However, because of a computer glitch Best Buy is unable to accept Visa today. Where should you buy your television?

Solution

You want to know which store you should buy the television at. You must calculate the net price (N) for each of the stores.

What You Already Know

Step 1:

You know the list price for each of the stores. You also know the discount available from Visa. Thus,

$L_{BestBuy} = \$5,571.99$ with no discounts since Visa cannot be used there

$L_{Visions} = \$5,599.99, d = 0.01$, since you can use your Visa card there

How You Will Get There

Step 2:

List price is known, so skip this step.

Step 3:

Apply Formula 6.1.

Perform

Step 3:

Best Buy: No discounts apply, so the list price equals the net price and $N = \$5,571.99$.

Visions: $N = \$5,599.99 \times (1 - 0.01) = \$5,599.99 \times 0.99 = \$5,543.99$

The net price for Visions is $\$5,543.99$. You save $\$5,571.99 - \$5,543.99 = \$28.00$ by purchasing your TV at Visions.

✓ Example 6.1.7: Understanding the Price

An advertisement claims that at 60% off, you are saving $18. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

Solution

You are looking for how much you should pay after the discounts (N), and the single equivalent percentage that represents the two discounts (d_{equiv} or just d).

What You Already Know

Step 1:

You know the discount amount for the first discount only, as well as the two discount rates:

$D\$_1 = \$18, d_1 = 0.60, \text{ and } d_2 = 0.20.$

How You Will Get There

Step 2:

Calculate the list price by applying Formula 6.2a and rearranging for L.

Step 3:

To calculate the net price, apply Formula 6.3.

Step 4:

To calculate the single equivalent discount, apply Formula 6.4.

Perform

Step 2:

$$\begin{aligned} \$18.00 &= L \times 0.60 \\ L &= \$18.00 \div 0.6 \\ &= \$30.00 \end{aligned}$$

Step 3:

$N = \$30 \times (1-0.60) \times (1-0.20) = \$30 \times 0.40 \times 0.80 = \$9.60$

Step 4:

$d_{equiv} = 1-(1-0.60) \times (1-0.20) = 1-(0.40)(0.80) = 1-0.32 = 0.68 \text{ or } 68\%$

You should pay $9.60 for the item, which represents a 68% savings.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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6.2: Markup- Setting the Regular Price (Need to Stay in Business)

As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that $4.99 seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself doesn't cost anywhere near that much. But then gazing around the café, you notice the carefully applied colour scheme, the comfortable seating, the high-end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably. Where did the money to pay for all of this come from? You smile as you realize your $4.99 pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup**. This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

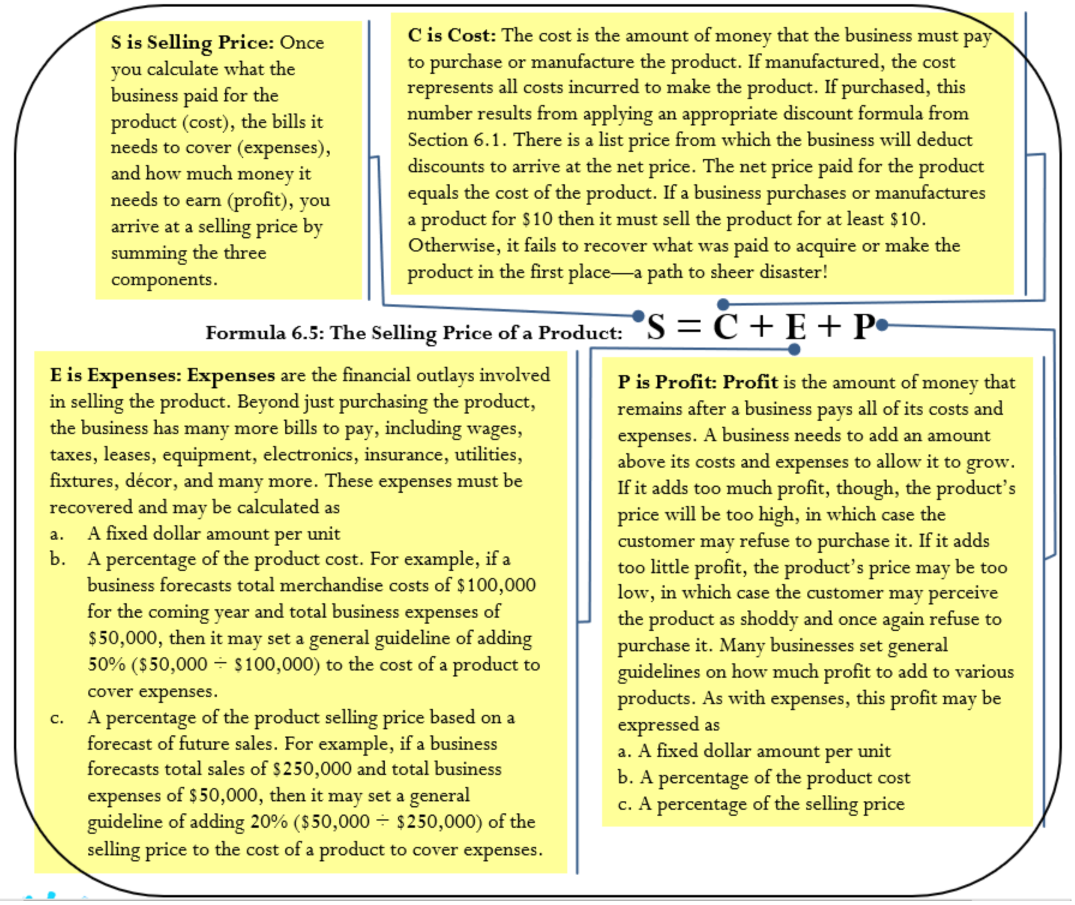
The Components in a Selling Price

Before you learn to calculate markup, you first have to understand the various components of a selling price. Then, in the next section, markup and its various methods of calculation will become much clearer.

When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product. Ultimately, you also need to make some money, or profit, as a result of the whole process.

The Formula

Most people think that marking up a product must be a fairly complex process. It is not. Formula 6.5 illustrates the relationship between the three components of cost, expenses, and profits in calculating the selling price.



S is Selling Price: Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.

C is Cost: The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount formula from Section 6.1. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for $10 then it must sell the product for at least $10. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!

Formula 6.5: The Selling Price of a Product: $S = C + E + P$

E is Expenses: Expenses are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases, equipment, electronics, insurance, utilities, fixtures, décor, and many more. These expenses must be recovered and may be calculated as

a. A fixed dollar amount per unit

b. A percentage of the product cost. For example, if a business forecasts total merchandise costs of $100,000 for the coming year and total business expenses of $50,000, then it may set a general guideline of adding 50% ($50,000 ÷ $100,000) to the cost of a product to cover expenses.

c. A percentage of the product selling price based on a forecast of future sales. For example, if a business forecasts total sales of $250,000 and total business expenses of $50,000, then it may set a general guideline of adding 20% ($50,000 ÷ $250,000) of the selling price to the cost of a product to cover expenses.

P is Profit: Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products. As with expenses, this profit may be expressed as

a. A fixed dollar amount per unit

b. A percentage of the product cost

c. A percentage of the selling price

Formula 6.5

How It Works

Follow these steps to solve pricing scenarios involving the three components:

Step 1: Four variables are involved in Formula 6.5. Identify the known variables. Note that you may have to calculate the product's cost by applying the single or multiple discount formulas (Formula 6.1 and Formula 6.3, respectively). Pay careful attention to expenses and profits to capture how you calculate these amounts.

Step 2: Apply Formula 6.5 and solve for the unknown variable.

Assume a business pays a net price of \$75 to acquire a product. Through analyzing its finances, the business estimates expenses at \$25 per unit, and it figures it can add \$50 in profit. Calculate the selling price.

Step 1: The net price paid for the product is the product cost. The known variables are C = \$75, E = \$25, and P = \$50.

Step 2: According to Formula 6.5, the unit selling price is $S = C + E + P$ = \$75 + \$25 + \$50 = \$150.

Important Notes

In applying Formula 6.5 you must adhere to the basic rule of linear equations requiring all terms to be in the same unit. That is, you could use Formula 6.5 to solve for the selling price of an individual product, where the three components are the unit cost, unit expenses, and unit profit. When you add these, you calculate the unit selling price. Alternatively, you could use Formula 6.5 in an aggregate form where the three components are total cost, total expenses, and total profit. In this case, the selling price is a total selling price, which is more commonly known as total revenue. But you cannot mix individual components with aggregate components.

Things To Watch Out For

The most common mistake in working with pricing components occurs in the "Understand" portion of the PUPP model. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.

? Exercise $6.2.1$: Give It Some Thought

a. What is the relationship between net price and cost?

Answer

a. Cost, expenses, and profit. They are expressed either per unit or as a total.
b. A specific dollar amount, a percentage of cost, or a percentage of the selling price.
c. The net price paid for a product is the same as the cost of the product.

✓ Example $6.2.1$: Setting a Price on Fashion in Dollars

Mary's Boutique purchases a dress for resale at a cost of \$23.67. The owner determines that each dress must contribute \$5.42 to the expenses of the store. The owner also wants this dress to earn \$6.90 toward profit. What is the regular selling price for the dress?

Solution

You are looking for the regular selling price for the dress, or S.

What You Already Know

Step 1:

The unit cost of the dress and the unit expense and the unit profit are all known: $C = \$23.67, E = \$5.42, P = \$6.90$

How You Will Get There

Step 2:

Apply Formula 6.5.

Perform

Step 2:

$S = \$23.67 + \$5.42 + \$6.90 = \$35.99$

Mary’s Boutique will set the regular price of the dress at $35.99.

✓ Example $6.2.2$: Setting the Price Using Percentage of Cost

John’s Discount Store just completed a financial analysis. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. John’s Discount Store purchases Chia Pets from its supplier for an MSRP of $19.99 less a trade discount of 45%. What will be the regular selling price for the Chia Pets?

Solution

You are looking for the regular selling price for the Chia pets, or S.

What You Already Know

Step 1:

The list price, discount rate, expenses, and profit are known:

L=$ 19.99, d=0.45

E = 20% of cost, or $0.20C$

P = 15% of cost, or $0.15C$

How You Will Get There

Step 1 (continued):

Although the cost of the Chia Pets is not directly known, you do know the MSRP (list price) and the trade discount. The cost is equal to the net price. Apply Formula 6.1.

Step 2:

To calculate the selling price, apply Formula 6.5.

Perform

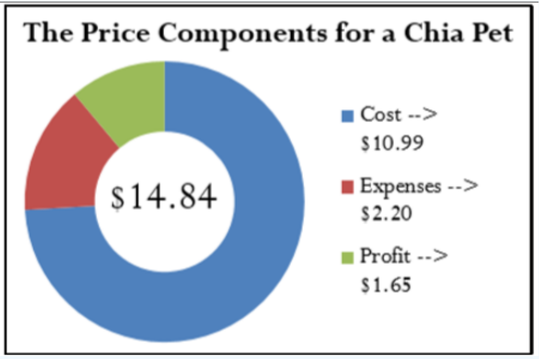
Step 1:

$N = \$19.99 \times (1 - 0.45) = \$19.99 \times 0.55 = \$10.99 = C$

Step 2:

$$\begin{aligned} S &= \$10.99 + 0.20C + 0.15C \\ S &= \$10.99 + 0.20(\$10.99) + 0.15(\$10.99) \\ S &= \$10.99 + \$2.20 + \$1.65 = \$14.84 \end{aligned}$$

John’s Discount Store will sell the Chia Pet for $14.84.



✓ Example $6.2.3$: Setting the Price Using Percentage of Selling Price

Based on last year's results, Benthal Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthal Appliance purchases a fridge for $1,200, what is the regular unit selling price?

Solution

You are looking for the regular unit selling price for the fridge, or S.

What You Already Know

Step 1:

The cost, expenses, and profit for the fridge are known:

E = 30% of S, or $0.3S$

P = 25% of S, or $0.25S$

C = $1,200.00

How You Will Get There

Step 2:

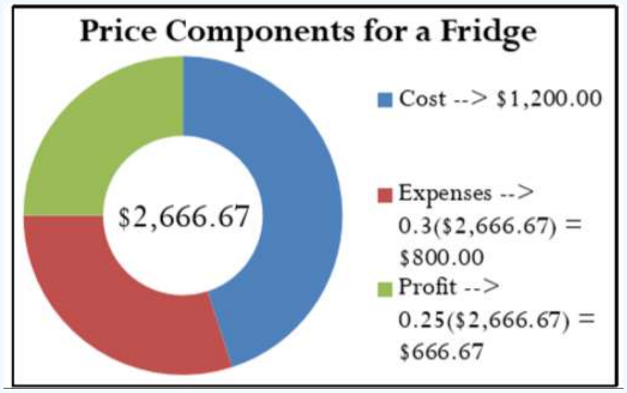
Apply Formula 6.5.

Perform

Step 2:

$$\begin{aligned} S &= \$1,200.00 + 0.3S + 0.25S \\ S &= \$1,200.00 + 0.55S \\ S - 0.55S &= \$1,200.00 \\ 0.45S &= \$1,200.00 \\ S &= \$2,666.67 \end{aligned}$$

Benthal Appliance should set the regular selling price of the fridge at $2,666.67.



✓

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of an MP3 player that has a regular selling price of $39.99?

Solution

You are looking for the cost of the MP3 player, or C.

What You Already Know

Step 1:

The expenses, profits, and the regular unit selling price are as follows:

S = \$39.99, P = 15% of S, or $0.15S$

E = 30% of cost, or $0.3C$

How You Will Get There

Step 2:

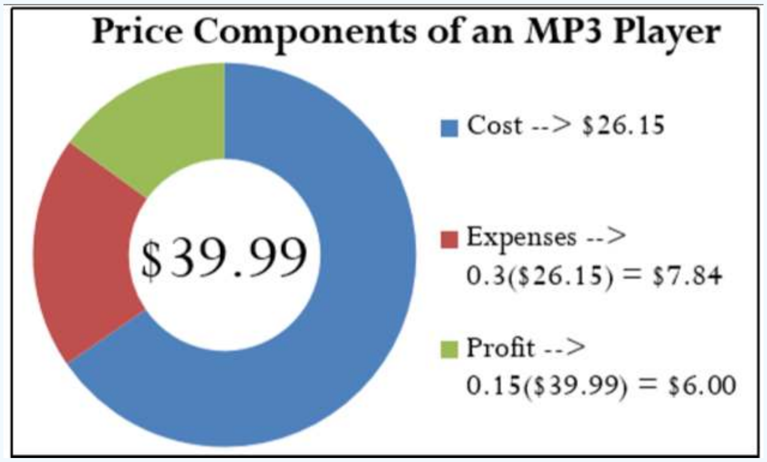
Apply Formula 6.5, rearranging for C.

Perform

Step 2:

$$\begin{aligned} \$39.99 &= C + 0.3C + 0.15(\$39.99) \\ \$39.99 &= 1.3C + \$6.00 \\ \$33.99 &= 1.3C \\ \$26.15 &= C \end{aligned}$$

The cost of the MP3 Player is \$26.15.



Example $6.2.5$: Determining Profitability

Peak of the Market considers setting the regular unit selling price of its strawberries at \$3.99 per kilogram. If it purchases these strawberries from the farmer for \$2.99 per kilogram and expenses average 40% of product cost, does Peak of the Market make any money?

Solution

In asking whether Peak of the Market makes money, you are looking for the profit, or P.

What You Already Know

Step 1:

The cost, expenses, and proposed regular unit selling price for the strawberries are as follows:

S = \$3.99, C = \$2.99, E = 40% of cost, or $0.4C$

How You Will Get There

Step 2:

Apply Formula 6.5, rearranging for P.

Perform

Step 2:

$$\begin{aligned} \$3.99 &= \$2.99 + 0.4(\$2.99) + P \\ \$3.99 &= \$4.19 + P \\ -\$0.20 &= P \end{aligned}$$

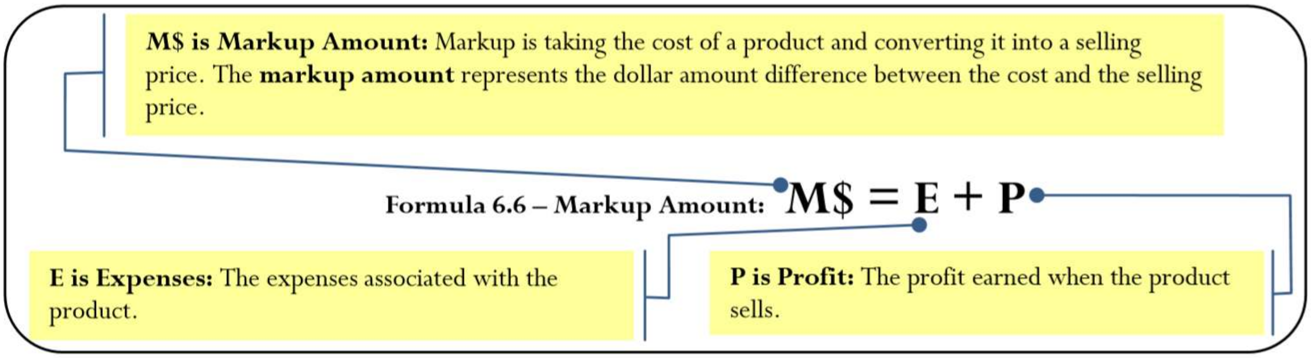
The negative sign on the profit means that Peak of the Market would take a loss of $0.20 per kilogram if it sells the strawberries at $3.99. Unless Peak of the Market has a marketing reason or sound business strategy for doing this, the company should reconsider its pricing.

Calculating the Markup Dollars

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and comparable, many companies combine expenses and profit together into a single quantity, either as a dollar amount or a percentage. This section focuses on the markup as a dollar amount.

The Formula

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together as expressed in Formula 6.6.



M$ is Markup Amount: Markup is taking the cost of a product and converting it into a selling price. The **markup amount** represents the dollar amount difference between the cost and the selling price.

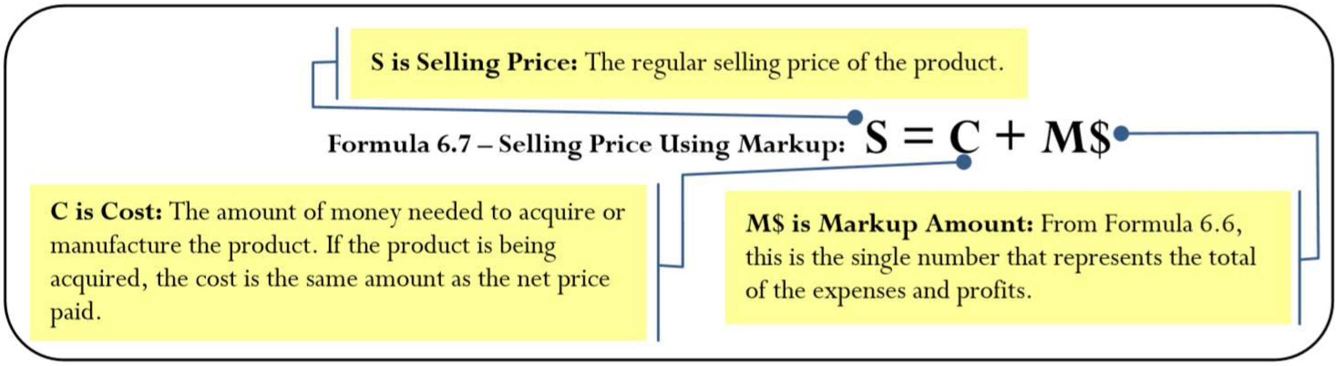
Formula 6.6 – Markup Amount: $M\$ = E + P$

E is Expenses: The expenses associated with the product.

P is Profit: The profit earned when the product sells.

Formula 6.6

Note that since the markup amount ($M\$$) represents the expenses (E) and profit (P) combined, you can substitute the variable for markup amount into Formula 6.5 to create Formula 6.7, which calculates the regular selling price.



S is Selling Price: The regular selling price of the product.

Formula 6.7 – Selling Price Using Markup: $S = C + M\$$

C is Cost: The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

M$ is Markup Amount: From Formula 6.6, this is the single number that represents the total of the expenses and profits.

Formula 6.7

How It Works

Follow these steps when you work with calculations involving the markup amount:

Step 1: You require three variables in either Formula 6.6 or Formula 6.7. At least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 6.6 or Formula 6.7 for the unknown variable.

Recall from Example 6.2.4 that the MP3 player's expenses are $7.84, the profit is $6.00, and the cost is $26.15. Calculate the markup amount and the selling price.

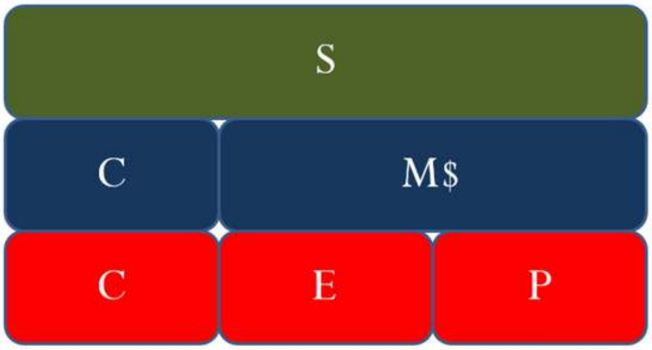
Step 1: The known variables are E = \$7.84, P = \$6.00, and C = \$26.15.

Step 2: According to Formula 6.6, the markup amount is the sum of the expenses and profit, or $M\$F = \$7.84 + \$6.00 = \$13.84$.

Step 2 (continued): Applying Formula 6.7, add the markup amount to the cost to arrive at the regular selling price, resulting in $S = \$26.15 + \$13.84 = \$39.99$

Paths To Success

You might have already noticed that many of the formulas in this chapter are interrelated. The same variables appear numerous times but in different ways. To help visualize the relationship between the various formulas and variables, many students have found it helpful to create a markup chart, as shown to the right.



This chart demonstrates the relationships between Formula 6.5, Formula 6.6, and Formula 6.7. It is evident that the selling price (the green line) consists of cost, expenses, and profit (the red line representing Formula 6.5); or it can consist of cost and the markup amount (the blue line representing Formula 6.7). The markup amount on the blue line consists of the expenses and profit on the red line (Formula 6.6).

✓ Example $6.2.6$: Markup as a Dollar Amount

A cellular retail store purchases an iPhone with an MSRP of \$779 less a trade discount of 35% and volume discount of 8%. The store sells the phone at the MSRP.

a. What is the markup amount?
b. If the store knows that its expenses are 20% of the cost, what is the store’s profit?

Solution

First of all, you need to calculate the markup amount ($M\$$). You also want to find the store’s profit (P).

What You Already Know

Step 1:

The smartphone MSRP and the two discounts are known, along with the expenses and selling price:

$L = \$270, d_1 = 0.35, d_2 = 0.08$

E = 20% of cost, or $0.2C$, S = \$779

How You Will Get There

Step 1 (continued):

Calculate the cost of the iPhone by applying Formula 6.3.

Step 2:

Calculate the markup amount using Formula 6.7.

Step 2 (continued):

Calculate the profit by applying Formula 6.6, rearranging for P.

Perform

Step 1 (continued):

$$N = \$779.00 \times (1 - 0.35) \times (1 - 0.08)$$
$$N = \$779.00 \times 0.65 \times 0.92 = \$465.84 = C$$

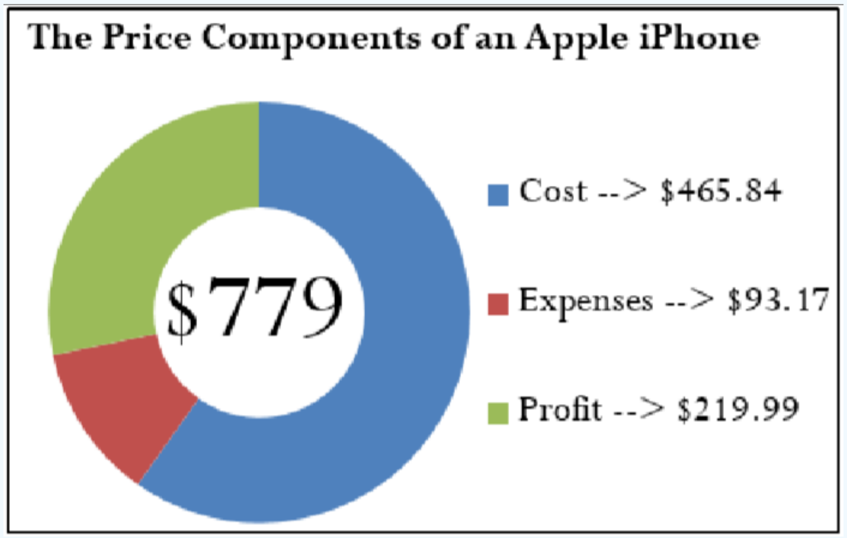
Step 2:

$$\$779.00 = \$465.84 + M\$$$
$$\$313.16 = M\$$$

Step 2 (continued):

$$\$313.16 = 0.2(\$465.84) + P$$
$$\$313.16 = \$93.17 + P$$
$$\$219.99 = P$$

The markup amount for the iPhone is $313.16. When the store sells the phone for $779.00, its profit is $219.99.



Calculating the Markup Percent

It is important to understand markup in terms of the actual dollar amount; however, it is more common in business practice to calculate the markup as a percentage. There are three benefits to converting the markup dollar amount into a percentage:

1. Easy comparison of different products having vastly different price levels and costs, to help you see how each product contributes toward the financial success of the company. For example, if a chocolate bar has a 50¢ markup included in a selling price of $1, while a car has a $1,000 markup included in a selling price of $20,000, it is difficult to compare the profitability of these items. If these numbers were expressed as a percentage of the selling price such that the chocolate bar has a 50% markup and the car has a 5% markup, it is clear that more of every dollar sold for chocolate bars goes toward list profitability.
2. Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products. For example, if all products are to be marked up by 50% of cost, an item with a $100 cost can be quickly converted into a selling price of $150.
3. An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product. For example, if an item selling for $25 includes a markup on selling price of 40% (which is $10), then you can determine that the cost is 60% of the selling price ($15) and that $10 of every $25 item sold goes toward list profits.

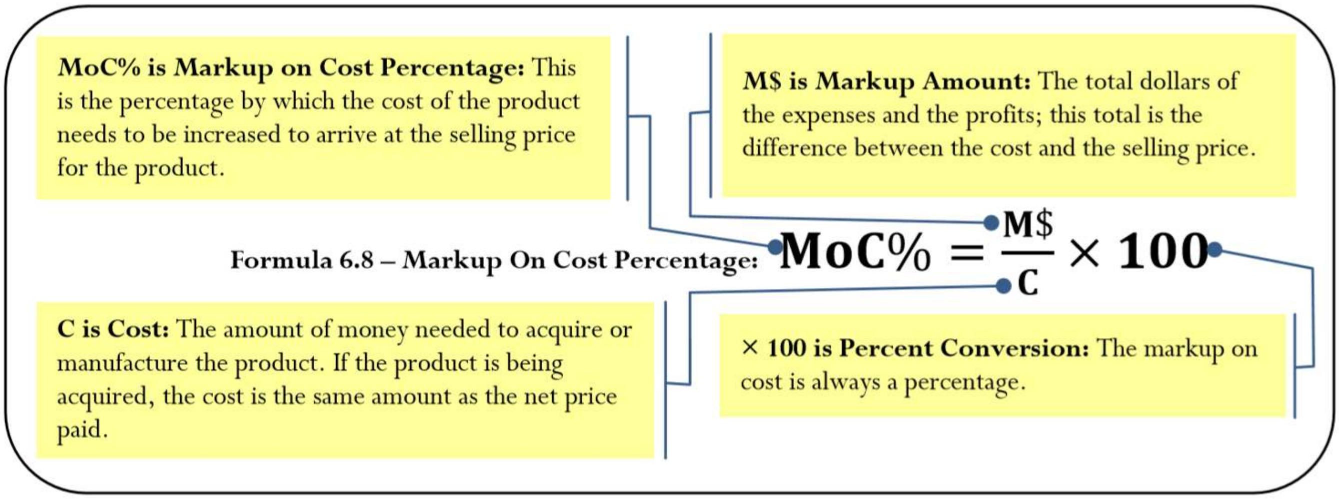
You can translate the markup dollars into a percentage using two methods, which express the amount either as a percentage of cost or as a percentage of selling price:

1. **Method 1: Markup as a Percentage of Cost**. This method expresses the markup rate using cost as the base. Many companies use this technique internally because most accounting is based on cost information. The result, known as the markup on cost percentage, allows a reseller to convert easily from a product's cost to its regular unit selling price.
2. **Method 2: Markup as a Percentage of Selling Price**. This method expresses the markup rate using the regular selling price as the base. Many other companies use this method, known as the markup on selling price percentage, since it allows for quick

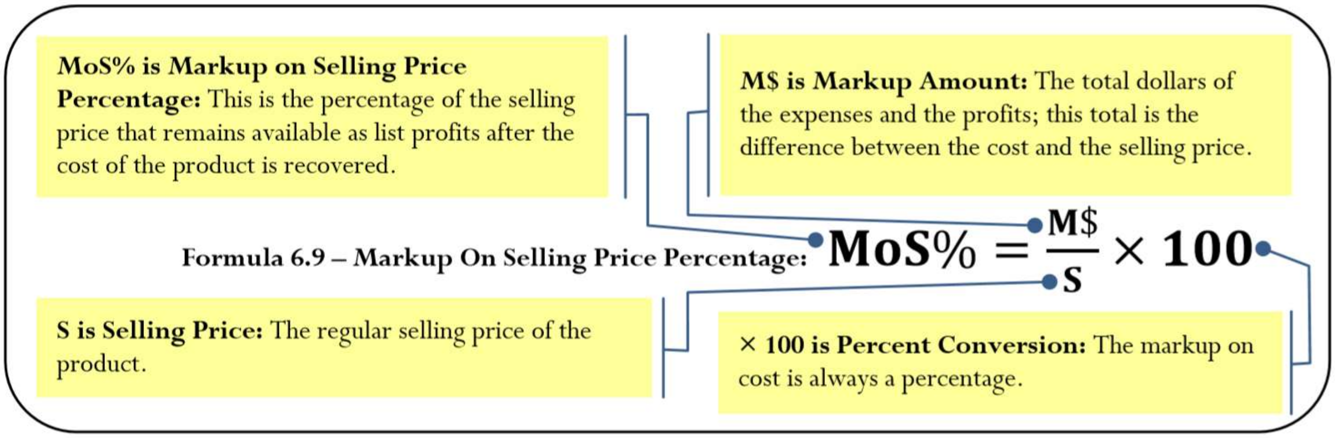
understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the list profit margin.

The Formula

Both formulas are versions of Formula 2.2 on rate, portion, and base. The markup on cost percentage is expressed in Formula 6.8, while the markup on selling price percentage is expressed in Formula 6.9.


MoC% is Markup on Cost Percentage: This is the percentage by which the cost of the product needs to be increased to arrive at the selling price for the product.
M$ is Markup Amount: The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.
Formula 6.8 – Markup On Cost Percentage: $MoC\% = \frac{M\$}{C} \times 100$
C is Cost: The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.
× 100 is Percent Conversion: The markup on cost is always a percentage.

Formula 6.8


MoS% is Markup on Selling Price Percentage: This is the percentage of the selling price that remains available as list profits after the cost of the product is recovered.
M$ is Markup Amount: The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.
Formula 6.9 – Markup On Selling Price Percentage: $MoS\% = \frac{M\$}{S} \times 100$
S is Selling Price: The regular selling price of the product.
× 100 is Percent Conversion: The markup on cost is always a percentage.

Formula 6.9

How It Works

Refer back to the steps in calculating markup. The steps you must follow for calculations involving markup percent are almost identical to those for working with markup dollars:

Step 1: Three variables are required in either Formula 6.8 or Formula 6.9. For either formula, at least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 6.8 or Formula 6.9 for the unknown variable.

Continuing to work with Example $6.2.4$, recall that the cost of the MP3 player is $26.15, the markup amount is $13.84, and the selling price is $39.99. Calculate both markup percentages.

Step 1: The known variables are C = $26.15, $M\$$ = $13.84, and S = $39.99.

Step 2: To calculate the markup on cost percentage, apply Formula 6.8:

$$MoC\% = \frac{\$13.84}{\$26.15} \times 100 = 52.9254\%$$

In other words, you must add 52.9254% of the cost on top of the unit cost to arrive at the regular unit selling price of $39.99.

Step 2 (continued): To calculate the markup on selling price percentage, apply Formula 6.9:

$$MoS\% = \frac{\$13.84}{\$39.99} \times 100 = 34.6087\%$$

In other words, 34.6087% of the selling price represents list profits after the business recovers the $26.15 cost of the MP3 player.

Important Notes

Businesses are very focused on profitability. Your Texas Instruments BAII Plus calculator is programmed with the markup on selling price percentage. The function is located on the second shelf above the number three. To use this function, open the window by pressing 2nd 3. You can scroll between lines using your ↑ and ↓ arrows. There are three variables:

- CST is the cost. Use the symbol C.
- SEL is the selling price. Use the symbol S.
- MAR is the markup on selling price percentage. Use the symbol MoS%.

As long as you know any two of the variables, you can solve for the third. Enter any two of the three variables (you need to press ENTER after each), making sure the window shows the output you are seeking, and press CPT.

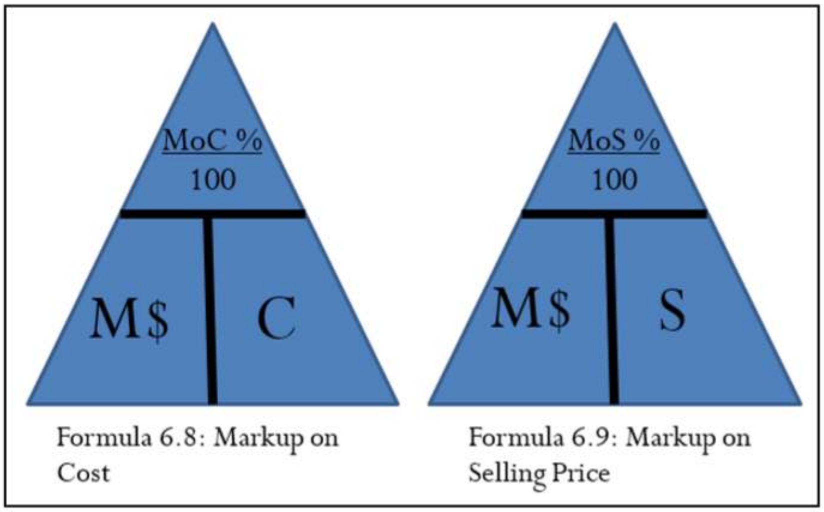
Things To Watch Out For

Merchandising involves many variables. Nine formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. The calculation of discounts is no different than going to Walmart and finding your favorite CD on sale. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money. And you have worked with percentages since elementary school.

Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, “How does knowing that new variable help solve any other formula?”

You do not need to get frustrated. Just be systematic and relate the question to what you already know.

Paths To Success


MoC %
100
M$
C
Formula 6.8: Markup on Cost
MoS %
100
M$
S
Formula 6.9: Markup on Selling Price

The triangle method simplifies rearranging both Formula 6.8 and Formula 6.9 to solve for other unknown variables as illustrated in the figure above.

Sometimes you need to convert the markup on cost percentage to a markup on selling price percentage, or vice versa. Two shortcuts allow you to convert easily from one to the other:

$$MoC\% = \frac{MoS\%}{1 - MoS\%} \qquad MoS\% = \frac{MoC\%}{1 + MoC\%}$$

Notice that these formulas are very similar. How do you remember whether to add or subtract in the denominator? In normal business situations, the $MoC\%$ is always larger than the $MoS\%$. Therefore, if you are converting one to the other you need to identify whether you want the percentage to become larger or smaller.

- To calculate $MoC\%$, you want a larger percentage. Therefore, make the denominator smaller by subtracting $MoS\%$ from 1.
- To calculate $MoS\%$, you want a smaller percentage. Therefore, make the denominator larger by adding $MoC\%$ to 1.

? Exercise 6.2.2: Give It Some Thought

Answer the following true/false questions.

a. If you know the markup on selling price percentage and the cost, you can calculate a selling price.

Answer

a. False. The markup amount is a portion of the selling price and therefore is less than 100%.
b. False. The markup amount plus the cost equals the selling price. It must be less than the selling price.
c. True. A cost can be doubled or tripled (or increased even more) to reach the price.
d. True. The base for markup on cost percentage is smaller, which produces a larger percentage.
e. True. You could combine Formula 6.7 and Formula 6.8 to arrive at the selling price.
f. True. You could convert the $MoS\%$ to a $MoC\%$ and solve as in the previous question.

✓ Example 6.2.7: Markup as a Percentage

A large national retailer wants to price a Texas Instruments BAII Plus calculator at the MSRP of $39.99. The retailer can acquire the calculator for $17.23.

a. What is the markup on cost percentage?
b. What is the markup on selling price percentage?

Solution

You have been asked to solve for the markup on cost percentage ($MoC\%$) and the markup on selling price percentage ($MoS\%$).

What You Already Know

Step 1:

The regular unit selling price and the cost are provided:

S = $39.99, C = $17.23

How You Will Get There

Step 1 (continued):

You need the markup dollars. Apply Formula 6.7, rearranging for $M\$$.

Step 2:

To calculate markup on cost percentage, apply Formula 6.8.

Step 2 (continued):

To calculate markup on selling price percentage, apply Formula 6.9.

Perform

Step 1 (continued):

$$\$39.99 = \$17.23 + M\$$$

$$\$22.76 = MS$$

Step 2:

$$MoC\% = \frac{\$22.76}{\$17.23} \times 100 = 132.0952\%$$

Step 2 (continued):

$$MoS\% = \frac{\$22.76}{\$39.99} \times 100 = 56.9142\%$$

Calculator Instructions

CST	SEL	MAR
17.23	39.99	Answer: 56.9142

The markup on cost percentage is 132.0952%. The markup on selling price percentage is 56.9142%.

Break-Even Pricing

In running a business, you must never forget the “bottom line.” In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! As revealed in Chapter 5, 15% of new businesses will not make it past their first year, and 49% fail in their first five years. This number becomes even more staggering with an 80% failure rate within the first decade.[1] Do not be one of these statistics! With your understanding of markup, you now know what it takes to break even in your business. Recall from Chapter 5 that break-even means that you are earning no profit, but you are not losing money either. Your profit is zero.

The Formula

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the regular unit selling price must exactly cover your costs and expenses when the profit is zero. In other words, if Formula 6.5 is modified to calculate the selling price at the break-even point (S_{BE}) with $P = 0$, then

$$S_{BE} = C + E$$

This is not a new formula. It just summarizes that at break-even there is no profit or loss, so the profit (P) is eliminated from the formula.

How It Works

The steps you need to calculate the break-even point are no different from those you used to calculate the regular selling price. The only difference is that the profit is always set to zero.

Recall Example $6.2.4$ that the cost of the MP3 player is \$26.15 and expenses are \$7.84. The break-even price (S_{BE}) is \$26.15 + \$7.84 = \$33.99. This means that if the MP3 player is sold for anything more than \$33.99, it is profitable; if it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

✓ Example $6.2.8$: Knowing Your Break-Even Price

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John’s last garage sale shopping spree, he only found one item—a Nintendo Wii that was sold to him for \$100. John’s vehicle expenses (for gas, oil, wear/tear, and time) amounted to \$40. eBay charges a \$2.00 insertion fee, a flat fee of \$2.19, and a commission of 3.5% based on the selling price less \$25. What is John’s minimum list price for his Nintendo Wii to ensure that he at least covers his expenses?

Solution

You are trying to find John’s break-even selling price (S_{BE}).

What You Already Know

Step 1:

John’s cost for the Nintendo Wii and all of his associated expenses are as follows:

C = \$100.00, E (vehicle) = \$40.00

$E\ (\text{insertion}) = \$2.00$

$E(\text{flat}) = \$2.19$

$E\ (\text{commission}\ ) = 3.5\%\,(S_{BE} - \$25.00)$

How You Will Get There

Step 1 (continued):

You have four expenses to add together that make up the E in the formula.

Step 2:

Formula 6.5 states $S = C + E + P$. Since you are looking for the break-even point, then P is set to zero and $S_{BE} = C + E$.

Perform

Step 1 (continued):

$$\begin{aligned}
E &= \$40.00 + 52.00 + 52.19 + 3.5\%\,(S_{BE} - \$25.00)\\
E &= \$44.19 + 0.035\,(S_{BE} - \$25.00)\\
&= \$44.19 + 0.035S_{BE} - \$0.875\\
E &= \$43.315 + 0.0355S_{BE}
\end{aligned}$$

Step 2:

$$\begin{aligned}
S_{BE} &= \$100.00 + \$43.315 + 0.035S_{BE}\\
S_{BE} &= \$143.315 + 0.035S_{BE}\\
S_{BE} - 0.035S_{BE} &= \$143.315\\
0.965S_{BE} &= \$143.315\\
S_{BE} &= \$148.51
\end{aligned}$$

At a price of \$148.51 John would cover all of his costs and expenses but realize no profit or loss. Therefore, \$148.51 is his minimum price.

References

1. Statistics Canada, “Failure Rates for New Firms,” *The Daily*, February 16, 2000, www.statcan.gc.ca/daily-quotidien/000216/dq000216b-eng.htm.

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6.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)

Flashy signs in a retail store announce, “40% off, today only!” Excitedly you purchase three tax-free products with regular price tags reading $100, $250, and $150. The cashier processing the transaction informs you that your total is $325. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is $500. If they are 40% off, you should receive a $200 deduction and pay only $300. The cashier apologizes for the mistake and corrects your total.

Although most retail stores use automated checkout systems, these systems are ultimately programmed by human beings. A computer system is only as accurate as the person keying in the data. A study by the Competition Bureau revealed that 6.3% of items at various retail stores scanned incorrectly. The average error spread is up to 13% around the actual product’s price![1] Clearly, it is important for you as a consumer to be able to calculate markdowns.

Businesses must also thoroughly understand markdowns so that customers are charged accurately for their purchases. Businesses must always comply with the Competition Act of Canada, which specifically defines legal pricing practices. If your business violates this law, it faces severe penalties.

The Importance of Markdowns

A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** to distinguish it from the selling price.

Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances:

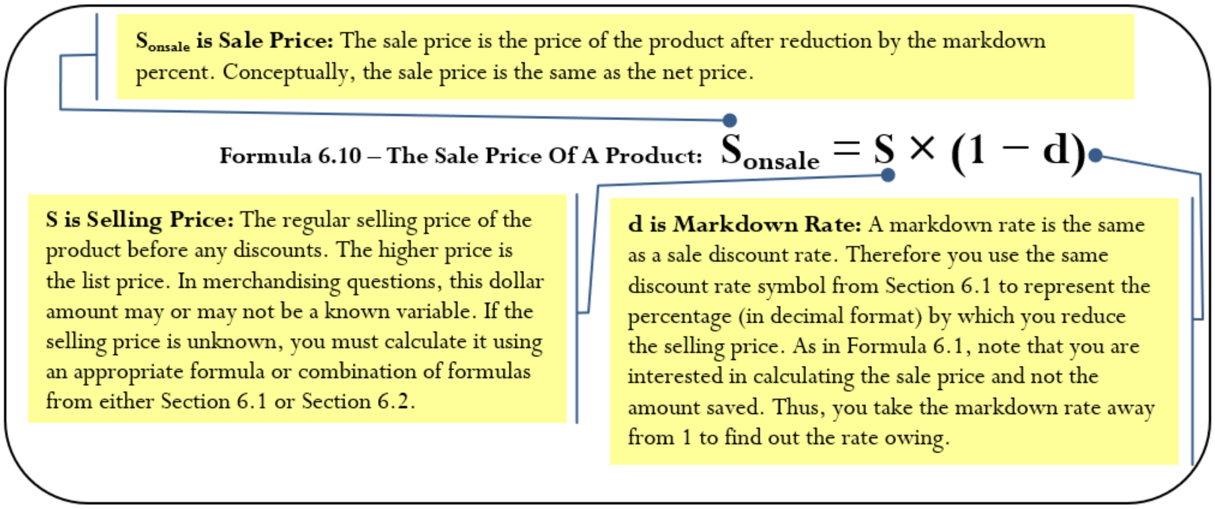
1. **Clearing Out Excess or Unwanted Inventory**. In these situations, the business thought it could sell 100 units; however, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.
2. **Clearing Out Damaged or Discontinued Items**. Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.
3. **Increasing Sales Volumes**. Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.
4. **Promoting Add-On Purchases**. Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable. Like many others, you may have walked into Target to buy one item but left with five instead.

The Formula

Markdowns are no different from offering a discount. Recall from Section 6.1 that one of the types of discounts is known as a sale discount. The only difference here lies in choice of language. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising. Formula 6.1, introduced in Section 6.1, calculates the net price for a product after it receives a single discount:

$$N=L\times(1-d)$$

Formula 6.10 adapts this formula for use in markdown situations.



S_{onsale} is Sale Price: The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.

Formula 6.10 – The Sale Price Of A Product: $S_{onsale} = S \times (1 - d)$

S is Selling Price: The regular selling price of the product before any discounts. The higher price is the list price. In merchandising questions, this dollar amount may or may not be a known variable. If the selling price is unknown, you must calculate it using an appropriate formula or combination of formulas from either Section 6.1 or Section 6.2.

d is Markdown Rate: A markdown rate is the same as a sale discount rate. Therefore you use the same discount rate symbol from Section 6.1 to represent the percentage (in decimal format) by which you reduce the selling price. As in Formula 6.1, note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

Formula 6.10

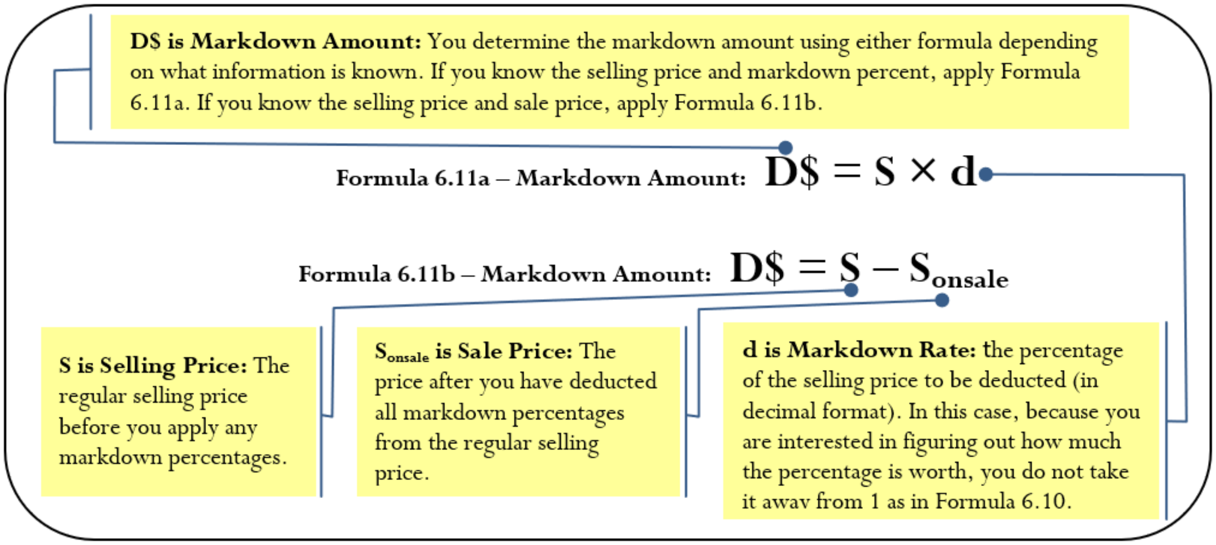
In markdown situations, the selling price and the sale price are different variables. The sale price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend Formula 6.10 in the same manner that Formula 6.3 calculated multiple discounts.

If you are interested in the markdown amount in dollars, recall that Formula 6.2 calculates the discount amount in dollars. Depending on what information is known, the formula has two variations:

$$\text{Formula 6.2a}: D\$ = L \times d$$

$$\text{Formula 6.2b}: D\$ = L - N$$

Formulas 6.11a and 6.11b adapt these formulas to markdown situations.



D\$ is Markdown Amount: You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 6.11a. If you know the selling price and sale price, apply Formula 6.11b.

Formula 6.11a – Markdown Amount: $D\$ = S \times d$

Formula 6.11b – Markdown Amount: $D\$ = S - S_{onsale}$

S is Selling Price: The regular selling price before you apply any markdown percentages.

S_{onsale} is Sale Price: The price after you have deducted all markdown percentages from the regular selling price.

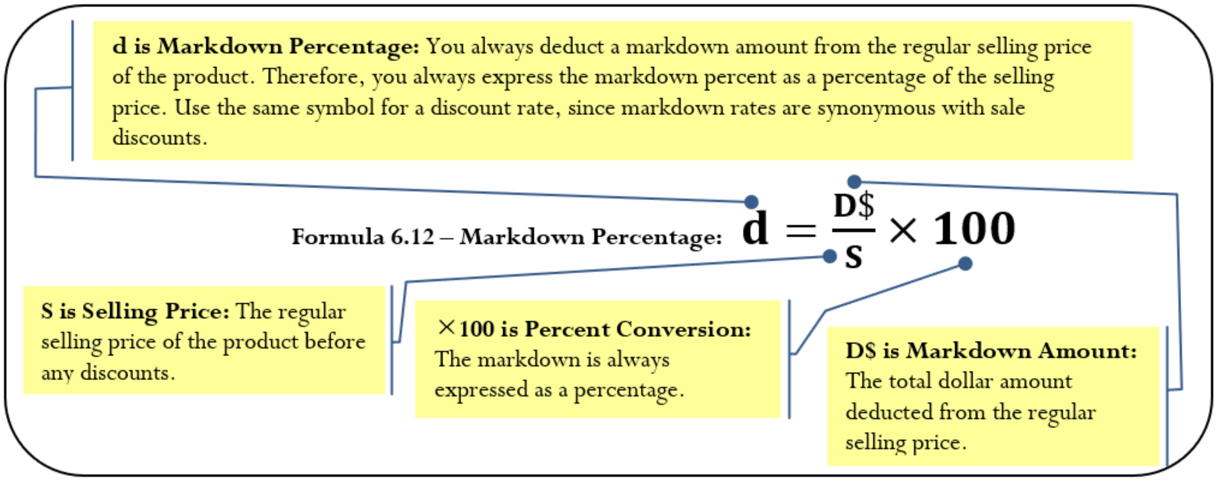
d is Markdown Rate: the percentage of the selling price to be deducted (in decimal format). In this case, because you are interested in figuring out how much the percentage is worth, you do not take it awav from 1 as in Formula 6.10.

Formula 6.11a and Formula 6.11b

The final markdown formula reflects the tendency of businesses to express markdowns as percentages, facilitating easy comprehension and comparison. Recall Formula 6.9 from Section 6.2, which calculated a markup on selling price percent:

$$MoS\% = \frac{M\$}{S} \times 100$$

Formula 6.12 adapts this formula to markdown situations.


d is Markdown Percentage: You always deduct a markdown amount from the regular selling price of the product. Therefore, you always express the markdown percent as a percentage of the selling price. Use the same symbol for a discount rate, since markdown rates are synonymous with sale discounts.
Formula 6.12 – Markdown Percentage: d = D$ / S × 100
S is Selling Price: The regular selling price of the product before any discounts.
×100 is Percent Conversion: The markdown is always expressed as a percentage.
D$ is Markdown Amount: The total dollar amount deducted from the regular selling price.

Formula 6.12

How It Works

Follow these steps to calculate a markdown:

Step 1: Across all three markdown formulas, the four variables consist of the selling price (S), sale price (S_{onsale}), markdown dollars ($D\$$), and markdown rate (d). Identify which variables are known. Depending on the known information, you may have to calculate the selling price using a combination of discount and markup formulas.

Step 2: Apply one or more of Formula 6.10, Formula 6.11a, Formula 6.11b, and Formula 6.12 to calculate the unknown variable(s). In the event that multiple markdown rates apply, extend Formula 6.10 to accommodate as many markdown rates as required.

Recall from Section 6.2 the example of the MP3 player with a regular selling price of $39.99. Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Step 1: The selling price and markdown percent are $S = \$39.99$ and $d = 0.10$, respectively.

Step 2: Apply Formula 6.10 to calculate the sale price, resulting in $S_{onsale} = \$39.99 \times (1 - 0.10) = \$35.99$.

Step 2 (continued): You could use either of Formulas 6.11a or 6.11b to calculate the markdown amount since the selling price, sale price, and markdown percent are all known. Arbitrarily choosing Formula 6.11a, you calculate a markdown amount of $D\$ = \$39.99 \times 0.10 = \$4.00$.

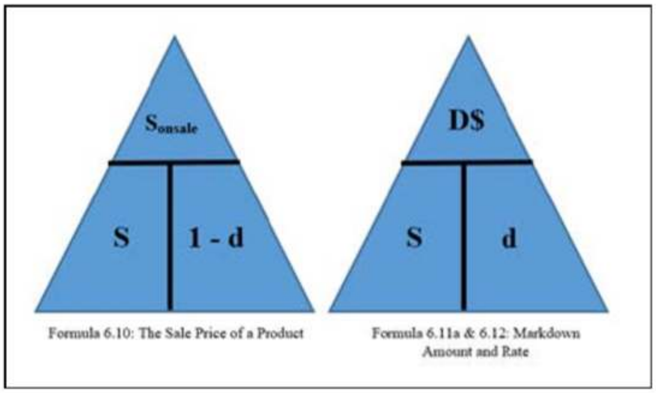
Therefore, if the retailer has a 10% off sale on the MP3 players, it marks down the product by $4.00 and retails it at a sale price of $\$35.99$.

Things To Watch Out For

Just as in Section 6.2, avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like. Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

Paths To Success

Three of the formulas introduced in this section can be solved for any variable through algebraic manipulation when any two variables are known. Recall that the triangle technique helps you remember how to rearrange these formulas, as illustrated here.


S_{onsale}
S
1 - d
D$
S
d
Formula 6.10: The Sale Price of a Product
Formula 6.11a & 6.12: Markdown Amount and Rate

Example $6.3.1$: Determining the Sale Price and Markdown Amount

The MSRP for the “Guitar Hero: World Tour” video game is $189.99. Most retail stores sell this product at a price in line with the MSRP. You have just learned that a local electronics retailer is selling the game for 45% off. What is the sale price for the video game and what dollar amount is saved?

Solution

There are two unknown variables. The first is the video game's sale price (S_{onsale}). The second is the markdown amount ($D\$$) that is realized at that sale price.

What You Already Know

Step 1:

The regular selling price for the video game and the markdown rate are known:

S = $189.99, d = 0.45

How You Will Get There

Step 2:

Calculate the sale price by applying Formula 6.10.

Step 2 (continued):

Calculate the markdown amount by applying Formula 6.11b.

Perform

Step 2:

$$\begin{aligned} S_{onsale} &= \$189.99 \times (1-0.45) \\ &= \$189.99 \times 0.55 \\ &= \$104.49 \end{aligned}$$

Step 2 (continued):

$$D\$ = \$189.99 - \$104.49 = \$85.50$$

The sale price for the video game is $104.49. When purchased on sale, “Guitar Hero: World Tour” is $85.50 off of its regular price.

Example $6.3.2$: Markdown Requiring Selling Price Calculation

A reseller acquires an Apple iPad for $650. Expenses are planned at 20% of the cost, and profits are set at 15% of the cost. During a special promotion, the iPad is advertised at $100 off. What is the sale price and markdown percent?

Solution

The unknown variables for the iPad are the sale price (S_{onsale}) and the markdown rate (d).

What You Already Know

Step 1:

The pricing elements of the iPad along with the markdown dollars are known:

C = \$650, $E = 0.2C$, $P = 0.15C$ $D\$ = \$100$

How You Will Get There

Step 1 (continued):

Calculate the selling price of the product by applying Formula 6.5.

Step 2:

Calculate the markdown percent by applying Formula 6.12:

Step 2 (continued):

Calculate the sale price by applying Formula 6.11b, rearranging for S_{onsale}.

Perform

Step 1 (continued):

$$S = \$650 + 0.2(\$650) + 0.15(\$650) = \$877.50$$

Step 2:

$$d = \frac{\$100}{\$877.50} \times 100 = 11.396\%$$

Step 2 (continued):

$$\begin{aligned} \$100 &= \$877.50 - S_{onsale} \\ S_{onsale} &= \$777.50 \end{aligned}$$

When the iPad is advertised at \$100 off, it receives an 11.396% markdown and it will retail at a sale price of \$777.50.

Never-Ending Sales

Have you noticed that some companies always seem to have the same item on sale all of the time? This is a common marketing practice. Recall the third and fourth circumstances for markdowns. Everybody loves a sale, so markdowns increase sales volumes for both the marked-down product and other regularly priced items.

For example, Michaels has a product line called the Lemax Village Collection, which has seasonal display villages for Christmas, Halloween, and other occasions. When these seasonal product lines come out, Michaels initially prices them at the regular unit selling price for a short period and then reduces their price. For Michaels, this markdown serves a strategic purpose. The company's weekly flyers advertising the Lemax Village Collection sale attract consumers who usually leave the store with other regularly priced items.

The Formula

If an item is on sale all the time, then businesses plan the pricing components with the sale price in mind. Companies using this technique determine the unit profitability of the product at the sale price and not the regular selling price. They adapt Formula 6.5 as follows:

$$S = C + E + P \quad \text{becomes} \quad S_{onsale} = C + E + P_{onsale}$$

where P_{onsale} represents the planned profit amount when the product is sold at the sale price. This is not a new formula, just a new application of Formula 6.5.

How It Works

Under normal circumstances, when businesses set their selling and sale prices they follow a three-step procedure:

1. Determine the product's cost, expenses, and profit amount.

2. Set the regular selling price of the product.
3. If a markdown is to be applied, determine an appropriate markdown rate or amount and set the sale price.

However, when a product is planned to always be on sale, businesses follow these steps instead to set the sale price and selling price:

Step 1: Set the planned markdown rate or markdown dollars. Determine the pricing components such as cost and expenses. Set the profit so that when the product is marked down, the profit amount is achieved. Alternatively, a planned markup on cost, markup on selling price, or even markup dollars may be set for the sale price.

Step 2: Calculate the sale price of the product. If cost, expenses, and profit are known, apply the adapted version of Formula 6.5. Alternatively, adapt and apply any of the other markup formulas (Formula 6.6 through Formula 6.9) with the understanding that the result is the sale price of the product and not the regular selling price.

Step 3: Using the known markdown rate or markdown amount, set the regular selling price by applying any appropriate markdown formula (Formula 6.10 through Formula 6.12).

Assume for the Michael's Lemax Village Collection that most of the time these products are on sale for 40% off. A particular village item costs $29.99, expenses are $10.00, and a planned profit of $8.00 is achieved at the sale price. Calculate the sale price and the selling price.

Step 1: The known variables at the sale price are $C = \$29.99, E = \$10.00, P = \$8.00$, and $d = 0.40$.

Step 2: Adapting Formula 6.5, the sale price is $S_{onsale} = C + E + P_{onsale} = \$29.99 + \$10.00 + \$8.00 = \$47.99$. This is the price at which Michael's plans to sell the product.

Step 3: However, to be on sale there must be a regular selling price. Therefore, if the 40% off results in a price of $\$47.99$, apply Formula 6.10 and rearrange to get the selling price: $S = \$47.99 \div (1 - 0.40) = \$79.98$. Therefore, the product's selling price is $79.98, which, always advertised at 40% off, results in a sale price of $47.99. At this sale price, Michael's earns the planned $8.00 profit.

Important Notes

You may ask, "If the product is always on sale, what is the importance of establishing the regular price?" While this textbook does not seek to explain the law in depth, it is worth mentioning that pricing decisions in Canada are regulated by the Competition Act. With respect to the discussion of never-ending sales, the Act does require that the product be sold at a regular selling price for a reasonable period of time or in reasonable quantity before it can be advertised as a sale price.

If you revisit the Michael's example, note in the discussion that the village initially needs to be listed at the regular selling price before being lowered to the sale price.

? Exercise $6.3.1$: Give It Some Thought

a. Under normal circumstances, arrange from smallest to largest: regular selling price, cost, and sale price.

Answer

a. Below cost, since the 40% markdown is off of the selling price, which is a larger value.
b. The profit will be increased by the markdown amount.
c. Cost, sale price, regular selling price.

✓ Example $6.3.3$: Setting the Price in a Never-Ending Sale

An electronics retailer has 16GB USB sticks on sale at 50% off. It initially priced these USB sticks for a short period of time at regular price, but it planned at the outset to sell them at the sale price. The company plans on earning a profit of 20% of the cost when the product is on sale. The unit cost of the USB stick is $22.21, and expenses are 15% of the cost.

a. At what price will the retailer sell the USB stick when it is on sale?
b. To place the USB stick on sale, it must have a regular selling price. Calculate this price.
c. If the USB stick is purchased at the regular selling price during the initial time period, how much profit is earned?

Solution

a. This company plans on always having the product on sale, so the pricing needs to be set for the sale price, or S_{onsale}.
b. You need the regular selling price, or S.
c. Solve for the profit at the regular selling price, or P.

What You Already Know

Step 1:

You know the unit cost, the retailer's associated expenses, its planned profit at the sale price, and the markdown rate:

$d = 0.50, C = \$22.21, E = 0.15C, P_{onsale} = 0.2C$

How You Will Get There

Step 2:

To solve part (a), apply the adapted version of Formula 6.5:

$$S_{onsale} = C + E + P_{onsale}$$

Step 3:

After you know the sale price, solve part (b) by applying Formula 6.10, rearranging for S.

Step 4:

Solving part (c) requires applying Formula 6.5.

Perform

Step 2:

$$\begin{aligned} S_{onsale} &= \$22.21 + 0.15(\$22.21) + 0.2(\$22.21) \\ &= \$22.21 + \$3.33 + \$4.44 \\ &= \$29.98 \end{aligned}$$

Step 3:

$$\begin{aligned} \$29.98 &= S \times (1 - 0.5) \\ \$29.98 &= S \times 0.5 \\ \$59.96 &= S \end{aligned}$$

Step 4:

$$\begin{aligned} \$59.96 &= \$22.21 + \$3.33 + P \\ \$59.96 &= \$25.54 + P \\ \$34.42 &= P \end{aligned}$$

The USB stick is on sale for \$29.98, letting the company achieve its profit of \$4.44 per unit. During the initial pricing period, the USB stick sells for \$59.96 (its regular selling price). If a consumer actually purchases a USB stick during the initial pricing period, the electronics store earns a profit of \$34.42 per unit (which is a total of the \$4.44 planned profit plus the planned markdown amount of \$29.96).

References

1. Competition Bureau, Fair Business Practices Branch, *Price Scanning Report*, Table B, page 5, 1999, www.competitionbureau.gc.ca/epic/site/cb-bc.nsf/en/01288e.html.

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6.4: Merchandising (How Does It All Come Together?)

Running a business requires you to integrate all of the concepts in this chapter. From discounts to markups and markdowns, all the numbers must fit together for you to earn profits in the long run. It is critical to understand how pricing decisions affect the various financial aspects of your business and to stay on top of your numbers.

Merchandising does not involve difficult concepts, but to do a good job of it you need to keep track of many variables and observe how they relate to one another. Merchandising situations in this section will apply all of the previously discussed pricing concepts. Next, the concept of maintained markup helps you understand the combined effect of markup and markdown decisions on list profitability. Finally, you will see that coupons and rebates influence expenses and profitability in several ways beyond the face value of the discounts offered.

The Complete Pricing Picture

Up to this point, you have examined the various components of merchandising as separate topics. Your study of basic product pricing has included the following:

1. Taking an MSRP or list price and applying discounts to arrive at a product’s cost
2. Marking up a product by adding expenses and profits to the cost to arrive at a regular selling price
3. Marking down a product by applying a discount and arriving at a sale price
4. Working with various percentages in both markup and markdown situations to either simplify calculations or present a clearer pricing picture

Now is the time to tie all of these merchandising concepts together. Section 6.2 introduced the concept of a markup chart to understand the relationship between various markup components. The figure below extends this figure to include all of the core elements of merchandising.

The figure reveals the following characteristics:

1. You calculate the cost of a product by applying a discount formula. The net price, N, is synonymous with cost, C.
2. When you mark a product down, not only is the profit reduced but the markup dollars are also reduced. The following relationships exist when a product is on sale:

$$M\$_{onsale} = E + P_{onsale} \quad \text{OR} \quad M\$_{onsale} = M\$ - D\$$$

$$P_{onsale} = P - D\$$$

3. At break-even (remember, P or P_{onsale} is zero) a sale product has the following relationships:

$$\begin{aligned} M\$_{onsale} &= E \\ S_{onsale} &= C + E \\ D\$ &= P \end{aligned}$$

None of these relationships represent new formulas. Instead, these reflect a deeper understanding of the relationships between the markup and markdown formulas.

How It Works

Follow these steps to solve complete merchandising scenarios:

Step 1: It is critically important to correctly identify both the known and unknown merchandising variables that you are asked to calculate.

Step 2: For each of the unknown variables in step 1, examine Formula 6.1 through Formula 6.12. Locate which formulas contain the unknown variable. You must solve one of these formulas to arrive at the answer. Based on the information provided, examine these formulas to determine which formula may be solvable. Write out this formula, identifying which components you know and which components remain unknown. For example, assume that in step 1 you were asked to solve for the expenses, or E. In looking at the formulas, you find this variable appears only in Formula 6.5 and Formula 6.6, meaning that you must use one of these two formulas to calculate the expenses. Suppose that, in reviewing the known variables from step 1, you already have the markup amount and the profit. In this case, Formula 6.6 would be the right formula to calculate expenses with.

Step 3: Note the unknown variables among all the formulas written in step 2. Are there common unknown variables among these formulas? These common variables are critical variables. Solving for these common unknowns is the key to completing the question. Note that these unknown variables may not directly point to the information you were requested to calculate, and they do not resolve the merchandising scenario in and of themselves. However, without these variables you cannot solve the scenario. For example, perhaps in step 1 you were requested to calculate both the markup on cost percent ($MoC\%$) and the markup on selling price percent ($MoS\%$). In step 2, noting that these variables are found only in Formula 6.8 and Formula 6.9, you wrote them down. Examination of these formulas revealed that you know the cost and selling price variables. However, in both formulas the markup dollars remain unknown. Therefore, markup dollars is the critical unknown variable. You must solve for markup dollars using Formula 6.7, which will then allow you to calculate the markup percentages.

Step 4: Apply any of the merchandising formulas from Formula 6.1 to Formula 6.12 to calculate the unknown variables required to solve the formulas. Your goal is to identify all required variables and then solve the original unknown variables from step 1.

Things To Watch Out For

Before proceeding, take a few moments to review the various concepts and formulas covered earlier in this chapter. A critical and difficult skill is now at hand. As evident in steps 2 through 4 of the solving process, you must use your problem-solving skills to figure out which formulas to use and in what order. Here are three suggestions to help you on your way:

1. Analyze the question systematically. Conscientiously use the Plan, Understand, Perform, and Present model.
2. If you are unsure of how the pieces of the puzzle fit together, try substituting your known variables into the various formulas. You are looking for
   a. Any solvable formulas with only one unknown variable or
   b. Any pair of formulas with the same two unknowns, since you can solve this system using your algebraic skills of solving two linear equations with two unknowns (covered in Section 2.5).
3. Merchandising has multiple steps. Think through the process. When you solve one equation for an unknown variable, determine how knowing that variable affects your ability to solve another formula. As mentioned in step 3, there is usually a critical unknown variable. Once you determine the value of this variable, a domino effect allows you to solve any other remaining formulas.

✓ Example $6.4.1$: A Merchandising Situation

A skateboard shop stocks a Tony Hawk Birdhouse Premium Complete Skateboard. The MSRP for this board is $82. If the shop is going to stock the board, it must order at least 25 units. The known information is as follows:

- The supplier offers a retail trade discount of 37% with a quantity discount of 12% for orders that exceed 15 units.
- Typical expenses associated with selling this premium skateboard amount to 31% of the regular selling price.
- The skateboard shop wants each of its products to have profitability of 13% of the regular selling price.
- In the event that the skateboards do not sell by the end of the season, the skateboard shop always holds an end-of-season clearance sale where it marks down its products to the break-even selling price.

Using this information about the skateboard, determine the following:

a. Cost
b. Regular unit selling price
c. Markup on cost percentage
d. Markup on selling price percentage
e. Sale price
f. Markdown percentage

Solution

In order, you are looking for C, S, $MoC\%$, $MoS\%$, S_{onsale}, and d(markdown).

What You Already Know

Step 1:

The list price, two discounts (note the store qualifies for both since it meets the minimum quantity criteria), expense and profit information, as well as information on the sale price are known:

$$\begin{aligned}L &= \$82.00\\ d_1 &= 0.37\\ d_2 &= 0.12\\ E &= 0.31S\\ P &= 0.13S\end{aligned}$$

$$S_{onsale} = \text{break-even price}$$

The unknown variables are: C, S, $MoC\%$, $MoS\%$, S_{onsale}, and d(markdown).

How You Will Get There

Step 2:

a. Cost and net price are synonymous. Net price appears in Formula 6.1, Formula 6.2b, and Formula 6.3. Since you know list price and two discounts, calculate cost through Formula 6.3.
b. Selling price appears in Formula 6.5, Formula 6.7, Formula 6.9, Formula 6.10, Formula 6.11a, and Formula 6.11b. Known variables include expenses and profit. You will know the cost once you have solved part (a). Choose Formula 6.5.
c. Markup on cost percent appears only in Formula 6.8. You will know the cost once you have solved part (a). $M\$$ is an unknown variable.
d. Markup on selling price percent appears only in Formula 6.9. You will know the selling price once you have solved parts (a) and (b). $M\$$ is an unknown variable.
e. A break-even price uses an adapted Formula 6.5. You will know the cost once you have solved part (a) and determined the expenses based on the selling price in part (b).
f. Markdown percent appears in Formula 6.10, Formula 6.11a, and Formula 6.12. You will know the regular price once you have solved part (b). You will know the sale price once you have solved part (e). Choose Formula 6.12.

Step 3:

From the above rationalization, the only unknown variables are cost and markup dollars. Once you know these two variables you can complete all other formulas. Calculate cost as described in step 2a. To calculate markup dollars, use Formula 6.6, Formula 6.7, Formula 6.8, and Formula 6.9. Once you know the cost and selling price from parts 2(a) and 2(b), you can apply Formula 6.7, rearranging for markup dollars.

Perform

Step 4:

a. $C = N = \$82 \times (1–0.37) \times (1–0.12) = \$45.46$

b. $$\begin{aligned}S &= \$45.46 + 0.31S + 0.13S\\ 0.56S &= \$45.46\\ S &= \$81.18\end{aligned}$$

c. As determined in step 3, you need markup dollars first.

$$\begin{aligned}\$81.18 &= \$45.46 + M\$\\ \$35.72 &= M\$\end{aligned}$$

Now Solve:

$$MoC\% = \frac{\$35.72}{\$45.46} \times 100 = 78.5746\%$$

d. $MoS\% = \dfrac{\$35.72}{\$81.18} \times 100 = 44\%$

e. $S_{onsale} = S_{BE} = \$45.46 + 0.31(\$81.18) = \$70.63$

f.

$$\begin{aligned}\$70.63 &= \$81.18 \times (1-d)\\ 0.87 &= 1-d\\ 0.13 &= d\end{aligned}$$

Calculator Instructions

For part (d) only, use 2nd Profit

CST	SEL	MAR
45.46	81.18	Answer: 44

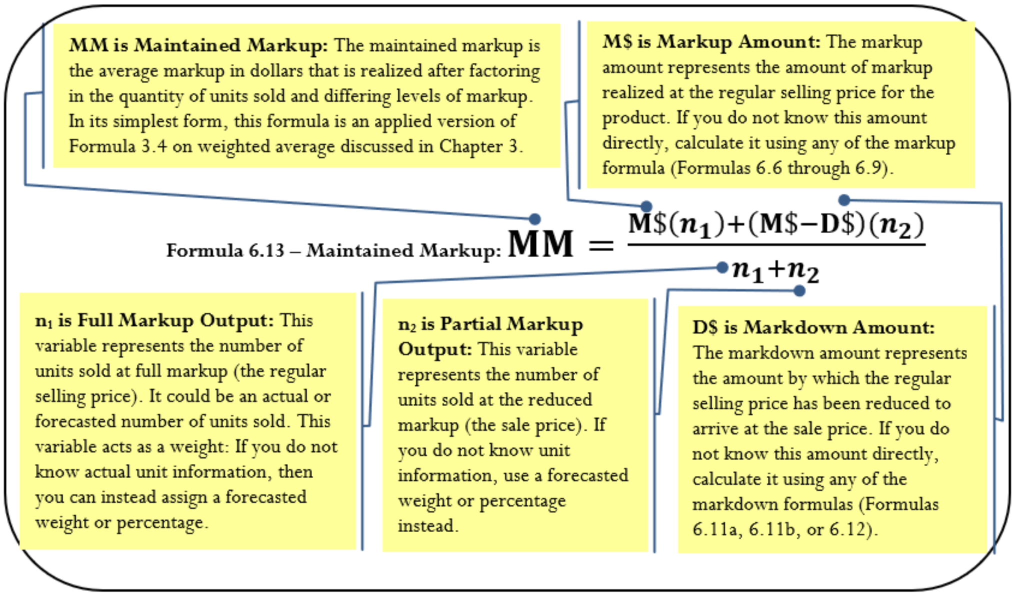
The cost of the skateboard is \$45.46. The regular selling price for the skateboard is \$81.18. This is a markup on cost percent of 78.5746% and a markup on selling price percent of 44%. To clear out the skateboards, the sale price is \$70.63, representing a markdown of 13%.

Maintained Markup

A price change results in a markup change. Whenever a selling price is reduced to a sale price, it has been demonstrated that the amount of the markup decreases (from $M\$$ to $M\$_{onsale}$) by the amount of the discount ($D\$$). This means that some of the products are sold at full markup and some at a partial markup. For businesses, the strategic marking up of a product must factor in the reality that they do not sell every product at the same level of markup. Therefore, if a business must achieve a specified level of markup on its products, it must set its price such that the average of all markups realized equals its markup objective. The average markup that is realized in selling a product at different price points is known as the maintained markup.

The Formula

To understand the formula, assume a company has 100 units for sale. From past experience, it knows that 90% of the products sell for full price and 10% sell at a sale price because of discounts, damaged goods, and so on. At full price, the product has a markup of \$10, and 90 units are sold at this level, equaling a total markup of \$900. On sale, the product is discounted \$5, resulting in a markup of \$5 with 10 units sold, equaling a total markup of \$50. The combined total markup is \$950. Therefore, the maintained markup is $\$950 \div 100 = \$9.50$ per unit. If the company's goal is to achieve a \$10 maintained markup, this pricing structure will not accomplish that goal. Formula 6.13 summarizes the calculations involved with maintained markup.



MM is Maintained Markup: The maintained markup is the average markup in dollars that is realized after factoring in the quantity of units sold and differing levels of markup. In its simplest form, this formula is an applied version of Formula 3.4 on weighted average discussed in Chapter 3.

M\$ is Markup Amount: The markup amount represents the amount of markup realized at the regular selling price for the product. If you do not know this amount directly, calculate it using any of the markup formula (Formulas 6.6 through 6.9).

Formula 6.13 – Maintained Markup: $$\mathbf{MM} = \frac{\mathbf{M\$}(n_1)+(\mathbf{M\$}-\mathbf{D\$})(n_2)}{n_1+n_2}$$

n_1 is Full Markup Output: This variable represents the number of units sold at full markup (the regular selling price). It could be an actual or forecasted number of units sold. This variable acts as a weight: If you do not know actual unit information, then you can instead assign a forecasted weight or percentage.

n_2 is Partial Markup Output: This variable represents the number of units sold at the reduced markup (the sale price). If you do not know unit information, use a forecasted weight or percentage instead.

D\$ is Markdown Amount: The markdown amount represents the amount by which the regular selling price has been reduced to arrive at the sale price. If you do not know this amount directly, calculate it using any of the markdown formulas (Formulas 6.11a, 6.11b, or 6.12).

Formula 6.13

Important Notes

Formula 6.13 assumes that only one price reduction applies, implying that there is only one regular price and one sale price. In the event that a product has more than one sale price, you can expand the formula to include more terms in the numerator. To expand the formula for each additional sale price required:

1. Extend the numerator by adding another $(\mathrm{M\$}-\mathrm{D\$})(n_x)$, where the $\mathrm{D\$}$ always represents the total difference between the selling price and the corresponding sale price and n_x is the quantity of units sold at that level of markup.
2. Extend the denominator by adding another n_x as needed.

For example, if you had the regular selling price and two different sale prices, Formula 6.13 appears as:

$$MM=\frac{M\$\left(n_1\right)+\left(M\$-D\$_1\right)\left(n_2\right)+\left(M\$-D\$_2\right)\left(n_3\right)}{n_1+n_2+n_3}$$

How It Works

Follow these steps to calculate the maintained markup:

Step 1: Identify information about the merchandising scenario. You must either know markup dollars at each pricing level or identify all merchandising information such as costs, expenses, profits, and markdown information. Additionally, you must know the number of units sold at each price level. If unit information is not available, you will identify a weighting based on percentages.

Step 2: If step 1 already identifies the markup dollars and discount dollars, skip this step. Otherwise, you must use any combination of merchandising formulas (Formula 6.1 to Formula 6.12) to calculate markup dollars and discount dollars.

Step 3: Calculate maintained markup by applying Formula 6.13.

? Exercise $6.4.1$: Give It Some Thought

A company's pricing objective is to maintain a markup of $5. It knows that some of the products sell at regular price and some of the products sell at a sale price. When setting the regular selling price, will the markup amount be more than, equal to, or less than $5?

Answer

> More than $5. In order to average to $5, the markup at regular price must be higher than $5 and the markup at sale price must be lower than $5.

✓ Example $6.4.2$: What Level of Markup Is Maintained?

A grocery store purchases a 10 kg bag of flour for $3.99 and resells it for $8.99. Occasionally the bag of flour is on sale for $6.99. For every 1,000 units sold, 850 are estimated to sell at the regular price and 150 at the sale price. What is the maintained markup on the flour?

Solution

You are looking to calculate the maintained markup (MM).

What You Already Know

Step 1:

The cost, selling price, and sale price information along with the units sold at each price are known:

$C=\$3.99, S=\$8.99, S_{onsale}=\$6.99, n_1=850, n_2=150$

How You Will Get There

Step 2:

To calculate markup dollars, apply Formula 6.7, rearranging for $M\$$. To calculate markdown dollars, apply Formula 6.11b.

Step 3: Calculate maintained markup by applying Formula 6.13.

Perform

Step 2:

Markup amount:

$$\begin{gathered}\$8.99=\$3.99+C\\ C=\$5.00\end{gathered}$$

Markdown amount:

$$D\$=\$8.99-\$6.99=\$2.00$$

Step 3:

$$MM = \frac{\$5(850) + (\$5 - \$2)(150)}{850 + 150} = \$4.70$$

On average, the grocery store will maintain a $4.70 markup on the flour.

✓ Example 6.4.3: Setting Prices to Achieve the Maintained Markup

Maplewood Golf Club has set a pricing objective to maintain a markup of $41.50 on all 18-hole golf and cart rentals. The cost of providing the golf and rental is $10. Every day, golfers before 3 p.m. pay full price while “twilight” golfers after 3 p.m. receive a discounted price equalling $30 off. Typically, 25% of the club’s golfers are “twilight” golfers. What regular price and sale price should the club set to meet its pricing objective?

Solution

The golf club needs to calculate the regular selling price (S) and the sale price (S_{onsale}).

What You Already Know

Step 1:

The cost information along with the maintained markup, markdown amount, and weighting of customers in each price category are known:

$C = \$10, MM = \$41.50, D\$ = \$30, n_1 = 100\% - 25\% = 75\%, n_2 = 25\%$

How You Will Get There

Reverse steps 2 and 3 since you must solve Formula 6.13 first and use this information to calculate the selling price and sale price.

Step 3:

Apply Formula 6.13, rearranging for $M\$$.

Step 2:

Calculate the selling price by applying Formula 6.7.

Step 2 (continued):

Calculate the sale price by applying Formula 6.11b, rearranging for (S_{onsale}).

Perform

Step 3:

$$\begin{aligned}\$41.50 &= \frac{M\$(0.75) + (M\$ - \$30)(0.25)}{0.75 + 0.25} \\ \$41.50 &= 0.75M\$ + 0.25M\$ - \$7.50 \\ \$49 &= M\$\end{aligned}$$

Step 2:

$$S = \$10 + \$49 = \$59$$

Step 2 (continued):

$$\begin{aligned}\$30 &= \$59 - S_{onsale} \\ S_{onsale} &= \$29\end{aligned}$$

In order to achieve its pricing objective, Maplewood Golf Club should set a regular price of $59 and a “twilight” sale price of $29.

The Pricing Impact of Coupons and Mail-in Rebates

Consumers sometimes need a nudge to help them decide on a purchase. Have you ever been thinking about buying a product but put it off for a variety of reasons? Then, while browsing online one day, you suddenly come across a coupon. You print it off and hurry down to your local store, because the coupon expires in the next few days. You just participated in a price adjustment.

A marketing price adjustment is any marketing activity executed by a member of the distribution channel for the purpose of altering a product's price. This includes an alteration to the regular unit selling price or a sale price. For example, Sony could distribute a rebate giving its consumers $50 off the price of one of its televisions. Alternatively, Superstore could run a special promotion where customers receive $30 off if they purchase at least $250 in merchandise.

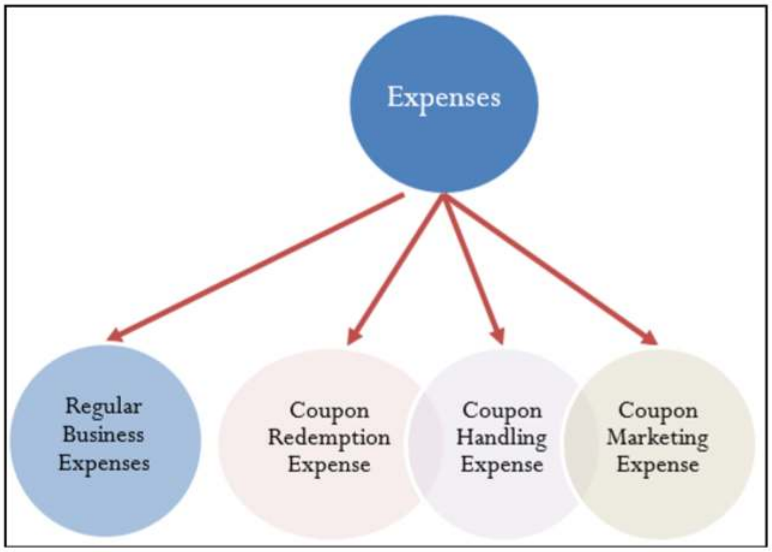
When making price adjustments, businesses must understand the full impact on the bottom line. Many inexperienced businesses distribute coupons for $1 off, assuming that the only cost to be accounted for is the $1. This is untrue for most price adjustments.

These marketing price adjustments reduce the prices that you pay as a consumer. Certainly, some price adjustments are more advantageous than others. Although they come in many types, this section focuses on two of the most widely used price adjustments: manufacturer coupons and mail-in rebates.

Manufacturer Coupons

A coupon is a promotion that entitles a consumer to receive a certain benefit, most commonly in the form of a price reduction. Coupons are typically offered by manufacturers to consumers.

The decision of a manufacturer to issue a coupon affects the expenses associated with the product in three ways.



1. **Coupon Redemption Expense**. The coupon redemption expense is the dollar amount price reduction that the coupon offers, also known as the face value of the coupon. The manufacturer must pay out this amount to the reseller as a reimbursement for redeeming the coupon, increasing the expenses accordingly. For example, if a coupon is for $3 off, then the coupon redemption expense is $3.
2. **Coupon Handling Expense**. Most commonly, retailers are the resellers that redeem manufacturer coupons. The resulting coupon handling expense for the manufacturer is known as a handling fee or handling charge, payable to the retailer. The manufacturer offers financial compensation to the retailer for redeeming the coupon, filling out the paperwork, and submitting it to manufacturer for reimbursement. Ultimately, the retailer should not be out any money for participating in the promotion. For example, in the small print a coupon may indicate that there is a $0.09 handling fee. This amount is payable to the retailer upon the manufacturer's receipt of the redeemed coupon.
3. **Coupon Marketing Expense**. Coupons do not just magically appear in consumer hands. Companies must pay for the creation, distribution, and redemption of the coupons, which is known as the coupon marketing expense:
    - Coupons are designed, turned into print-ready artwork, and printed. This incurs design, labor, and material costs.
    - Coupons are commonly distributed separately from the product in media such as magazines, flyers, and newspapers. This requires the purchase of ad space.

- Coupons are commonly redeemed through coupon clearinghouses that handle coupon redemption requests. Manufacturers must pay the fees charged by the clearinghouse.

The manufacturer issuing the coupon estimates how many coupons it expects consumers will redeem and assigns the average of the coupon marketing costs to each unit based on the forecasted volume.

It should be evident that the decision to issue coupons results in expenses much higher than the face value of the coupon. All three of the expenses listed above are added to the product's regular expenses. The end result is that the unit expenses increase while unit profits decrease.

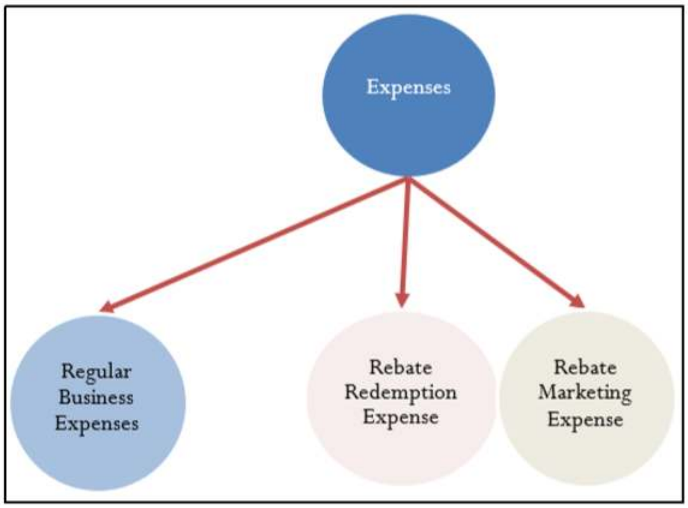
As a consumer, you can appreciate the immediacy of the price reduction a coupon offers you at the cash register. Unlike with mail-in rebates, there is nothing further you need to do.

Mail-In Rebates

Have you ever purchased an item that came with a mail-in rebate but not submitted the rebate because it was too much hassle? Was your rebate refused because you failed to meet the stringent requirements? If you answered yes to either of these questions, then you have joined the ranks of many other consumers. In fact, marketers know you will either not mail in your rebates (some estimates are that 50% to 98% of mail-in rebates go unredeemed), or you will fail to send the required supporting materials. From a pricing point of view, this means that more product is sold at the regular price! That is why marketers use mail-in rebates.

Mail-in rebates are similar to coupons; however, there is a distinct difference in how the price adjustment works. A mail-in rebate is a refund at a later date after the purchase has already been made, whereas coupons instantly reward consumers with a price reduction at the cash register. Mail-in rebates commonly require consumers to fill in an information card as well as meet a set of stringent requirements (such as cutting out UPC codes and providing original receipts) that must be mailed in to the manufacturer at the consumer's expense (see the photo for an example). After a handling time of perhaps 8 to 12 weeks and if the consumer has met all requirements, the manufacturer issues a cheque in the amount of the rebate and sends it to the consumer.

With coupons there are three associated expenses: coupon redemption, handling, and marketing. In the case of rebates, since the participants in the channel are not involved with the rebate redemption, no handling expenses are paid out to the retailers.

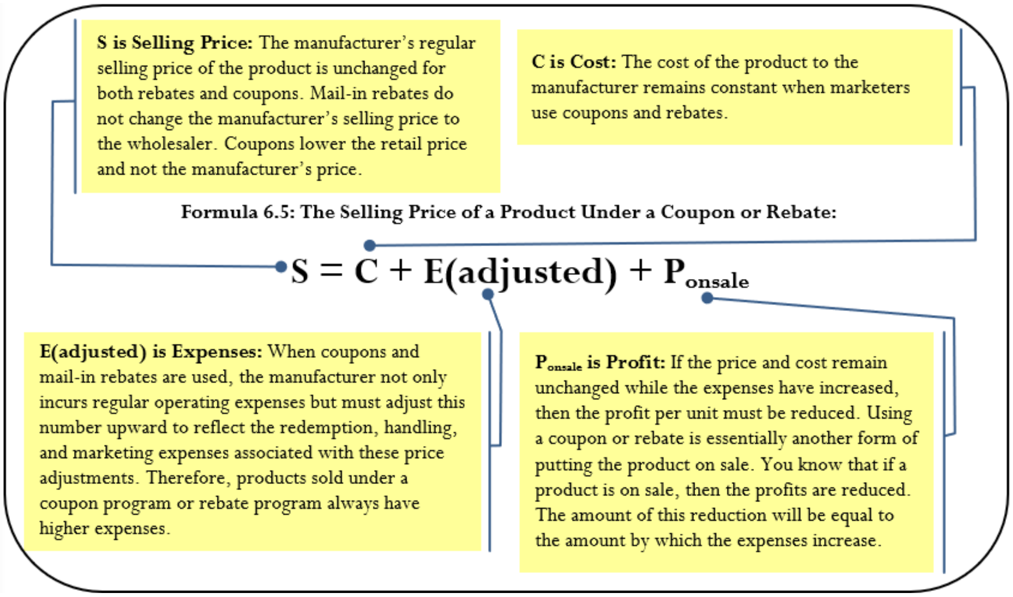


1. **Rebate Redemption Expense**. The rebate redemption expense is the dollar amount price reduction that the rebate offers, also known as the face value of the rebate. This amount must be paid out to the consumer, commonly in the form of a cheque. For example, if a rebate is for $50, then this is the amount sent to the consumer. This increases expenses, but often not by the full $50. Recall that though many consumers purchase the product because the rebate is offered, not all of them redeem the rebate. To calculate the unit expense many companies average the rebate redemption amount based on the number of rebates they expect to fulfill. Take, for example, a company that issues a $50 rebate on the purchase of a surround sound system. From past experience, it knows that only 5% of the rebates are actually redeemed properly and thus paid out. Therefore, the increase in expenses is not $50, but 5% of $50, or $2.50 per unit sold under the rebate program.
2. **Rebate Marketing Expense**. Many mail-in rebates today are distributed either on or in the packaging of the product, or they can be located online. This tends to minimize (but not eliminate) the marketing expenses associated with creation, distribution,

and redemption. The rebate marketing expense is commonly averaged across the increase in sales that results from the rebate issuance and not the number or rebates redeemed. Therefore, if a company has $100,000 in rebate marketing expenses, a direct increase in sales by 20,000 units, and only 1,000 rebates redeemed, the rebate marketing expense is $\$100,000 \div 20,000 = \$5.00$ per unit under the rebate program (note that the rebates redeemed is irrelevant to this calculation).

The Formula

With coupons and rebates, the focus is on how these marketing tools affect expenses and profit. You do not require any new formula; instead, Formula 6.5 is adapted to reflect the pricing impact of these tactics.



S is Selling Price: The manufacturer's regular selling price of the product is unchanged for both rebates and coupons. Mail-in rebates do not change the manufacturer's selling price to the wholesaler. Coupons lower the retail price and not the manufacturer's price.

C is Cost: The cost of the product to the manufacturer remains constant when marketers use coupons and rebates.

Formula 6.5: The Selling Price of a Product Under a Coupon or Rebate:

$$S = C + E(\text{adjusted}) + P_{\text{onsale}}$$

E(adjusted) is Expenses: When coupons and mail-in rebates are used, the manufacturer not only incurs regular operating expenses but must adjust this number upward to reflect the redemption, handling, and marketing expenses associated with these price adjustments. Therefore, products sold under a coupon program or rebate program always have higher expenses.

P_{onsale} is Profit: If the price and cost remain unchanged while the expenses have increased, then the profit per unit must be reduced. Using a coupon or rebate is essentially another form of putting the product on sale. You know that if a product is on sale, then the profits are reduced. The amount of this reduction will be equal to the amount by which the expenses increase.

How It Works

Follow these steps to calculate the selling price or any of its components when working with coupons or rebates:

Step 1: Identify the known components involved in the selling price, such as the cost, unit expense, profit, or selling price. Use any formulas needed to calculate any of these components. Also identify information about the coupon or rebate, including redemption, handling, and marketing expenses.

Step 2: Calculate the unit expense associated with the marketing pricing tactic.

a. **Coupons**. The unit redemption expense is equal to the face value of the coupon. The unit handling expense is the promised handling fee of the coupon. The unit marketing expense is a simple average of the total marketing expense divided by the expected number of coupon redemptions. Calculate the total coupon expenses by summing all three elements.
b. **Rebates**. The unit redemption expense is equal to the face value of the coupon multiplied by the redemption rate of the rebate. The unit marketing expense is equal to a simple average of the total marketing expense divided by the expected increase in sales as a result of the rebate. Calculate the total rebate expenses by summing both elements.

Step 3: Apply the required merchandising formulas to calculate the unknown variable(s). The modified Formula 6.5 presented earlier will most likely play prominently in these calculations.

Quick Coupon Example

Assume a manufacturer coupon is issued with a face value of $5. The handling charge is $0.15 per coupon. The marketing expenses are estimated at $150,000, and the company forecasts that 100,000 coupons will be redeemed. The regular profit on the product is $20. How much will the unit expenses of the product increase, and what is the unit profit under the coupon program?

Step 1: You know the following information: P = $20, Face Value = $5, Handling Charge = $0.15; Total Marketing Expenses = $150,000, and Forecasted Coupon Redemption = 100,000.

Step 2: The increase in expenses is a sum of the redemption, handling, and marketing expenses. The redemption expense is $5. The handling expense is $0.15. To get the marketing expense per unit, take the total expense and divide it by the number of redemptions expected. Therefore $\$150,000 \div 100,000 = \$1.50$. The total increase in expenses is then $\$5.00 + \$0.15 + \$1.50 = \$6.65$.

Step 3: If the expenses rise by $6.65 with the same cost and selling price, then the profits fall by the same amount. The unit profit on any product sold using the coupon is $\$20.00 - \$6.65 = \$13.35$.

Quick Rebate Example

Assume a manufacturer mail-in rebate is issued with a face value of $20. The marketing expenses are estimated at $300,000 and the company plans on 10% of the mail-in rebates being redeemed. It is forecasted that sales will rise by 37,500 units as a result of the mail-in rebate. The regular profit on the product is $25. How much will the unit expenses of the product increase, and what is the unit profit under the rebate program?

Step 1: You know the following information: P= $25, Face Value = $20, Total Marketing Expenses = $300,000, Rebate Redemption Rate = 10%, and Increase in Sales = 37,500.

Step 2: The increase in expenses is a sum of the redemption and marketing expenses. The redemption expense is $20.00 multiplied by the 10% redemption rate, or $2. To calculate the marketing expense per unit, take the total expense and divide it by the increase in sales. Therefore $\$300,000 \div 37,500 = \$8$. The total increase in expenses is then $\$2 + \$8 = \$10$.

Step 3: If the expenses rise by $10 with the same cost and selling price, then the profits fall by the same amount. The unit profit on any product sold using the coupon is $\$25 - \$10 = \$15$.

Things To Watch Out For

When calculating the per-unit expense, be careful to distinguish how to handle the various associated expenses for both manufacturer coupons and mail-in rebates, as summarized in the table below.

Marketing Tactic	Redemption Expense	Handling Expense	Marketing Expense
Coupon	Face value of the coupon	A flat rate per coupon	The total expense divided by the number of coupons redeemed
Mail-In Rebate	Face value of the mail-in rebate multiplied by the redemption rate	N/A	The total expense divided by the increase in sales attributed to the mail-in rebate

Paths To Success

It pays to be smart when you deal with coupons and mail-in rebates as a consumer. Two important considerations are timing and hassle.

1. **Timing**. Coupon amounts are usually smaller than rebate amounts. You may be offered a $35 coupon (sometimes called an “instant rebate,” but should not be confused with rebates) at the cash register, or a $50 mail-in rebate. If you choose the former, you get the price savings immediately. If you choose the latter, you must spend the money up front and then wait 8 to 12 weeks (or more) to receive your rebate cheque.
2. **Hassle**. Mail-in rebates are subject to very strict rules for redemption. Most require original proofs of purchase such as your original cash register receipts, UPC codes, model numbers, and serial numbers (which are difficult to find sometimes). You must photocopy these for your records in the event there is a rebate problem. Specific dates of purchase and deadlines for redemption must be met. Some rebates, particularly computer software upgrades, even require you to send your old CD-ROM in with your purchase, substantially raising your postage and envelope costs. Does all that extra time and cost justify the rebate? For example, Benylin ran a sales promotion offering an instant coupon for $1; alternatively you could purchase two products and mail-in two UPC codes for a $3 rebate. When factoring in costs for postage, envelope, photocopying, and time, the benefit from the mail-in rebate is marginal at best.

? Exercise $6.4.2$: Give It Some Thought

What happens to the unit profit when a coupon or rebate is issued under the following scenarios:

a. The cost remains the same and the selling price increases

Answer

a. Profit decreases by the total unit expenses associated with the coupon or rebate.
b. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and the amount of the increased cost together.
c. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and increases by the amount of the price increase.
d. Profit decreases by the total unit expenses associated with the coupon or rebate and the amount of the increased cost together.

✓ Example $6.4.4$: The Impact of a Coupon

When the DVD edition of the Bee Movie was released in retail stores, DreamWorks Animation executed a coupon marketing program. Assume the unit cost of the DVD is \$2.50, the normal expenses associated with the DVD amount to \$1.25, and that DreamWorks sells it directly to retailers for \$10. The coupon offered a \$3 savings on the DVD to consumers and promised the retailer an \$0.08 handling fee per coupon redeemed. The estimated costs of creating, distributing, and redeeming the coupons are \$285,000, and the projected number of coupons redeemed is 300,000.

a. What is the unit profit of the DVD when it sold without the coupon?
b. What unit profit did DreamWorks realize for those DVDs sold under its coupon promotion?

Solution

You need to first calculate the profit (P) when no coupon was available. Second, you will calculate the impact of the coupon on the profit when the DVD is discounted under the coupon program (P_{onsale}).

What You Already Know

Step 1:

The unit cost of the DVD along with its normal associated expenses and regular unit selling price are known. The coupon details are:

$C=\$2.50, E=\$1.25, S=\$10.00$

Face Value = \$3

Handling Fee = \$0.08

Total Marketing Expense = \$285,000

Number of Redeemed Coupons = 300,000

How You Will Get There

Step 2:

When the coupon is issued, the expenses associated with the DVD rise to accommodate the coupon redemption expense, the handling expense, and the marketing expenses.

Step 3:

Calculate the profit normally realized on the DVD by applying Formula 6.5, rearranging for P.

Step 3 (continued):

Recalculate the profit under the coupon program by adapting Formula 6.5, rearranging for (P_{onsale}).

Perform

Step 2:

$$\text{Unit coupon expense } = \$3 + \$0.08 + (\$285,000 \div 300,000) = \$4.03$$

Step 3:

$$\begin{aligned}\$10.00 &= \$2.50 + \$1.25 + P \\ \$6.25 &= P\end{aligned}$$

Step 3 (continued):

$$\begin{aligned}\$10.00 &= \$2.50 + (\$1.25 + \$4.03) + P_{onsale} \\ \$10.00 &= \$7.78 + P_{onsale} \\ \$2.22 &= P_{onsale}\end{aligned}$$

Without the coupon promotion a Bee Movie DVD generates \$6.25 in profit. Under the coupon promotion, DreamWorks achieves a reduced profit per DVD of \$2.22.

✓ Example $6.4.5$: The Impact of a Mail-In Rebate

Computer manufacturers commonly issue rebates on items such as printers and monitors. Assume LG decided to distribute its new 22” widescreen LCD monitor (MSRP: \$329.99) with a prepackaged mail-in rebate included inside the box. The rebate is \$75. The company predicts a 25% rebate redemption rate. Its unit cost to produce the monitor is \$133.75, with expenses equaling 35% of cost. It sells the monitor directly to retailers for a regular unit selling price of \$225. The increased marketing expenses for promotion and rebate redemption equal \$8.20 per monitor. What is the profit for a monitor sold through the rebate program?

Solution

You are looking for the profit ((P_{onsale})) for a monitor sold under the rebate program. This profit must factor in the rebate face value, redemption rate, and marketing expenses.

What You Already Know

Step 1:

The unit cost, unit expenses, and regular unit selling price are known. The rebate details are:

$C = \$133.75, E = 0.35C, S = \$225.00$

Face Value = \$75

Redemption rate = 25%

Unit Marketing Expenses = \$8.20

How You Will Get There

Step 2:

When the rebate is issued, the expenses associated with the monitor rise to accommodate the rebate redemption expenses and the marketing expenses.

Step 3:

Apply an adjusted Formula 6.5, rearranging for (P_{onsale}).

Step 2:

$$\text{Unit rebate expense } = \$75 \times 25\% + \$8.20 = \$18.75 + 8.20 = \$26.95$$

Step 3:

$$\begin{aligned}\$225.00 &= \$133.75 + (\$46.81 + \$26.95) + P_{onsale} \\ \$225.00 &= \$207.51 + P_{onsale} \\ \$17.49 &= P_{onsale}\end{aligned}$$

> The rebate redemption expense for the LG monitor is \$18.75. LG averages a profit of \$17.49 per monitor sold under its mail-in rebate program.

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6.S: Summary

Key Concepts Summary

6.1: Figuring Out the Cost: Discounts (How Much?)

- The relationship between distribution and pricing
- Some of the types of discounts available to businesses and consumers
- How to calculate the net price when only one discount is involved
- How to calculate the net price when multiple discounts are involved
- Converting multiple discounts into single discounts

6.2: Markup: Setting the Regular Price (Need to Stay in Business)

- The three components that compose a selling price
- Calculating the markup in dollars and the relationship to pricing components
- Calculating the markup as a percentage under two different approaches
- Determining the price point where all costs are paid but no profits are earned—the break-even point

6.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)

- How to take a regular selling price and make it into a sale price by applying markdowns
- How to plan the merchandising of products that are always on sale

6.4: Merchandising (How Does It All Come Together?)

- Combining discounts, markup, and markdown into complete merchandising situations
- Calculating the maintained markup when a product was sold at both a regular selling price and a sale price
- How coupons and mail-in rebates impact product pricing

The Language of Business Mathematics

cost

An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

coupon

A promotion that entitles a consumer to receive certain benefits, usually in the form of a reduction of the selling price for a product.

coupon handling expense

A handling charge that is paid to channel members for redeeming a coupon.

coupon marketing expense

The expense associated with creating, distributing, and redeeming a coupon.

coupon redemption expense

The face value price reduction offered by a coupon.

discount

A reduction in the price of a product.

expenses

A business’s financial outlays incurred in the selling of a product.

list price

A price for a product that has been published or advertised in some way.

loyalty discount

A discount given from a seller to a purchaser for repeat business.

mail-in rebate

A refund that occurs after a product has been purchased.

maintained markup

The average level of markup that is maintained across all units sold at various price levels including the selling price and the sale price(s).

manufacturer's suggested retail price (MSRP)

A recommended product retail price that a manufacturer sets for a retailer based on market research.

markdown

A reduction from the regular selling price of a product resulting in a new lower sale price.

marketing price adjustment

Any marketing activity executed by a member of the distribution channel for the purposes of altering a product’s price.

markup

The process of taking a product’s cost and increasing it by a certain amount to arrive at a selling price.

markup amount

The dollar amount of the expenses and profit combined together into a single number; it represents the difference between the price and cost in dollars.

markup on cost percentage

The markup dollars expressed as a rate using cost as the base.

markup on selling price percentage

The markup dollars expressed as a rate using the regular selling price as the base.

net price

The price of the product after a discount is removed from the list price.

profit

The amount of money that remains after a business pays all of its costs and expenses.

quantity discount

A discount for purchasing larger quantities of a certain product.

rebate marketing expense

The expense associated with creating, distributing, and redeeming a rebate.

rebate redemption expense

The face value amount that a consumer will receive as a refund if a submitted rebate fulfills all rebate conditions.

sale discount

A temporary discount lowering the price from a product’s regular selling price.

sale price

A price for a product after a markdown that is lower than its regular selling price.

seasonal discount

A discount offered to consumers and businesses for purchasing products out of season.

single equivalent discount

A single discount rate that is equal to a series of multiple rate discounts.

trade discount

A discount offered to businesses only based on the type of business and its location in the distribution system.

The Formulas You Need to Know

Symbols Used

C = cost

d = discount rate or markdown rate

$d_1, d_2, ...d_n$ = multiple discount rates, where the subscript represents each discount up to a count of n discounts

$D\$$ = discount amount or markdown amount

E = expenses

L = list price

$M\$$ = markup amount

MM = maintained markup

$MoC\%$ = markup on cost percentage

$MoS\%$ = markup on selling price percentage

n = number of pieces of data, which in this chapter is the level of output

N = net price

P = profit

P_{onsale} = planned profit amount when a product is sold at the sale price

S = regular selling price

S_{BE} = the selling price at the break-even point

S_{onsale} = sale price

Formulas Introduced

Formula 6.1 Single Discount: $N = L \times (1 - d)$

Formula 6.2a Discount Amount: $D\$ = L \times d$

Formula 6.2b Discount Amount: $D\$ = L - N$

Formula 6.3 Multiple Discounts: $N = L \times (1 - d_1) \times (1 - d_2) \times ... \times (1 - d_n)$

Formula 6.4 Single Equivalent Discount: $d_{equiv} = 1 - (1 - d_1) \times (1 - d_2) \times ... \times (1 - d_n)$

Markup Formulas

Formula 6.5 The Selling Price of a Product: $S = C + E + P$

Formula 6.6 Markup Amount: $M\$ = E + P$

Formula 6.7 The Selling Price of a Product Using the Markup Amount: $S = C + M\$$

Formula 6.8 Markup on Cost Percentage: $MoC\% = \dfrac{M\$}{C} \times 100$

Formula 6.9 Markup on Selling Price Percentage: $MoS\% = \dfrac{M\$}{S} \times 100$

Formula 6.13 Maintained Markup: $MM = \frac{M\$\left(n_1\right) + \left(M\$ - D\$\right)\left(n_2\right)}{n_1 + n_2}$

Markdown Formulas

Formula 6.10 The Sale Price of a Product: $S_{onsale} = S \times (1 - d)$

Formula 6.11a Markdown Amount: $D\$ = S \times d$

Formula 6.11b Markdown Amount: $D\$ = S - S_{onsale}$

Formula 6.12 Markdown Percentage: $d = \frac{D\$}{S} \times 100$

Technology

Calculator

The following calculator functions were introduced in this chapter:

Markup on Selling Price Percentage

2^{nd} Profit to access this feature.

Enter two of the three variables by pressing Enter after each input and using and ¯ to scroll through the display. The variables are:

$\text{CST} = \text{ The cost of the item}$

$\text{SEL} = \text{ The selling price of the item}$

MAR = The markup on selling price percentage (in % format)

Press CPT on the unknown (when it is on the screen display) to compute.

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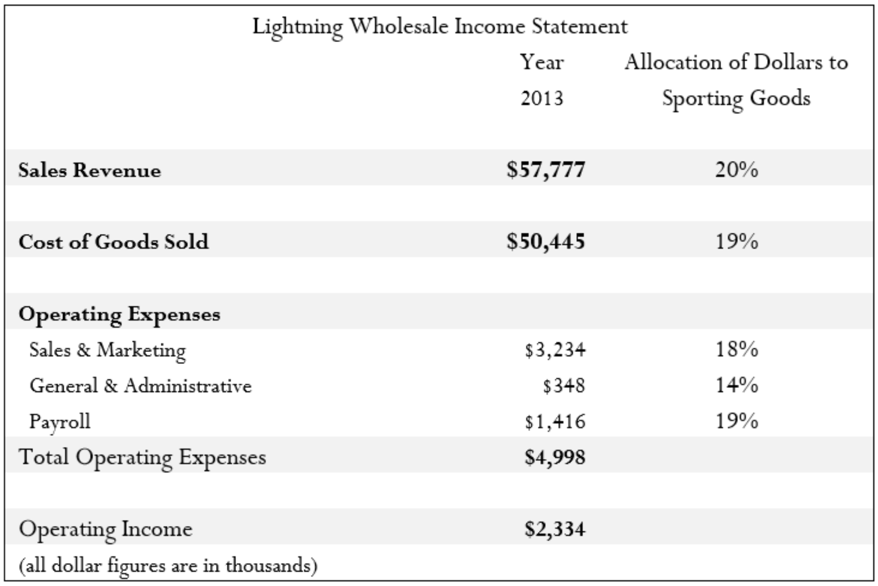
6.CS: Case Study (Making a Profitable Product Selection)

The Situation

Lightning Wholesale is looking at changing its ski product line for 2014. It wants to carry only two brands of skis. It has received four offers from four different ski manufacturers. The details of the offers are in the first table. It will select the two brands that offer the highest total profit. The accounting department has analyzed 2013 sales and provided the income statement summarized in the second table, which specifies the percentage of each category that is attributed to the sporting goods department.

The Data

	Nordica	Fischer	Ogasaka	Atomic
List Price	$799.95	$914.99	$829.95	$839.99
Discounts:				
Retail	40%	40%	40%	40%
Wholesale	5%	11%	6%	7%
Seasonal	3%	5%		
Sale	5%		5%	10%
Loyalty	3%		5%	
Quantity	3%	4%	5%	2%



Lightning Wholesale Income Statement

	Year 2013	Allocation of Dollars to Sporting Goods
Sales Revenue	**$57,777**	20%
Cost of Goods Sold	**$50,445**	19%
Operating Expenses		
Sales & Marketing	$3,234	18%
General & Administrative	$348	14%
Payroll	$1,416	19%
Total Operating Expenses	**$4,998**	
Operating Income	**$2,334**	

(all dollar figures are in thousands)

Important Information

- All of Lightning Wholesale's retail customers plan to sell the skis at the list price.
- Lightning Wholesale prices all of its skis on the assumption that all of its retailers use the standard industry markup of 40% of the regular selling price.
- Skis fall into the sporting goods category for Lightning Wholesale.
- Expenses are assigned to products based on the percentage of cost method for each category.
- Forecasted sales for each brand (if carried by Lightning Wholesale) in 2014 are as follows:
  - Nordica = 380 units
  - Fischer = 270 units

- Ogasaka = 310 units
- Atomic = 300 units

Your Tasks

1. Based strictly on financial considerations, recommend which two brands Lightning Wholesale should carry in 2014.
    a. Using the appropriate table, calculate the net price (cost) of each brand of skis.
    b. To determine the expenses for each brand of skis:
        i. Using the appropriate table, calculate the dollar values assigned to sporting goods for each row by multiplying the company total by the allocation percentage.
        ii. Total the dollar amount of the expenses.
        iii. Convert the total expenses into a percentage of cost.
        iv. Calculate the expenses for each brand using the percentage of cost.
    c. Using the appropriate table, calculate the wholesale price (the amount at which Lightning Wholesale will sell the skis to a retailer).
    d. For each brand, calculate the profit per unit.
    e. Calculate the total profitability for each brand.
2. At the end of the year, if any inventories are left over, Lightning Wholesale clears out all ski products at break-even.
    a. Determine the break-even prices for the two recommended brands.
    b. Determine the markdown percentage that can be advertised for each recommended brand.

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6.E: Marketing Applications (Exercises)

6.1 Exercises

Round all money to two decimals and percentages to four decimals in each of the following questions.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. "N/A" indicates that the particular variable is not applicable in the question.

List Price or MSRP	First Discount	Second Discount	Third Discount	Net Price	Equivalent Single Discount Rate	Total Discount Amount
1. $980.00	42%	N/A	N/A	?	N/A	?
2. ?	25%	N/A	N/A	$600.00	N/A	?
3. $1,975.00	25%	15%	10%	?	?	?
4. ?	18%	4%	7%	$366.05	?	?

Applications

5. A wholesaler of stereos normally qualifies for a 35% trade discount on all electronic products purchased from its manufacturer. If the MSRP of a stereo is $399.95, what net price will the wholesaler pay?
6. Mary is shopping at the mall where she sees a sign that reads, "Everything in the store is 30% off, including sale items!" She wanders in and finds a blouse on the clearance rack. A sign on the clearance rack states, "All clearance items are 50% off." If the blouse is normally priced at $69.49, what price should Mary pay for it?
7. A distributor sells some shoes directly to a retailer. The retailer pays $16.31 for a pair of shoes that has a list price of $23.98. What trade discount percent is the distributor offering to its retailers?
8. A retailer purchases supplies for its head office. If the retailer pays $16.99 for a box of paper and was eligible for a 15% volume discount, what was the original MSRP for the box of paper?
9. Mountain Equipment Co-op has purchased a college backpack for $29 after discounts of 30%, 8%, and 13%. What is the MSRP for the backpack? What single discount is equivalent to the three discounts?
10. Walmart purchased the latest CD recorded by Selena Gomez. It received a total discount of $10.08 off the MSRP for the CD, which represents a discount percent of 42%.
    a. What was the MSRP?
    b. What was the net price paid for the CD?
11. Best Buy just acquired an HP Pavilion computer for its electronics department. The net price on the computer is $260.40 and Best Buy receives discounts of 40% and 38%.
    a. What single discount is equivalent to the two discounts?
    b. What is the list price?
    c. What is the total discount amount?
12. TELUS retails a Samsung cellphone at the MSRP of $399.99. TELUS can purchase the phone from its supplier and receive a 20% trade discount along with a 5% volume discount.
    a. What is the single equivalent percent discount?
    b. What net price does TELUS pay for the phone?
    c. How much of a discount in dollars does this represent?
13. 13. A wholesaler offers the following discounts: 10% seasonal discount for all purchases made between March 1 and May 1, 15% cumulative quantity discount whenever more than 5,000 units are purchased in any month, 5% loyalty discount for customers who have made regular purchases every month for at least one year, and a 33% trade discount to any retailer. Ed's Retail Superstore makes a purchase of 200 watches, MSRP $10, from the wholesaler on April 29. This month alone, Ed's has ordered more than 5,000 watches. However, Ed's has purchased from the wholesaler for only the past six months. Determine the total price that Ed's should pay for the watches.

14. If a distributor is eligible for a 60% trade discount, 5% volume discount, and 3% seasonal discount, what single equivalent discount rate would it be eligible to receive? If the trade discount is applied first and equals a trade discount of $48, calculate the net price for the item.

Challenge, Critical Thinking, & Other Applications

As mentioned in one of the "Paths to Success" sections, discount percentages share a commonality with negative percent changes (Section 3.1**). Use the formulas from this chapter to solve questions 15–17 involving percent change.**

15. A human resource manager needs to trim labour costs in the following year by 3%. If current year labour costs are $1,231,498, what are the labour costs next year?
16. At an accounting firm, the number of accountants employed is based on the ratio of 1:400 daily manual journal entries. Because of ongoing increases in automation, the number of manual journal entries declines at a constant rate of 4% per year. If current entries are 4,000 per year, how many years and days will it take until the firm needs to lay off one accountant? (Hint: An accountant is laid off when the number of journal entries drops below 3,600.)
17. An economist is attempting to understand how Canada reduced its national debt from 1999 to 2008. In 1999, Canada's national debt was $554.143 billion. In 2008, the national debt stood at $457.637 billion. What percentage had the national debt been reduced by during this time period?
18. Sk8 is examining an invoice. The list price of a skateboard is $109.00, and the invoice states it received a trade discount of 15% and quantity discount of 10% as well as a loyalty discount. However, the amount of the loyalty discount is unspecified.
    a. If Sk8 paid $80.88 for the skateboard, what is the loyalty discount percent?
    b. If the loyalty discount is applied after all other discounts, what amount of loyalty dollars does Sk8 save per skateboard?
19. Currently, a student can qualify for up to six different tuition discounts at a local college based on such factors as financial need or corporate sponsorships. Mary Watson just applied to the college and qualifies for all six discounts: 20%, 15%, 23%, 5%, 3%, and 1%.
    a. She is confused and wants the college to tell her what single discount percent she is receiving. What should the college tell her?
    b. If her total list tuition comes to $6,435.00, how much should she pay?
20. Sumandeep is very loyal to her local hairstylist. Because she is loyal, her hairstylist gives her three different discounts: 10%, 5%, and 5%. These discounts amount to $14.08 in savings.
    a. What was the list price her hairstylist charged her?
    b. What amount did she pay her hairstylist?
    c. If her hairstylist increases prices by 5%, what are the list price, net price, and total discount amount?

6.2 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–8, solve for the unknown variables (identified with a ?) based on the information provided.

Regular Unit Selling Price	Cost	Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price
1. ?	$188.42	$48.53	$85.00	?	?	?	?
2. $999.99	?	30% of C	23% of C	?	?	?	?
3. ?	?	?	10% of S	$183.28	?	155%	?
4. $274.99	?	20% of S	?	?	?	?	35%
5. ?	?	45% of C	?	$540.00	$1,080.00	?	?
6. ?	$200 less 40%	?	15% of S	?	?	68%	?
7. ?	?	$100.00	?	$275.00	?	?	19%

Regular Unit Selling Price	Cost	Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price
8. ?	?	15% of C	12% of S	?	$253.00	?	?

Applications

9. If a pair of sunglasses sells at a regular unit selling price of $249.99 and the markup is always 55% of the regular unit selling price, what is the cost of the sunglasses?
10. A transit company wants to establish an easy way to calculate its transit fares. It has determined that the cost of a transit ride is $1.00, with expenses of 50% of cost. It requires $0.75 profit per ride. What is its markup on cost percentage?
11. Daisy is trying to figure out how much negotiating room she has in purchasing a new car. The car has an MSRP of $34,995.99. She has learned from an industry insider that most car dealerships have a 20% markup on selling price. What does she estimate the dealership paid for the car?
12. The markup amount on an eMachines desktop computer is $131.64. If the machine regularly retails for $497.25 and expenses average 15% of the selling price, what profit will be earned?
13. Manitoba Telecom Services (MTS) purchases an iPhone for $749.99 less discounts of 25% and 15%. MTS's expenses are known to average 30% of the regular unit selling price.
    a. What is the regular unit selling price if a profit of $35 per iPhone is required?
    b. What are the expenses?
    c. What is the markup on cost percentage?
    d. What is the break-even selling price?
14. A snowboard has a cost of $79.10, expenses of $22.85, and profit of $18.00.
    a. What is the regular unit selling price?
    b. What is the markup amount?
    c. What is the markup on cost percentage?
    d. What is the markup on selling price percentage?
    e. What is the break-even selling price? What is the markup on cost percentage at this break-even price?

Challenge, Critical Thinking, & Other Applications

15. A waterpark wants to understand its pricing better. If the regular price of admission is $49.95, expenses are 20% of cost, and the profit is 30% of the regular unit selling price, what is the markup amount?
16. Sally works for a skateboard shop. The company just purchased a skateboard for $89.00 less discounts of 22%, 15%, and 5%. The company has standard expenses of 37% of cost and desires a profit of 25% of the regular unit selling price. What regular unit selling price should Sally set for the skateboard?
17. If an item has a 75% markup on cost, what is its markup on selling price percentage?
18. A product received discounts of 33%, 25%, and 5%. A markup on cost of 50% was then applied to arrive at the regular unit selling price of $349.50. What was the original list price for the product?
19. Mountain Equipment Co-op (MEC) wants to price a new backpack. The backpack can be purchased for a list price of $59.95 less a trade discount of 25% and a quantity discount of 10%. MEC estimates expenses to be 18% of cost and it must maintain a markup on selling price of 35%.
    a. What is the cost of backpack?
    b. What is the markup amount?
    c. What is the regular unit selling price for the backpack?
    d. What profit will Mountain Equipment Co-op realize?
    e. What happens to the profits if it sells the backpack at the MSRP instead?
20. Costco can purchase a bag of Starbucks coffee for $20.00 less discounts of 20%, 15%, and 7%. It then adds a 40% markup on cost. Expenses are known to be 25% of the regular unit selling price.
    a. What is the cost of the coffee?
    b. What is the regular unit selling price?
    c. How much profit will Costco make on a bag of Starbucks coffee?
    d. What markup on selling price percentage does this represent?

e. Repeat questions (a) through (d) if the list price changes to $24.00.

6.3 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

Regular Selling Price	Markdown Amount	Markdown Percent	Sale Price
1. $439.85	?	35%	?
2. ?	$100.00	?	$199.95
3. $1,050.00	?	?	$775.00
4. $28,775.00	$3,250.00	?	?
5. ?	?	33%	$13,199.95
6. ?	$38.33	12%	?

Applications

7. A pair of Nike athletic shoes is listed at a regular selling price of $89.99. If the shoes go on sale for 40% off, what is the sale price?
8. During its special Bay Days, The Bay advertises a Timex watch for $39.99 with a regular price of $84.99. Calculate the markdown percent and markdown amount.
9. For spring break you are thinking about heading to Tulum, Mexico. In planning ahead, you notice that a one-week stay at the Gran Bahia Principe Tulum, regularly priced at $2,349 for air and six nights all inclusive, offers an early-bird booking discount of $350. What markdown percentage is being offered for booking early?
10. A Heritage Infusio deep frying pan is advertised at 70% off with a sale price of $39.99. What is the frying pan's regular selling price, and what markdown amount does this represent?
11. A mass merchandiser uses its Lagostina cookware product line as a marketing tool. The cookware is always on sale at an advertised price of 45% off. The cost of the cookware is $199.99, expenses are $75, and the planned profit at the sale price is $110. Calculate the sale price and selling price for the cookware.
12. Quicky Mart regularly sells its Red Bull sports drink for $2.99 per can. Quicky Mart noticed that one of its competitors down the street sells Red Bull for $1.89. What markdown percentage must Quicky Mart advertise if it wants to match its competitor?
13. A hardware store always advertises a Masterdesigner 75-piece screwdriver set at 80% off for a sale price of $17.99.
    a. If the cost of the set is $10 and expenses are 30% of the sale price, what is the planned profit when the product is on sale?
    b. What profit is earned if the product actually sells at its regular selling price?
14. A campus food outlet is advertising a “Buy one, get one 25% off” deal. The 25% off comes off the lower-priced item. If you purchase a chicken dinner for $8.99 and your friend gets the burger combo for $6.99, what is the markdown percentage on the total price?
15. Blast’em Stereos purchases a stereo system for $1,900 less two discounts of 40% and 18%. The store uses this product to draw customers to the store and always offers the stereo on sale at 25% off. When the stereo is on sale, it plans on expenses equaling 30% of the cost and a profit of 20% of the sale price.
    a. What is the sale price for the stereo?
    b. How much profit does Blast’em make when the stereo sells at the sale price?
    c. By law, this stereo must sell at the regular selling price for a period of time before going on sale. What is the regular selling price?
    d. What profit does Blast’em earn if a customer purchases the stereo during this initial period?

Challenge, Critical Thinking, & Other Applications

16. Frigid Boards purchases one of its snowboards for $395 less a retail trade discount of 15% and a loyalty discount of 4%. Its markup on selling price percentage on all snowboards is 21%. At the end of the season, any leftover snowboards are marked

down by 10%. What is the sale price for the snowboard?

17. An HP LaserJet printer has an MSRP of $399.95. It is subject to trade discounts of 30% and 23%. The LaserJet is a featured item for a computer store and is always on sale. The store plans to sell the LaserJet for a sale price that allows it to cover expenses equalling 15% of cost and realize a profit of $35.00.
    a. What is the sale price?
    b. If the MSRP is the regular unit price of the printer, what rate of markdown can the computer store advertise?
    c. What markup on selling price percentage is realized at the sale price?
18. The Brick advertises that when you purchase a queen-size Tempur-Pedic mattress set for $2,499.97 it will give you a 51" 3-D plasma television with a 3-D starter kit included. The value of this gift is $1,199.99. What markdown percent does this represent?
19. A Maytag 27 cubic foot refrigerator retails for $2,400.00 at Landover Appliance Centre. The company, which is celebrating its 30th anniversary this coming weekend, features the fridge for 30% off. The markup on selling price percentage on the fridge at the regular unit selling price is 53%.
    a. What is the sale price?
    b. At the sale price, what is the markup on selling price percentage?
    c. If the expenses are 15% of the regular selling price, what is the profit when the fridge is on sale?
20. Dreger Jewellers is selling a diamond bracelet. It uses this bracelet in its promotions and almost always has it on sale. The cost of the bracelet is $2,135 less discounts of 20% and 30%. When the bracelet is on sale for 25% off, the expenses are 15% of cost and the profit is 20% of cost.
    a. What is the sale price?
    b. What is the bracelet's regular selling price?
    c. If the bracelet sells at the regular selling price, what are the markup amount and the markup on cost percent?

6.4 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

Regular Unit Selling Price	Cost	Unit Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price	Markdown Amount	Markdown Percent	Sale Price
1. $100.00	?	24% of C	?	?	?	100%	50%	?	33%	?
2. ?	?	?	$20.00	$30.00	?	?	?	$13.00	?	$37.00
3. ?	$56.25	?	10% of S	?	?	33⅓%	?	?	?	$64.99
4. $99.99	$43.00	?	?	?	$78.50	?	15%	?	10%	?

For questions 5–8, solve for the unknown variables (identified with a ?) based on the information provided. N/A indicates that the particular variable is not applicable in the question.

Regular Unit Selling Price	Unit Cost	Regular Unit Expenses	Coupon or Rebate Handling Expense per Unit	Coupon or Rebate Redemption Expense per Unit	Coupon or Rebate Marketing Expense per Unit	Profit w/o Coupon or Rebate per Unit	Profit with Coupon or Rebate per Unit
5. $9.85	$4.73	35% of C	$0.25	$2.00	$0.75	?	?
6. $47.65	$23.50	15% of S	N/A	$10.00	$3.50	?	?

Regular Unit Selling Price	Unit Cost	Regular Unit Expenses	Coupon or Rebate Handling Expense per Unit	Coupon or Rebate Redemption Expense per Unit	Coupon or Rebate Marketing Expense per Unit	Profit w/o Coupon or Rebate per Unit	Profit with Coupon or Rebate per Unit
7. $6.30	?	$0.95	$0.45	?	$1.15	$3.05	$0.45
8. ?	$38.25	40% of C	N/A	?	$2.25	$21.40	$4.15

Applications

9. A box of glossy inkjet photo paper costs $9.50. The markup on cost is 110%. The profit is 31.5% of the cost. The paper occasionally goes on sale for 20% off.
   a. What are the regular unit selling price, expenses, profit, markup amount, and markup on selling price percentage?
   b. When the paper goes on sale, what are the sale price, markdown amount, markup dollars, and markup on selling price percentage?
10. John's Bakery is thinking of distributing a $0.50 coupon for one of its least popular brands of bread in hopes of increasing its sales. The bread costs $0.43 to produce, and expenses are 10% of the regular unit selling price. The bread retails for $1.99 per loaf. John's Bakery estimates that the coupon marketing expenses are $0.25 per loaf sold, and coupon handling expenses would be an additional $0.10 per loaf sold.
    a. What is John's Bakery profit when the loaf is sold at the regular unit selling price without the coupon?
    b. If John's Bakery uses the coupon, what is the profit?
    c. If the bread still doesn't sell well, John's Bakery will clear it out while still allowing consumers to use the coupon. What is the break-even selling price for a loaf in this situation?
11. An airline needs to maintain an average markup of $230 on a daily return flight from Toronto to Vancouver. The Boeing 737-800 has a capacity of 166 passengers, and the cost per passenger is $175. Typically 90% of tickets are sold at full fare while 10% are sold on standby at a sale price equaling $200 off the regular price. What regular and sale prices should the airline set to maintain the average markup?
12. A mail-in rebate form is printed on the side of every High Liner box of breaded fish. It allows the consumer to mail in the UPC along with the form and receive a $2 rebate. If a 700 g box of fish is normally sold to grocery stores for $3.50, costs are 40% of the regular unit selling price, and expenses are 20% of cost, what is the expected profit under the rebate program? Assume that only 25% of rebates are actually redeemed and that rebate marketing expenses are $0.15 per box.
13. Lay's Potato Chips is thinking of distributing a coupon on its 454 g bag of regular crinkled chips. Currently, the cost of producing a bag of chips is $0.49, with typical expenses of $0.23 per bag. Lay's sells the bag to retailers for $1.75 per bag. The coupon is in the amount of $0.50, while retailers are offered $0.12 for handling expenses. Lay's forecasts an increase in sales of 300,000 bags with the coupon, incurring total marketing expenses of $100,000. Will running this coupon promotion be profitable for Lay's? What is its profit (or loss) per unit if Lay's goes forward with the coupon?
14. LG is thinking of placing a $30 mail-in rebate on its 19" widescreen computer monitor. The cost to produce a monitor is $67.40, with typical expenses amounting to 50% of the cost. The monitor is sold to retailers for $119.00. Based on previous rebate programs, LG knows that only 40% of those who purchased a monitor with a rebate will actually redeem it. The marketing expense associated with the rebate totals $350,000, resulting in increased sales of 50,000 monitors. Is the rebate program profitable for LG? What is its profit (or loss) per monitor if it proceeds with the rebate offer?
15. A car dealership acquired a used Ford Mustang for $15,000. Its expenses on used vehicles are 15% of the cost, and it prices the vehicle with a 30% markup on cost percentage. After two months, if the car doesn't sell, it marks the car down by 10%.
    a. What are the expenses?
    b. What is the dollar amount of the markup?
    c. What is the regular selling price?
    d. What is the markup on selling price percentage?
    e. What is the sale price?
    f. What is the amount of the markdown?
    g. What is the profit earned if the car is sold at the sale price?

    h. What is the maintained markup supposing the dealership was able to acquire eight used Ford Mustangs at the same cost, where five cars are sold at regular price and three are sold at the sale price?

Challenge, Critical Thinking, & Other Applications

16. Target is trying to determine whether it would be more profitable to use a coupon or a mail-in rebate when promoting a new supersoaker water gun. The toy costs $14.94 and retails for $29.95. Expenses are 30% of cost. The coupon is for $4, handling expenses are $0.50 per unit, and the marketing expenses are $1 per unit. The rebate is for $7.75, with 45% of the rebate forms expected to be redeemed. Rebate marketing expenses are $1.25 per unit.
    a. If Target forecasts that sales will increase 10,000 units under the coupon program and 8,000 units under the rebate program, should it use the coupon or the mail-in rebate?
    b. How much more profit is earned based on your recommended option?
17. A hair salon has limited shelf space to carry shampoo products. The store's pricing policies have a markup on cost of 220%, and expenses equal 20% of cost. Two suppliers have approached the salon with the following offers:
    a. Supplier #1: The shampoo has a list price of $29 with discounts of 60% and 20%. Over a six-month period, 150 bottles are expected to be sold at full price and 50 bottles at a 25% markdown during a special promotion.
    b. Supplier #2: The shampoo has a list price of $37 with discounts of 70% and 10%. Over a six-month period, 120 bottles are expected to be sold at full price and 75 bottles at a 20% markdown during a special promotion.

    Based on this information, recommend which supplier the salon should purchase the shampoo from. Show calculations to support your answer.

18. The Home Depot purchases garden sheds for $1,000 less discounts of 40% and 20% and then sells them at a 100% markup on cost percentage. Seasonally, these sheds sell for full price initially followed by a reduced price and then are blown out at 50% off the regular price. Past experience indicates that 60% of the sheds sell at full price, 30% at the reduced price, and 10% at the blowout price. What should the reduced price be set at to achieve a maintained markup of $360?
19. CCM sells ice skates to Sport Chek for $179.99 less a trade discount of 40% and a quantity discount of 15%. The ice skates cost $27.60 to manufacture, and associated expenses are 35% of the regular unit selling price. CCM runs a mail-in rebate program on the ice skates where consumers receive $50 on a pair of skates. CCM estimates that 40% of purchasers will redeem the rebate. Rebate marketing expenses are estimated at $6 per pair sold. What is CCM's profitability under the rebate program?
20. Canadian Tire purchases a camping tent from its supplier at an MSRP of $79.99 less discounts of 40% and 25%. It plans to sell the tent at the MSRP, and its expenses are 20% of the selling price. At the end of the season, remaining tents are sold at 30% off.
    a. What are the expenses?
    b. What is the profit?
    c. What is the dollar amount of the markup?
    d. What is the markup on selling price percentage?
    e. What is the markup on cost percentage?
    f. What is the sale price?
    g. What is the amount of the markdown?
    h. What is the profit earned when tents are sold at the sale price?
    i. What is the maintained markup if 90% of tents are sold at the regular price and 10% of tents are sold at the sale price?

Review Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

1. The Gap is purchasing an Eminem graphic T-shirt for its stores. If the list price for the T-shirt is $34.50 and the Gap can receive a 40% trade discount, what net price will the Gap pay?
2. Nike has been approached by a retailer who wants to carry their Shox brand. The retailer needs a trade discount of 30% and its wholesaler needs 15%. The retailer also wants a 5% quantity discount. If the MSRP for the shoes is $195.00, what is the wholesale (net) price that Nike will charge?
3. If the regular price of a product is $775.00 and it is eligible to receive discounts of 14%, 8%, and 5%, what single discount percentage is equal to the multiple discount percentages? Show calculations that show how the single and multiple discounts

produce the same net price.

4. Subway wants to price a new 12 inch sandwich. The cost of all the ingredients is \$1.23. Expenses are 200% of cost, and Subway wants to earn a profit of 187% of cost. What is the regular selling price of the new sandwich?
5. If the cost of a Bose Wave music system is \$27.82 and the list price is \$49.99, determine the markup amount. Express the markup both as a percentage of cost and percentage of selling price.
6. Rogers Communications is thinking about having a special promotion on its Digital One Rate 250 Plan, which offers 250 minutes of cellular calling. If the current price is \$85.00 per month and Rogers wants to put it on special for \$75.00 per month, what is the percentage markdown?
7. If a \$1 coupon is distributed for a product that normally has a profit of \$4 per unit, how much is the profitability reduced if coupon handling expenses are \$0.30 per unit and coupon marketing expenses are \$1.10 per unit? What is the profit under the coupon program?
8. At a local retailer, a swimsuit costs \$10 and sells for \$30. At the end of the season, the swimsuit price is reduced by 30%. If 820 swimsuits are sold at the regular price and 145 swimsuits are sold at the sale price, what is the maintained markup?

Applications

9. An iMac has an MSRP of \$774.99 with available trade discounts of 20% to the retailer and an additional 15% to the wholesaler.
   a. What price should a retailer pay for the product? What is the dollar value of the discount?
   b. What price should a wholesaler pay for the product? How much less than the retailer's price is this?
   c. Overall, what single discount percent does the wholesaler receive?
10. The marketing manager for Tim Hortons is attempting to price a cup of coffee. She knows that the cost of the coffee is \$0.14 per cup, and expenses are 30% of the regular selling price. She would like the coffee to achieve an 88.24% markup on selling price.
    a. What is the regular selling price for a cup of coffee?
    b. What is the profit per cup?
    c. What is the markup on cost percentage?
11. A grocery store head office is thinking of distributing a \$3 in-store coupon for one of its 454 g bags of coffee. The cost of the coffee to one of the franchised retail stores is \$6.34, and it is usually sold to consumers for \$12.99. A retail store incurs expenses of 25% of the regular selling price to stock, shelve, and sell the coffee. The franchisees are reimbursed \$0.25 per coupon redeemed for handling expenses, and the head office estimates that it incurs marketing expenses amounting to \$1.35 per coupon. Based strictly on the profit associated with the sale of this coffee, should the head office proceed with this promotion?
12. Jonathan needs a new business suit for a presentation in his communications course. He heads to Moores and finds a suit on sale for 33% off. The regular price of the suit is \$199.50.
    a. What is the sale price?
    b. What is the dollar amount of the markdown?
13. Birchwood Honda needs to clear out its Honda generators for an end-of-season sale. If the dealership pays \$319.00, has expenses of 15% of cost, and profits of 30% of the regular unit selling price, what markdown percentage can the dealership advertise if it wants to break even during the sale?
14. eMachines has decided to issue a \$50 mail-in rebate on one of its best-selling desktop computers. Retailers purchase the machines directly from eMachines for a list price of \$449.00 less a 40% trade discount. eMachines has profits of 35% of its regular unit selling price and expenses of 20% of its regular unit selling price.
    a. What is the profit per computer during the rebate, assuming rebate marketing expenses of \$13.50 per unit and an expected redemption rate of 35%?
    b. How much lower is the profit with the rebate than without the rebate?
    c. If 2,500 computers are sold at regular price and 1,025 computers are sold through the rebate program, what is the maintained markup?

Challenge, Critical Thinking, & Other Applications

15. A retailer purchasing the Halo 3 video game is eligible to receive a trade discount, quantity discount, and loyalty discount. If the net price of the game is \$22.94, the list price is \$39.99, the trade discount is 25%, and the quantity discount is 15%, what percentage is the loyalty discount?
16. An article in a business paper on October 22, 2008, reported that the benchmark S&P/TSX Composite Index ended down 558.92 points for the day, or 5.7%. It went on further to state that the index is down 30% for the year 2008. Rounding all point

values to two decimals, answer the following:

  a. What were the point values on October 21 and 22, respectively?
  b. What was the point value at the start of the year 2008?

17. Whistler Blackcomb Ski Resort purchased 400 pairs of skis for sale in its retail store. It sells the skis at an MSRP of $299.99. It is eligible to receive discounts of 35% and 20% on its purchase. Expenses average 15% of cost. If it sells 300 pairs of skis at the regular unit selling price and the other 100 skis are sold during a clearance sale for 20% off, what is the total amount of profit that Whistler Blackcomb earns? What is the maintained markup?
18. A company is examining its options for selling a product listed at $359.97. Both expenses and profits are 20% and 22% of the regular unit selling price, respectively. At this price, the company forecasts sales of 6,000 units. Consider the following two scenarios:

    a. Scenario A: If the company holds a sale and marks down the product by 10%, its market research predicts that sales will rise to 11,225 units.
    b. Scenario B: If the company issues a $60.00 mail-in rebate, it will have increased marketing expenses of $6.35 per unit. Only 45% of rebates are expected to be redeemed. Market research predicts that sales will rise to 10,500 units.

    Should the company leave the product at its regular price, conduct the sale, or issue the mail-in rebate?

19. Indigo is stocking a new hardcover book. The book has a cover price of $44.99 and Indigo is eligible to receive discounts of 25%, 8%, 4%, and 1%. Expenses are 20% of cost. The book sells for the cover price.

    a. What is the profit per book?
    b. Some new books are launched at 30% off the cover price. From a strictly financial perspective, would it be wise for Indigo to use this approach for this book? Why or why not?

20. Mary is shopping for a new George Foreman grill. While shopping at Polo Park Shopping Centre, she came across the following offers at three different retail stores:

    a. Offer #1: Regular price $149.99, on sale for 30% off.
    b. Offer #2: Regular price $169.99, on sale for 15% off plus an additional 30% off.
    c. Offer #3: Regular price $144.95, on sale for 10% off plus an additional 15% off. Mary also gets a 5% loyalty discount at this store.

    Which is the best offer for Mary?

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CHAPTER OVERVIEW

7: Accounting Applications

7.0: Introduction
7.1: Sales Taxes
7.2: Property Taxes
7.3: Exchange Rates and Currency Exchange
7.4: Invoicing - Terms of Payment and Cash Discounts
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7.0: Introduction

The price tag on the product is not necessarily what you pay for it. In 1789, Benjamin Franklin might have said it best when he wrote, "In this world nothing can be said to be certain, except death and taxes."1 Almost everything is taxed, whether a physical good, a service, or even property.

For instance, while shopping on eBay, you find a product listed in an eBay store located in your home province. Another product you need is listed by a seller in the United States. You need both products and the prices including shipping costs are reasonable, so for each you click the "Buy It Now" option. The Canadian invoice lists the product's pre-tax price along with two more lines indicating GST and PST amounts, all totaled at the bottom. The US invoice lists the price of the item in US currency. When paying this seller, PayPal converts the price from US currency to Canadian currency using an exchange rate that does not quite match the rate you have seen published in the newspaper, and you wonder why. If you go with this US seller, you know that the Canada Border Services Agency (customs) will charge for GST.

This scenario highlights three of this chapter's key topics: taxes, currency exchange, and invoicing.

1. If there is one thing you encounter in all aspects of your life, it is sales tax, which provides revenue to governments in Canada and abroad. In addition, anyone involved in the ownership of real estate must deal with property taxes. Corporate offices, production facilities, and warehouses sit on business-owned real estate, which is charged property taxes. Residential homeowners pay annual taxes to their municipalities. Even if you just rent property like an apartment, the apartment building owner is charged property taxes, which are passed on to tenants within the rental charges.
2. Canadian businesses and consumers operate in a global economy in which they commonly complete international transactions. Any payment will incorporate an exchange rate ensuring that the amount paid in Canadian currency is equivalent to the amount charged in the foreign currency.
3. Many transactions are completed through an invoicing procedure. Along with a list of the items purchased and their respective prices, invoices show terms of payment, consequences of failing to make the payment, and additional charges such as taxes and shipping.

There is no avoiding sales taxes, property taxes, currency exchange, and invoicing. As a consumer, you generally need to pay these costs. As a business, you not only pay these costs but you need to charge your customers appropriately and collect taxes on behalf of the government. This chapter explores the mathematics behind these concepts.

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7.1: Sales Taxes

On your recent cross-Canada road trip, you purchased from many different Tim Hortons' stores. At each store, your products retailed for \$6.99. When you review your credit card receipts after returning home from your trip, you notice that you paid different totals everywhere. In Alberta, they only added GST and your combo cost \$7.34. In British Columbia, they added both PST and GST, resulting in total cost of \$7.83. In Ontario, they added something called HST, resulting in a total cost of \$7.90. You find it interesting that the same combo came to different totals as you travelled across Canada.

Three Sales Taxes

A sales tax is a percent fee levied by a government on the supply of products. In Canada, there are three types of sales taxes: the goods and services tax (GST), provincial sales tax (PST), and the harmonized sales tax (HST). In this section you will learn the characteristics of each of these taxes and then the mathematics for calculating any sales tax.

Goods & Services Tax (GST)

The goods and services tax, better known as GST, is a national federal tax of 5% that applies to the purchase of most goods and services in Canada. Every province and territory has GST. The consumer ultimately bears the burden of this sales tax.

Businesses must collect GST on most of their sales and pay GST on most purchases in the daily course of operations. However, when remitting these taxes, businesses claim a credit with the federal government to recover the GST they paid on eligible purchases. The net result is that businesses do not pay the GST on these eligible purchases. While this may outrage some people, the logic is simple. If a business pays the GST, it becomes a cost of the business, which is then passed on to consumers as it is incorporated into retail prices. When the consumer purchases the product, the consumer would be charged the GST again! In essence, a consumer would be double-taxed on all purchases if businesses paid the GST.

Some goods and services are exempt from GST. While there are many complexities and nuances to the exemptions, generally items that are deemed necessities (such as basic groceries), essential services (such as health, legal aid, and childcare), and charitable activities are nontaxable. You can find a complete listing of exemptions on the Canada Revenue Agency website at www.cra.gc.ca.

Provincial Sales Tax (PST)

Province/Territory	PST
British Columbia	7%
Alberta	0%
Saskatchewan	5%
Manitoba	8%
Quebec	9.975%
Northwest Territories	0%
Yukon Territory	0%
Nunavut	0%

Provincial sales taxes, or PST, are provincially administered sales taxes that are determined by each individual provincial or territorial government in Canada. The table here lists the current PST rates in Canada.

Similar to GST, PST applies to the purchase of most goods and services in the province, and consumers bear the burden. For the same reasons as with GST, businesses typically pay the PST on purchases for non-resale items (such as equipment and machinery) and do not pay the PST on resale items. Businesses are responsible for collecting PST on sales and remitting the tax to the provincial government. Individual provincial websites list the items and services that are exempt from PST.

Harmonized Sales Tax (HST)

The harmonized sales tax, or HST, is a combination of GST and PST into a single number. Since most goods and services are subjected to both taxes anyway, HST offers a simpler method of collecting and remitting the sales tax—a business has to collect and remit only one tax instead of two. Because there are pros and cons to HST, not all provinces use this method of collection, as summarized in the table below.

Pros	Cons
• Items that are previously PST payable to a business are now refunded, lowering input costs and lowering consumer prices • Results in overall lower corporate taxes paid • Increases the competitiveness of businesses and results in job creation • Businesses only remit one tax and not two, resulting in financial and auditing savings	• Many items such as utilities, services, and children's clothing that are ineligible for PST become taxed at the full HST rate • Consumer cost of living increases • Tax-exempt items see prices rise because HST is being applied to services and goods such as transportation and gasoline

Province/Territory	HST
Ontario	13%
New Brunswick	13%
Nova Scotia	14%
Newfoundland and Labrador	13%
Prince Edward Island	14%

To understand HST you can separate it into its GST and PST components. For example, in the table to the left Ontario has an HST of 13%. If GST is 5% and the HST is 13%, then it is clear that the province has a PST of 8%. HST operates mostly in the same manner as GST, in that consumers generally bear the burden for paying this tax, and businesses both collect and pay the HST but have the HST reimbursed on eligible purchases when remitting taxes to the government.

The Formula

With respect to sales taxes, you usually calculate two things:

1. The dollar amount of the sales tax.
2. The price of a product including the sales tax.

Calculating the Sales Tax Amount

A sales tax is a percent rate calculated on the base selling price of the product. Therefore, if you are interested solely in the amount of the sales tax (the portion owing), apply Formula 2.2 on Rate, Portion, Base:

$$\text{Rate } = \frac{\text{Portion}}{\text{Base}} \text{ becomes Tax Rate } = \frac{\text{Tax Amount}}{\text{Price before Taxes}}$$

Rearranging this formula to solve for the tax amount gives the following:

$$\begin{gathered}\text{Tax Amount}\\ = \text{ Tax Rate } \times \text{ Price before Taxes (which is the same as Portion } = \text{ Rate } \times \text{ Base)}\end{gathered}$$

This is not a new formula. It applies the existing formula from Chapter 2.

Calculating a Price Including Tax

When calculating a selling price including the tax, you take the regular selling price and increase it by the sales tax percentage. This is a percent change calculation using Formula 3.1 from Chapter 3:

$$\Delta\% = \frac{New - Old}{Old} \times 100$$

The Old price is the price before taxes, the percent change is the sales tax percentage, and you need to calculate the New price including the tax. Rearranging the formula for New, you have

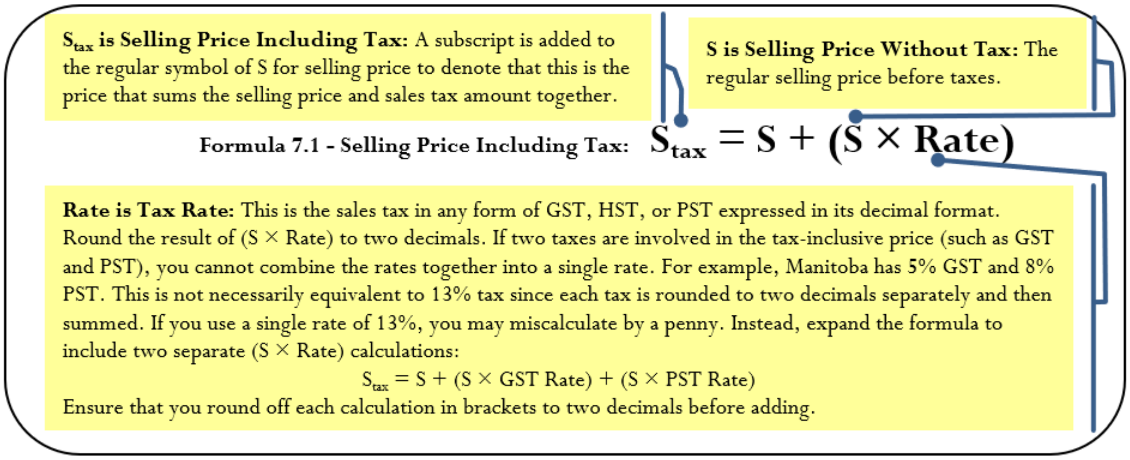
$$New = Old + Old\left(\frac{\Delta\%}{}\right)$$

$$\left(\quad 100 \right)$$

which you factor and rewrite as

$$New = Old \times \left(1 + \frac{\Delta\%}{100}\right) \quad \text{or} \quad New = Old + \left(Old \times \frac{\Delta\%}{100}\right)$$

This is a widespread application of the percent change formula. Since it can be tedious to keep rearranging Formula 3.1, Formula 7.1 expresses this relationship.



S_{tax} is Selling Price Including Tax: A subscript is added to the regular symbol of S for selling price to denote that this is the price that sums the selling price and sales tax amount together.

S is Selling Price Without Tax: The regular selling price before taxes.

Formula 7.1 - Selling Price Including Tax: $S_{tax} = S + (S \times Rate)$

Rate is Tax Rate: This is the sales tax in any form of GST, HST, or PST expressed in its decimal format. Round the result of (S × Rate) to two decimals. If two taxes are involved in the tax-inclusive price (such as GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST and 8% PST. This is not necessarily equivalent to 13% tax since each tax is rounded to two decimals separately and then summed. If you use a single rate of 13%, you may miscalculate by a penny. Instead, expand the formula to include two separate (S × Rate) calculations:

$$S_{tax} = S + (S \times \text{GST Rate}) + (S \times \text{PST Rate})$$

Ensure that you round off each calculation in brackets to two decimals before adding.

Formula 7.1

How It Works

Follow these steps to perform calculations involving sales taxes:

Step 1: Identify the pricing information. In particular, pay careful attention to distinguish whether the price is before taxes (S) or inclusive of taxes (S_{tax}). Also identify all applicable sales taxes, including GST, PST, and HST.

Step 2: Apply Formula 7.1 to solve for the unknown variable.

Step 3: If you need to find sales tax amounts, apply Formula 2.2 and rearrange for portion. Ensure that for the base you use the price before taxes.

Assume a $549.99 product is sold in British Columbia. Calculate the amount of the sales taxes and the price including the sales taxes.

Step 1: The price before taxes is S = $549.99. In British Columbia, GST is 5% and PST is 7% (from the PST Table).

Step 2: To calculate the price including the sales taxes, apply Formula 7.1:

$$S_{tax} = \$549.99 + (\$549.99 \times 5\%) + (\$549.99 \times 7\%) = \$615.99$$

Step 3: Applying the rearranged Formula 2.2 for the GST, you calculate:

$$\text{GST Tax Amount } = 5\% \times \$549.99 = \$27.50$$

Applying the same formula for the PST, you calculate:

$$\text{PST Tax Amount } = 7\% \times \$549.99 = \$38.50$$

(Note: You may notice that you could just pull these amounts from the interim calculations in step 2). Therefore, on a $549.99 item in British Columbia, $27.50 in GST and $38.50 in PST are owing, resulting in a price including sales taxes of $615.99.

Important Notes

To calculate a price including a single sales tax rate, use the percent change function (Δ%) on the calculator. You can review the full instructions for this function at the end of Chapter 3. When using this function, OLD is the price before taxes, NEW is the price after taxes, and %CH is the single tax rate in its percentage format. You do not use the #PD variable, which therefore defaults to 1.

Note that if more than one tax rate applies on the same base, you cannot use this calculator function because of the possibility of a rounding error, as explained above.

Paths To Success

You will often need to manipulate Formula 7.1. Most of the time, prices are advertised without taxes and you need to calculate the price including the taxes. However, sometimes prices are advertised including the taxes and you must calculate the original price of the product before taxes. When only one tax is involved, this poses no problem, but when two taxes are involved (GST and PST), combine the taxes into a single amount before you solve for S.

? Exercise 7.1.1: Give It Some Thought

On any given product selling for the same price, put the following provinces in order from highest price to lowest price including taxes (GST and PST, or HST): Alberta, Saskatchewan, British Columbia, Ontario, Prince Edward Island.

Answer

> PEI (14% HST), Ontario (13% HST), British Columbia (5% GST + 7% PST), Saskatchewan (5% GST + 5% PST), Alberta (5% GST, no PST)

✓ Example 7.1.1: Calculating Sales Taxes across Canada

Dell Canada lists a complete computer system on its Canadian website for $1,999.99. Calculate the price including taxes if the Canadian buyer is located in:

a. Alberta
b. Ontario
c. Quebec
d. British Columbia (BC)

Solution

Four answers are required. For each of the provinces listed, calculate the appropriate GST/PST or HST to add onto the price and arrive at the selling price including taxes (S_{tax}).

What You Already Know

Step 1:

The price of the computer and tax rates are known:

$S = \$1,999.99$

Alberta sales tax = 5% GST

Ontario sales tax = 13% HST

Quebec sales tax = 5% GST & 9.975% PST

BC sales tax = 5% GST & 7% PST

How You Will Get There

Step 2:

For all provinces, apply Formula 7.1.

a. For Alberta, the Rate is the 5% GST. Substitute the S and Rate to solve for S_{tax}.
b. For Ontario, the Rate is the 13% HST. Substitute the S and Rate to solve for S_{tax}.
c. For Quebec, both the 5% GST and 9.975% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax}.
d. For British Columbia, both the 5% GST and 7% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax}.

Perform

Step 2:

a. Alberta: $$\begin{aligned} S_{tax} &= \$1,999.99 + (\$1,999.99 \times 5\%) \\ &= \$2,099.99 \end{aligned}$$

b. Ontario: $$\begin{aligned} S_{tax} &= \$1,999.99 + (\$1,999.99 \times 13\%) \\ &= \$2,259.99 \end{aligned}$$

c. Quebec: $$\begin{aligned} S_{tax} &= \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 9.975\%) \\ &= \$2,299.49 \end{aligned}$$

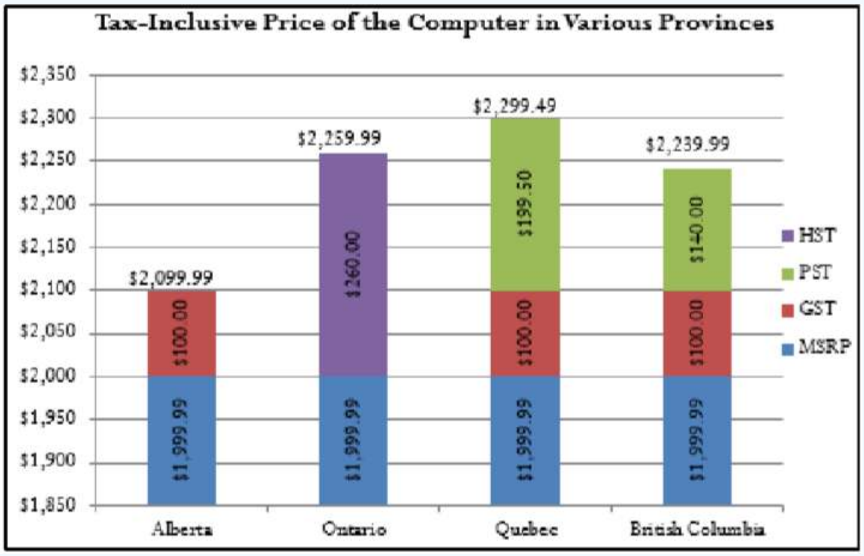
d. British Columbia: $$\begin{aligned} S_{tax} &= \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 7\%) \\ &= \$2,239.99 \end{aligned}$$

Calculator Instructions

All calculations use the $\Delta\%$ function.

Province	OLD	NEW	%CH	#PD
Alberta	1999.99	Answer: 2,099.9895	5	1
Ontario	1999.99	Answer: 2,259.9887	13	1
Quebec & BC	Not recommended to use a calculator since two taxes on the same base are involved and a rounding error could result.			

Depending on which province you reside in, the price for the computer from lowest to highest is Alberta at $2,099.99, British Columbia at $2,239.99, Ontario at $2,259.99, and Quebec at $2,299.49.



"The Brick is having its Midnight Madness sale! Pay no taxes on products purchased during this event!" While this is good marketing, it probably goes without saying that governments do not give up the sales taxes. Essentially The Brick is advertising a tax-inclusive price. Calculate GST and PST amounts for a product advertised at $729.95, including GST and PST, in Saskatchewan.

Solution

To calculate the PST and GST for Saskatchewan, you must calculate the price before taxes (S). Then calculate each tax amount based on the S.

What You Already Know

Step 1:

The price after taxes and the tax rates are as follows:

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$S_{tax} = \$729.95$

Tax rates = 5% GST and 5% PST

How You Will Get There

Step 2:

Apply Formula 7.1 using the combined PST and GST as the Rate to calculate the S.

Step 3:

Apply Formula 2.2 rearranged for Portion to calculate the tax amounts.

Perform

Step 2:

Remove the taxes: $$\begin{aligned} \$729.95 &= S + (S \times 10\%) \\ \$729.95 &= 1.1S \\ S &= \$663.59 \end{aligned}$$

Step 3:

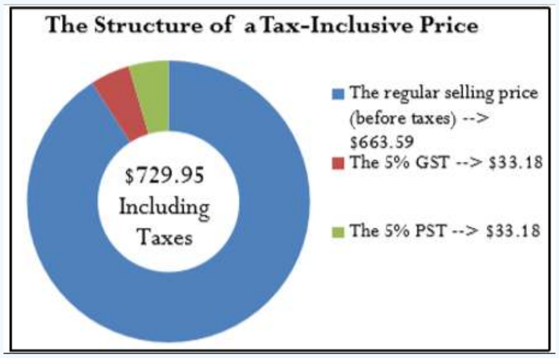
Calculating GST tax amount: 5% = GST Portion ÷ \$663.59 GST Portion = \$33.18

PST is also 5%, then PST Portion =\$33.18

Calculator Instructions

OLD	NEW	%CH	#PD
Answer: 663.59	729.95	10	1

When paying a tax-inclusive price of \$729.95 in Saskatchewan, the regular selling price of the item is \$663.59 with \$33.18 in each of GST and PST included.

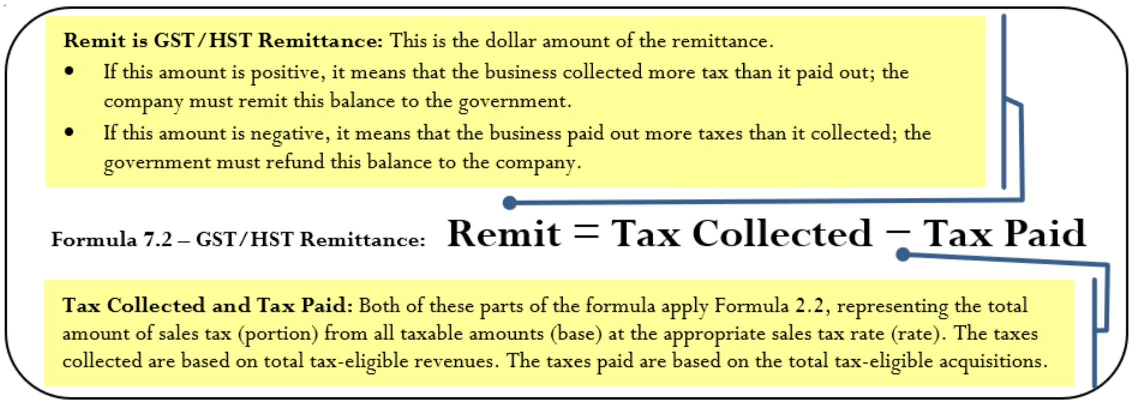


The GST/HST Remittance

When a business collects sales taxes, it is a go-between in the transaction. These sales tax monies do not belong to the business. On a regular basis, the business must forward this money to the government. This payment is known as a tax remittance.

The Formula

Generally speaking, a business does not pay sales taxes. As a result, the government permits a business to take all eligible sales taxes that it paid through its acquisitions and net them against all sales taxes collected from sales. The end result is that the business is reimbursed for any eligible out-of-pocket sales tax that it paid. Formula 7.2 expresses this relationship.



Remit is GST/HST Remittance: This is the dollar amount of the remittance.

- If this amount is positive, it means that the business collected more tax than it paid out; the company must remit this balance to the government.
- If this amount is negative, it means that the business paid out more taxes than it collected; the government must refund this balance to the company.

Formula 7.2 – GST/HST Remittance: $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$

Tax Collected and Tax Paid: Both of these parts of the formula apply Formula 2.2, representing the total amount of sales tax (portion) from all taxable amounts (base) at the appropriate sales tax rate (rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions.

Formula 7.2

How It Works

Follow these steps to complete a GST/HST remittance:

Step 1: Identify the total amounts of tax-eligible revenues and acquisitions upon which the sales tax is collected or paid, respectively. Identify the applicable sales tax rate of the GST or HST.

Step 2: Calculate the total taxes collected by applying Formula 2.2, where the sales tax is the rate and the total revenue is the base. Solve for portion.

Step 3: Calculate the total taxes paid by applying Formula 2.2, where the sales tax is the rate and the total acquisitions are the base. Solve for portion.

Step 4: Apply Formula 7.2 to calculate the tax remittance.

Assume a business has paid GST on purchases of $153,000. It has also collected GST on sales of $358,440. Calculate the GST remittance.

Step 1: Identifying the variables, you have:

Total Revenue = $358,440

Total Acquisitions = $153,000

GST Tax Rate = 5%

Step 2: Calculate taxes collected by applying Formula 2.2, where:

$$\text{GST collected } = 5\% \times \$358,440 = \$17,922$$

Step 3: Calculate taxes paid by applying Formula 2.2, where:

$$\text{GST paid } = 5\% \times \$153,000 = \$7,650$$

Step 4: To calculate the remittance, apply Formula 7.2 and calculate $\text{Remit } = \$17,922 - \$7,650 = \$10,272$. The business should remit a check for $10,272 to the government.

Paths To Success

A shortcut can help you calculate the GST/HST Remittance using Formula 7.2. If you do not need to know the actual amounts of the tax paid and collected, you can net GST/HST–eligible revenues minus acquisitions and multiply the difference by the tax rate:

$$\text{Remit } = (\text{Revenues } - \text{ Acquisitions}) \times \text{ Rate}$$

In the example above, $\text{Remit } = (\$358,440 - \$153,000) \times 5\% = \$10,272$. If this calculation produces a negative number, then the business receives a refund instead of making a remittance.

✓ Example 7.1.3: Calculating a GST/HST Remittance

An Albertan lumber company reported the following quarterly purchases and sales in its 2013 operating year:

Quarter	Purchases	Sales
Winter	\$1,316,000	\$911,500
Spring	\$1,266,200	\$1,473,300
Summer	\$1,053,700	\$1,381,700
Fall	\$878,100	\$1,124,600

Assuming all purchases and sales are eligible and subject to GST, calculate the GST remittance or refund for each quarter.

Solution

Calculate the GST tax remittance, or Tax Remit, for each of the quarters.

What You Already Know

Step 1:

From the information provided, the total purchases and sales for each quarter along with the GST = 5%, which is the Tax Rate, are known.

How You Will Get There

Step 2:

For each quarter, calculate the GST collected by rearranging and applying Formula 2.2.

Step 3:

Calculate the tax paid.

Step 4:

Apply Formula 7.2 for each quarter.

Perform

	Step 3		Step 3		Step 4
Quarter	**Sales**	**GST Collected (Sales $\times 5\%$)**	**Purchases**	**GST Paid (Purchases $\times 5\%$)**	**Remittance (Tax Collected - Tax Paid)**
Winter	\$911,500	\$45,575	\$1,316,000	\$65,800	\$45,575 - \$65,800 = -\$20,225
Spring	\$1,473,300	\$73,650	\$1,266,200	\$63,310	\$73,650 - \$63,310=\$10,340
Summer	\$1,381,700	\$69,085	\$1,053,700	\$52,685	\$69,085 - \$52,685=\$ 16,400
Fall	\$1,124,600	\$56,230	\$878,100	\$43,905	\$56,230 - \$43,905=\$ 12,325

In the Winter quarter, the company gets a GST refund of \$20,225. In the other three quarters, the company remits to the government payments of \$10,340, \$16,400, and \$12,325, respectively.

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7.2: Property Taxes

As you drive through your neighborhood, you pass a city crew repairing the potholes in the road. Hearing sirens, you check your rear-view mirror and pull to the side of the road as a police car and fire engine race by, heading toward some emergency. Pulling back out, you drive slowly through a public school zone, where you smile as you watch children playing on the gigantic play structure. A city worker mows the lawn.

Where does the municipality get the money to pay for all you have seen? No one owns the roads, schools are free, fire crews and police do not charge for their services, play structures have no admission, and parks are open to everyone. These are just some examples of what your municipality does with the money it raises through property taxes.

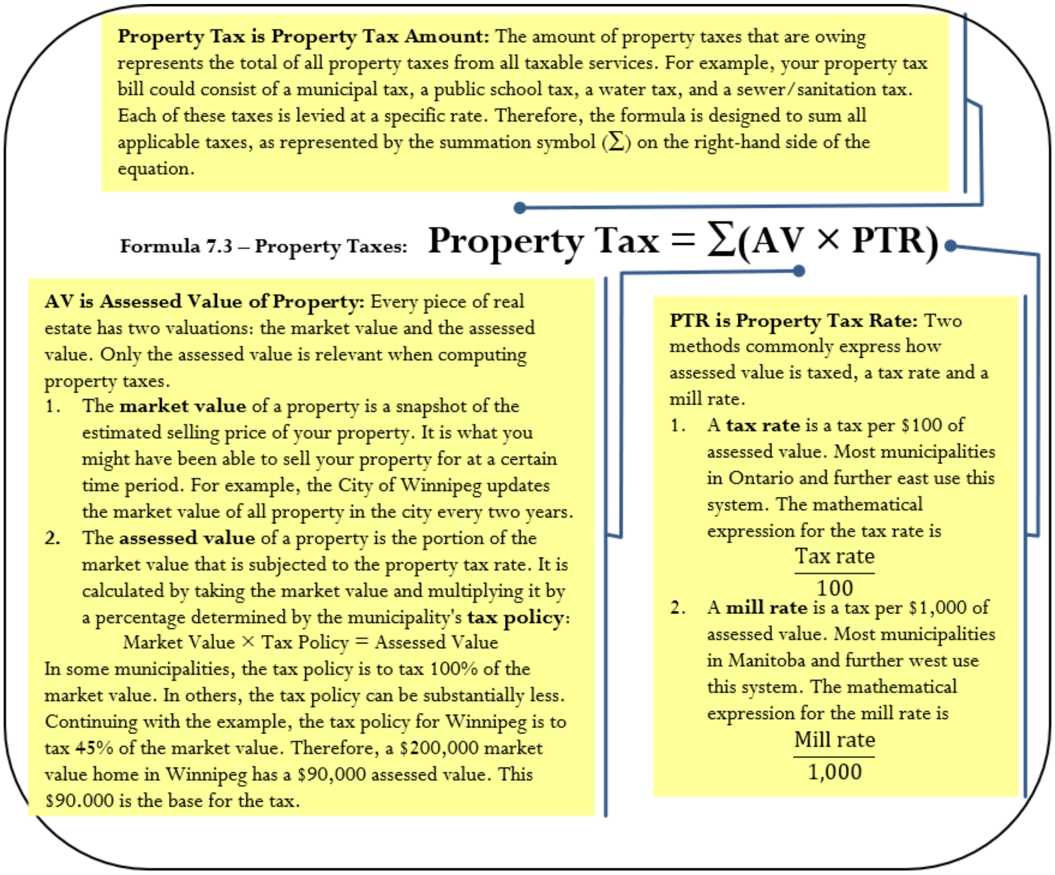
Property Taxation

Property taxes are annual taxes paid by real estate owners to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services. Every individual and every business pays property taxes. Even if you don't own property, you pay property taxes that are included in your rental and leasing rates from your landlord.

Property taxes are imposed on real estate owners by their municipal government along with any other bodies authorized to levy taxes. For example, in Manitoba each divisional school board is authorized to levy property taxes within its local school division boundaries.

The Formula

Since property taxes are administered at the municipal level and every municipality has different financial needs, there are a variety of ways to calculate a total property tax bill. Formula 7.3 is designed to be flexible to meet the varying needs of municipal tax calculations throughout Canada.



Property Tax is Property Tax Amount: The amount of property taxes that are owing represents the total of all property taxes from all taxable services. For example, your property tax bill could consist of a municipal tax, a public school tax, a water tax, and a sewer/sanitation tax. Each of these taxes is levied at a specific rate. Therefore, the formula is designed to sum all applicable taxes, as represented by the summation symbol ($\sum$) on the right-hand side of the equation.

Formula 7.3 – Property Taxes: $$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$$

AV is Assessed Value of Property: Every piece of real estate has two valuations: the market value and the assessed value. Only the assessed value is relevant when computing property taxes.

1. The **market value** of a property is a snapshot of the estimated selling price of your property. It is what you might have been able to sell your property for at a certain time period. For example, the City of Winnipeg updates the market value of all property in the city every two years.
2. The **assessed value** of a property is the portion of the market value that is subjected to the property tax rate. It is calculated by taking the market value and multiplying it by a percentage determined by the municipality's **tax policy**:

Market Value × Tax Policy = Assessed Value

In some municipalities, the tax policy is to tax 100% of the market value. In others, the tax policy can be substantially less. Continuing with the example, the tax policy for Winnipeg is to tax 45% of the market value. Therefore, a $200,000 market value home in Winnipeg has a $90,000 assessed value. This $90.000 is the base for the tax.

PTR is Property Tax Rate: Two methods commonly express how assessed value is taxed, a tax rate and a mill rate.

1. A **tax rate** is a tax per $100 of assessed value. Most municipalities in Ontario and further east use this system. The mathematical expression for the tax rate is $\frac{\text{Tax rate}}{100}$
2. A **mill rate** is a tax per $1,000 of assessed value. Most municipalities in Manitoba and further west use this system. The mathematical expression for the mill rate is $\frac{\text{Mill rate}}{1,000}$

Formula 7.3

How It Works

Follow these steps when working with calculations involving property taxes:

Step 1: Identify all known variables. This includes the market value, tax policy, assessed value, all property tax rates, and the total property taxes.

Step 2: If you know the assessed value, skip this step. Otherwise, calculate the assessed value of the property by multiplying the market value and the tax policy.

Step 3: Calculate either the tax amount for each property tax levy or the grand total of all property taxes by applying Formula 7.3.

Continuing with the Winnipeg example in which a home has a market value of $200,000, the tax policy of Winnipeg is to tax 45% of the market value. A Winnipegger receives a property tax levy from both the City of Winnipeg itself and the local school board. The mill rates are set at 14.6 and 16.724, respectively. Calculate the total property tax bill.

Step 1: The known variables are market value = $200,000, tax policy = 45%, City of Winnipeg mill rate = 14.6, and school board mill rate = 16.724.

Step 2: Calculate the assessed value by taking the market value of $200,000 and multiplying by the tax policy of 45%, or

Assessed Value $= \$200,000 \times 45\% = \$90,000$

Step 3: To calculate each property tax, apply Formula 7.3.

The City of Winnipeg Property Tax $= \$90,000 \times 14.6 \div 1,000 = \$1,314$

The school board Property Tax $= \$90,000 \times 16.724 \div 1,000 = \$1,505.16 = \$1,505.16$

Add these separate taxes together to arrive at

Total Property Tax $= \$1,314 + \$1,505.16 = \$2,819.16$

Important Notes

Mill rates are commonly expressed with four decimals and tax rates are expressed with six decimals. Although some municipalities use other standards, this text uses these common formats in its rounding rules. In addition, each property tax levied against the property owner is a separate tax. Therefore, you must round each property tax to two decimals before summing the grand total property tax.

Things To Watch Out For

The most common mistake is to use the wrong denominator in the tax calculation. Ensure that you read the question accurately, noting which term it uses, tax rate or mill rate. If neither appears, remember that Ontario eastward uses tax rates and Manitoba westward uses mill rates.

A second common mistake is to add multiple property tax rates together when the assessed value remains constant across all taxable elements. For example, if the assessed value of $250,000 is used for two tax rates of 2.168975 and 1.015566, you may be tempted to sum the rates, which would yield a rate of 3.184541. This does not always work and may produce a small error (a penny or two) since each tax is itemized on a tax bill. You must round each individual tax to two decimals before summing to the total property tax.

✓ Example $7.2.1$: Calculating a Property Tax

A residence has a market value of $340,000. The municipality's tax policy is set at 70%. Real estate owners have to pay three separate taxes: the municipality tax, a library tax, and an education tax. The tax rates for each are set at 1.311666, 0.007383, and 0.842988, respectively. Calculate the total property tax bill for the residence.

Solution

Calculate the total property tax for the residence by summing the assessed value multiplied by the respective tax rates.

What You Already Know

Step 1:

The house, tax policy, and tax rates are known:

Market Value = $340,000

Tax Policy = 70%

Municipal PTR = 1.311666

Library PTR = 0.007383

Education PTR = 0.842988

How You Will Get There

Step 2:

Calculate the assessed value (AV) by taking the market value and multiplying by the tax policy.

Step 3:

Apply Formula 7.3. Note that this municipality uses a tax rate, so the PTR is divided by 100.

Perform

Step 2:

$AV = \$340,000 \times 70\% = \$340,000 \times 0.7 = \$238,000$

Step 3:

Municipal Tax $= \$238,000 \times \dfrac{1.311666}{100} = \$3,121.77$

Library Tax $= \$238,000 \times \dfrac{0.007383}{100} = \$17.57$

Education Tax $= \$238,000 \times \dfrac{0.842988}{100} = \$2,006.31$

Property Tax $= \$3,121.77 + \$17.57 + \$2,006.31 = \$5,145.65$

The property owner owes the municipality $3,121.77, the library $17.57, and education $2,006.31 for a total property tax bill of $5,145.65.

Paths To Success

The collective property taxes paid by all of the property owners form either all or part of the operating budget for the municipality. Thus, if a municipality consisted of 1,000 real estate owners each paying $2,000 in property tax, the municipality’s operating income from property taxes is $2,000 × 1,000 = $2,000,000. If the municipality needs a larger budget from property owners, either the assessed values, the mill/tax rate, or some combination of the two needs to increase.

✓ Example $7.2.2$: Setting a New Mill Rate

A school board is determining next year's operating budget and calculates that it needs an additional $5 million. Properties in its municipality have an assessed value of $8.455 billion. The current mill rate for the school board is set at 6.1998. If the assessed property values are forecasted to rise by 3% next year, what mill rate should the school set?

Solution

Aim to calculate the new property tax rate (PTR) for next year's mill rate.

What You Already Know

Step 1 (Current Year):

The information about the school board and its municipality are known:

$AV = \$8.455$ billion

PTR (current mill rate) = 6.1998

Δ% (to AV next year) = 3%

Property tax increase needed is $5 million

How You Will Get There

Step 2 (Current Year):

Skip. Calculations are based on assessed values already.

Step 3 (Current Year):

Calculate the school board's current budget using Formula 7.3. The property tax collected is the board’s operating budget.

Step 1 (Next Year):

Increase the budget, or Property Tax, for next year. Also increase the current assessed value base by applying Formula 3.1 (percent change), rearranging to solve for New. The current AV is the Old value.

Step 2 (Next Year):

The assessed values are known. Skip this step.

Step 3 (Next Year):

Recalculate the new mill rate using the new Property Tax and new Assessed Value from step 1 (Next Year). Apply Formula 7.3 and solve for the mill rate in the PTR.

Step 3 (Current Year):

Property Tax $= \$8,455,000,000 \times \dfrac{6.1998}{1,000} = \$52,419,309$

Step 1 (Next Year):

Next Year Property Tax $= \$52,419,309 + \$5,000,000 = \$57,419,309$

$$\begin{aligned} 3\% &= \frac{\text{New } - \$8,455,000,000}{\$8,455,000,000} \times 100 \\ \$253,650,000 &= \text{New} - \$8,455,000,000 \\ \$8,708,650,000 &= \text{New} = \text{Next Year Assessed Value} \end{aligned}$$

Step 3 (Next Year):

$$\begin{aligned} \$57,419,309 &= \$8,708,650,000 \times \frac{\text{Mill Rate}}{1,000} \\ 6.5934 &= \text{ Mill Rate} \end{aligned}$$

Calculator Instructions

For step 1 (Next Year) only, use the percent change function ($\Delta\%$) to calculate the New Assessed Value.

OLD	NEW	%CH	#PD
8455000000	Answer: 8,708,650,000	3	1

The current budget for the school board is $52,419,309, which will be increased to $57,419,309 next year. After the board adjusts for the increased assessed values of the properties, it needs to set the mill rate at 6.5934 next year, which is an increase of 0.4736 mills.

Contributors and Attributions

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7.3: Exchange Rates and Currency Exchange

You have set aside $6,000 in Canadian funds toward hostel costs during a long backpacking trip through the United States, Mexico, and Europe. After searching on the Internet, you decide to use Hotwire.com to reserve your hostel rooms. The website quotes you the following amounts for each country: 1,980 US dollars, 21,675 Mexican pesos, and 1,400 euros. Have you allocated enough money to cover these costs?

Whether you are a consumer backpacking around the world on distant vacations, investing in international securities, or shopping online at global Internet sites, you must pay for your purchases in local currencies out of your Canadian currency accounts. Businesses are no different as they export and import products to and from other countries. With outsourcing on the rise, it is also common for business services such as call centers and advertising agencies to be located abroad along with manufacturing facilities. Large-scale companies may have operations in several countries throughout the world.

All of these transactions and operations require the conversion of Canadian currency to a foreign currency or vice versa. This section shows you the basics of currency conversion rates and then explores finer details such as charges for currency conversion and what happens when one currency gets stronger or weaker relative to another.

Exchange Rates

An exchange rate between two currencies is defined as the number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa. Since two currencies are involved in every transaction, two published exchange rates are available. Let's use Canada and the United States to illustrate this concept.

- The first exchange rate indicates what one dollar of Canada's currency becomes in US currency.
- The second exchange rate indicates what one dollar of US currency becomes in Canadian currency.

These two exchange rates allow Canadians to determine how many US dollars their money can buy and vice versa. These currency rates have an inverse relationship to one another: if 1 Canadian dollar equals 0.80 US dollars, then 1 US dollar equals $\frac{1}{\$0.80}=1.25$ Canadian dollars.

Most Canadian daily and business newspapers publish exchange rates in their financial sections. Although exchange rates are published in a variety of ways, a currency cross-rate table, like the table below, is most common. Note that exchange rates fluctuate all of the time as currencies are actively traded in exchange markets. Therefore, any published table needs to indicate the date and time at which the rates were determined. Also note that the cells where the same currency appears show no published rate as you never need to convert from Canadian dollars to Canadian dollars!

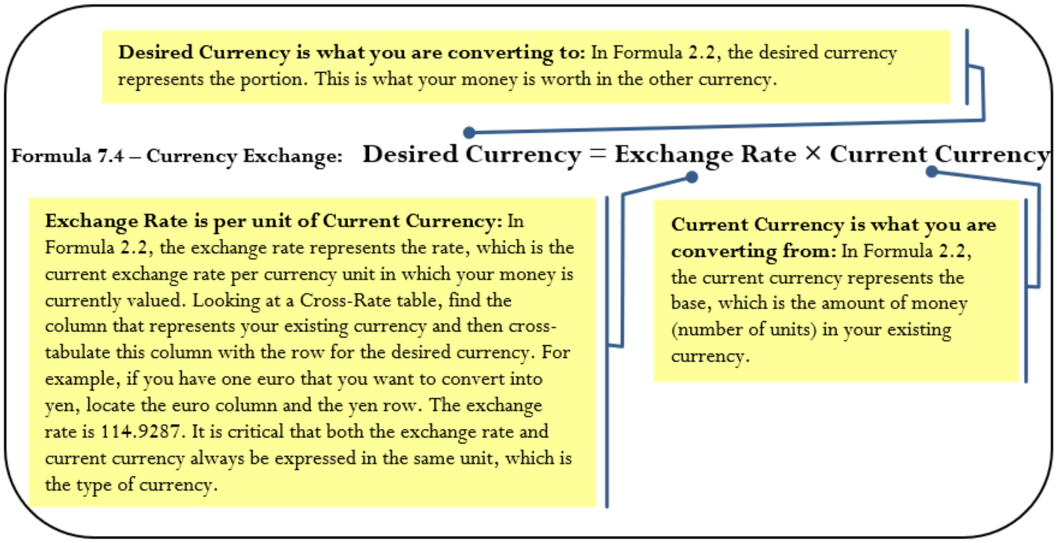
	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU$
Canadian Dollar (C$)	\	0.9787	1.4012	0.0122	0.0823	1.0360
US Dollar (US$)	1.0218	\	1.4317	0.0125	0.0841	1.0585
Euro (€)	0.7137	0.6985	\	0.0087	0.0588	0.7394
Japanese Yen (¥)	82.0233	80.2765	114.9287	\	6.7540	84.9747
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481	\	12.5814
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	\

In the table, all exchange rates have been rounded to four decimals. In true exchange markets, most exchange rates are expressed in 10 decimals or more such that currency exchanges in larger denominations are precisely performed. For the purposes of this textbook, we will use a four decimal standard to simplify the calculations while still illustrating the principles of currency exchange.

Technically, only half of the table is needed, since one side of the diagonal line is nothing more than the inverse of the other. For example, the euro is 0.7137 per $C\$$ on the bottom of the diagonal. The inverse, or $\frac{1}{0.7137} = 1.4011$ per €, is what is listed on top of the diagonal (the difference of 0.0001 is due to rounding to four decimals).

The Formula

The formula is yet again another adaptation of Formula 2.2 on Rate, Portion, and Base. Formula 7.4 expresses this relationship in the language of currency exchange.



Desired Currency is what you are converting to: In Formula 2.2, the desired currency represents the portion. This is what your money is worth in the other currency.

Formula 7.4 – Currency Exchange: $\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$

Exchange Rate is per unit of Current Currency: In Formula 2.2, the exchange rate represents the rate, which is the current exchange rate per currency unit in which your money is currently valued. Looking at a Cross-Rate table, find the column that represents your existing currency and then cross-tabulate this column with the row for the desired currency. For example, if you have one euro that you want to convert into yen, locate the euro column and the yen row. The exchange rate is 114.9287. It is critical that both the exchange rate and current currency always be expressed in the same unit, which is the type of currency.

Current Currency is what you are converting from: In Formula 2.2, the current currency represents the base, which is the amount of money (number of units) in your existing currency.

Formula 7.4

Important Notes

Currencies, exchange rates, and currency cross-tables all raise issues regarding decimals and financial fees.

1. **Decimals in Currencies**. Not all currencies in the world have decimals. Here in North America, MXN$, US$, and C$ have two decimals. Mexicans call those decimals centavos while Canadians and Americans call them cents. Australia and the European countries using the euro also have cents. However, the Japanese yen does not have any decimals in its currency. If you are unsure about the usage of decimals, perform a quick Internet search to clarify the issue.
2. **Financial Fees**. Technically, the rates in a cross-rate table are known as mid-rates. A mid-rate is an exchange rate that does not involve or provide for any charges for currency conversion. When you convert currencies, you need to involve a financial organization, which will charge for its services.
    a. **Sell Rates**. When you go to a bank and convert your domestic money to a foreign currency, the bank charges you a sell rate, which is the rate at which a foreign currency is sold. When you exchange your money, think of this much like a purchase at a store—the bank's product is the foreign currency and the price it charges is marked up to its selling price. The sell rate is always higher than the mid-rate in terms of $C\$$ per unit of foreign currency and thus is always lower than the mid-rate in terms of foreign currency per unit of $C\$$. For example, the exchange rate of $C\$$ per US$ is 0.9787 (you always look up the “per currency” column that you are purchasing). This means it will cost you $C\$$0.9787 to purchase US$1.00. The bank, though, will sell you this money for a sell rate that is higher, say $0.9987. This means it will cost you an extra $C\$$0.02 per US$\$$ to exchange the money. That $0.02 difference is the fee from the bank for its services, and it is how the bank makes a profit on the transaction.
    b. **Buy Rates**. When you go to a bank and convert your foreign currency back into your domestic money, the bank charges you a buy rate, which is the rate at which a foreign currency is purchased. The buy rate is always lower than the mid-rate in terms of $C\$$ per unit of foreign currency and thus is always higher than the mid-rate in terms of foreign currency per unit of C$. Using the same example as above, if you want to take your US$1.00 to the bank and convert it back to Canadian funds, the bank charges you a buy rate that is lower, say $0.9587. In other words, you receive $C\$$0.02 less per US$\$$. Again, the $0.02 difference is the bank's fee for making the currency exchange on your behalf.

How It Works

Follow these steps when performing a currency exchange:

Step 1: Identify all known variables. Specifically, identify the currency associated with any amounts. You also require the mid-rate. If buy rates and sell rates are involved, identify how these rates are calculated.

Step 2: If there are no buy or sell rates, skip this step. If buy and sell rates are involved, calculate these rates in the manner specified by the financial institution.

Step 3: Apply Formula 7.4 using the appropriate mid-rate, buy rate, or sell rate to convert currencies.

This section opened with your backpacking vacation to the United States, Mexico, and Europe, for which you were quoted prices of US$1,980, MXN$21,675, and €1,400 for hostels. Assume all purchases are made with your credit card and that your credit card company charges 2.5% on all currency exchanges. Can your C$6,000 budget cover these costs?

Step 1: There are three currency amounts: US$1,980, MXN$21,675, and €1,400. Using the cross-rate table, the Canadian exchange mid-rate per unit of each of these currencies is 0.9787 US$, 0.0823 MXN$, and 1.4012 €.

Step 2: Calculate the buy rates (since you are converting foreign currency into domestic currency) for each currency:

$$\text{US\$} = 0.9787(1.025) = 1.0032$$

$$\text{MXN\$} = 0.0823(1.025) = 0.0844$$

$$€ = 1.4012(1.025) = 1.4362$$

Step 3: Apply Formula 7.4 to each of these currencies:

$$\text{US\$: Desired Currency } = \text{US\$}1,980 \times 1.0032 = \text{C\$}1,986.34.$$

$$\text{MXN\$: Desired Currency } = \text{MXN\$}21,675 \times 0.0844 = \text{C\$}1,829.37.$$

$$€ : \text{Desired Currency } = € \ 1,400 \times 1.4362 = \text{C\$}2,010.68.$$

Putting the three amounts together, your total hostel bill is:

$$\$1,986.34 + \$1,829.37 + \$2,010.68 = \$5,826.39$$

Because this is under budget by $173.61, all is well with your vacation plans.

Things To Watch Out For

When working with currency exchange, probably the trickiest element is that you have to choose one of two inverse exchange rates depending on which way the money conversion is taking place. In any currency situation, it is important that you take the time to understand the basis on which the currency rate is being expressed. Typically, exchange rates are expressed on a per-unit basis in the country's domestic currency. For example, Canadians express the US dollar exchange rate on a per $C\$$ basis. From the cross-rate table, that exchange rate is 1.0218. In contrast, Americans express the Canadian dollar exchange rate on a per US$ basis, or 0.9787.

Paths To Success

Let the buyer beware when it comes to international transactions. If you have ever purchased and returned an item to an international seller, you may have noticed that you did not receive all of your money back. For most consumers, international purchases are made via credit cards. What most consumers do not know is that the credit card companies do in fact use buy and sell rates that typically charge 2.5% of the exchange rate when both buying and selling.

For example, if you purchase a US$2,000 item at the rates listed in the cross-rate table, your credit card is charged $\$2,000 \times 0.9787(1.025) = \$2,006.40$. If you return an item you do not want, your credit card is refunded $\$2,000 \times 0.9787(0.975) = \$1,908.40$. In other words, you are out $\$2,006.40 - \$1,908.40 = \$98!$ This amount represents your credit card company's charge for the currency conversion—a whopping 4.9% of your purchase price!

? Exercise 7.3.1: Give It Some Thought

In each of the following situations and using the cross-rate table, determine on a strictly numerical basis whether you would have more or fewer units of the target currency than of the original currency. You want to convert:

a. C$ into US$
b. AU$ into MXN$
c. MXN$ into ¥
d. C$ into €

Answer

a. More, since C$1 becomes US$1.0218
b. More, since AU$1 becomes MXN$12.5814
c. More, since MXN$1 becomes ¥6.7540
d. Less, since C$1 becomes €0.7137

✓ Example 7.3.1: A Straight Currency Conversion

Strictly using the mid-rates from the cross-rate table presented earlier, if you wanted to convert C$1,500 into MXN$ for your spring break vacation in Cancun, Mexico, how many Mexican pesos would you have?

Solution

Take Canadian currency and convert it into the desired Mexican currency.

What You Already Know

Step 1:

The amount and the exchange rate are known:

Current Currency = $1,500

Exchange Rate = 12.1445

How You Will Get There

Step 2:

No buy or sell rates are involved. Skip this step.

Step 3:

Apply Formula 7.4.

Perform

Step 3:

Desired Currency = 12.1445 × $1,500 = $18,216.75

You have $18,216.75 Mexican pesos available for your spring break vacation.

✓ Example 7.3.2: Calculating Profit Using Buy and Sell Rates

A Mexican manufacturer imports its parts from Canada and assembles its product in Mexico, then exports some of the finished product back to Canada to sell at retail. Suppose the total cost of the imported parts, purchased and paid for in Canadian funds, is C$5.50 per unit, assembly costs are MXN$24.54 per unit, and expenses are MXN$4.38 per unit. If the product is exported back to Canada at a selling price of C$9.95 (for which its Canadian customers pay in Canadian funds), what is the total profit in Mexican pesos on 10,000 units? The manufacturer's financial institution charges a sell rate 2% higher than the mid-rate and has a buy rate that is 3% lower than the mid-rate. Use the mid-rates from the cross-rate table.

Solution

Calculate the total profit (P) in MXN$ for the Mexican manufacturer. However, since money is being expressed in different currencies, you must first convert all amounts into Mexican pesos.

What You Already Know

Step 1:

You know the costs, expenses, price, and mid-rate:

$$C_{parts} = 10,000 \times C\$5.50 = C\$55,000$$

$$C_{assembly} = 10,000 \times \text{MXN}\$24.54 = \text{MXN}\$245,400$$

$$E = 10,000 \times \text{MXN}\$4.38 = \text{MXN}\$43,800$$

$$S = 10,000 \times C\$9.95 = C\$99,500$$

Mid-Rate = 12.1445 per $C\$$

Sell Rate = 2% higher

Buy Rate = 3% lower

Note the mid-rate used is “per $C\$$” since the manufacturer needs to purchase Canadian funds to pay its suppliers and also needs to sell its Canadian revenues to the bank.

How You Will Get There

Step 2:

Calculate the buy and sell rates.

Step 3 (Parts):

Convert the purchase of the parts in $C\$$ into $\text{MXN}\$$ using the sell rate. Apply and adapt Formula 7.4.

Step 3 (Sale):

Convert the sale of the product in $C\$$ into $\text{MXN}\$$ using the buy rate. Apply and adapt Formula 7.4.

Step 4:

All amounts are in $\text{MXN}\$$. Apply Formula 6.5: $S = C_{parts} + C_{assembly} + E + P$

Perform

Step 2:

Sell Rate = 2% higher than mid-rate = 12.1445 (1 + 0.02) = 12.3874

Buy Rate = 3% lower than mid-rate = 12.1445 (1 − 0.03) = 11.7802

Step 3 (Parts):

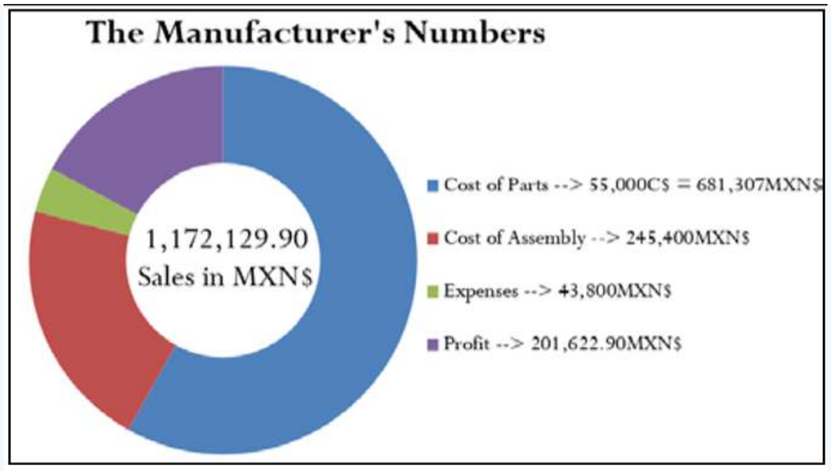
$$C_{parts} \text{ in MXN\$} = 12.3874 \times \$55,000 = \$681,307$$

Step 3 (Sale):

$$S \text{ in MXN\$} = 11.7802 \times \$99,500 = \$1,172,129.90$$

$$\begin{aligned} \$1,172,129.90 &= \$681,307 + \$245,400 + \$43,800 + P \\ \$1,172,129.90 &= \$970,507 + P \\ \$201,622.90 &= P \end{aligned}$$

After converting all Canadian currency into Mexican pesos and factoring in the financial institution's buy and sell rates, the manufacturer realizes a profit of $201,622.90 in Mexican pesos.



Currency Appreciation and Depreciation

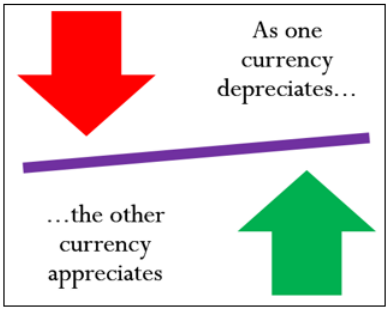
An analyst on Global News is discussing how the Canadian dollar has strengthened against the US dollar. Your first reaction is that a strong Canadian dollar ought to be good thing, so hearing that this change might hurt Canada's exports, you wonder how that could be.

Currencies are actively traded in the international marketplace, which means that exchange rates are changing all the time. As such, exchange rates rise and decline. A currency appreciates (or strengthens) relative to another currency when it is able to purchase more of that other currency than it could previously. A currency depreciates (or weakens) relative to another currency when it is able to purchase less of that other currency than it could previously. Take a look at two examples illustrating these concepts:

- EXAMPLE 1: If C$1 buys US$1.02 and the exchange rate rises to US$1.03, then your C$1 purchases an additional penny of the US dollars. Therefore, the Canadian currency appreciates, or strengthens, relative to the US dollar.
- EXAMPLE 2: Similarly, if the exchange rate drops to US$1.01, then your C$1 purchases one less penny of the US dollars. Therefore, the Canadian currency depreciates, or weakens, relative to the US dollar.

These concepts are particularly important to international business and global economies. Generally speaking, when a currency appreciates it has a positive effect on imports from the other country because it costs less money than it used to for domestic companies to purchase the same amount of products from the other country. However, the currency appreciation tends to also have a negative effect on exports to other countries because it costs the foreign companies more money to purchase the same amount of products from the domestic companies.

Important Notes

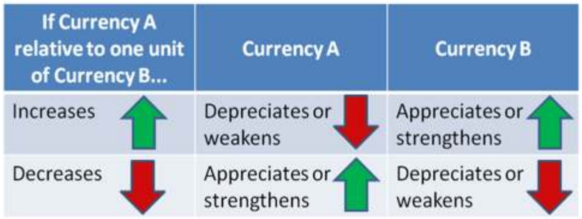


It is critical to observe that if currency A appreciates relative to currency B, then the opposite is true for currency B relative to currency A (currency B depreciates relative to currency A). The figure to the right illustrates this concept. Recalling Example 1 above, the Canadian currency appreciated, so the US currency depreciated. In Example 2, the Canadian currency depreciated, so the US currency appreciated.

How It Works

When you work with currency appreciation or depreciation, you still use the same basic steps as before. The additional skill you require in the first step is to adjust an exchange rate appropriately based on how it has appreciated or depreciated.

Things To Watch Out For



If Currency A relative to one unit of Currency B...	Currency A	Currency B
Increases ↑	Depreciates or weakens ↓	Appreciates or strengthens ↑
Decreases ↓	Appreciates or strengthens ↑	Depreciates or weakens ↓

It is very easy to confuse the two relative currencies, their values, and the concepts of appreciation and depreciation. For example, if the exchange rate increases between currency A relative to a unit of currency B, which exchange rate appreciated? If currency A has increased per unit of currency B, then it takes more money of currency A to buy one unit of currency B. As a result, currency B appreciates because a single unit of currency B can now buy more of currency A. For example, if the exchange rate is US$1.0218 relative to C$1 and it increases to US$1.0318 relative to C$1, then the Canadian dollar purchases one penny more. The figure helps you understand the relationships involved and provides a visual reminder of which direction everything is moving.

? Exercise 7.3.2: Give It Some Thought

a. If the exchange rate in terms of US dollars per unit of euros increases, which currency weakened?
b. If the Australian dollar weakens against the Canadian dollar, did the exchange rate increase or decrease in terms of Australian dollars per unit of Canadian dollars?
c. If the exchange rate in terms of yen per unit of the Mexican pesos decreases, which currency weakened?
d. If the British pound (£) appreciates against the US dollar, did the exchange rate increase or decrease in terms of pounds per unit of US dollars?

Answer

a. US dollars
b. Increase
c. Mexican pesos
d. Decrease

✓ Example 7.3.3: Effects of Currency Appreciation

A Canadian manufacturer requires parts from the United States. It purchases from its supplier in lots of 100,000 units at a price of US$7.25 per unit. Since the last time the manufacturer made a purchase, the Canadian dollar has appreciated 0.0178 from the previous mid-rate of US$1.0218 per C$. If the sell rate is 1.5% above the mid-rate, how have the manufacturer's costs changed?

Solution

Calculate the cost of the product both before and after the currency appreciation. The difference between the two numbers is the change in the manufacturer's costs.

What You Already Know

Step 1:

The following purchase and exchange rates are known:

Current Currency = Total Purchase = 100,000 × $7.25 = US$725,000

Mid-Rate = 1.0218

Sell Rate = 1.5% higher

Canadian appreciation = 0.0178

How You Will Get There

Step 2:

Calculate the old and new sell rates, factoring in the currency appreciation. Notice that the Current Currency is in US dollars but the sell rates are per Canadian dollar. You will need to invert the rates so that the exchange rate is expressed per US dollar to match the Current Currency.

Step 3:

Apply Formula 7.4 for each transaction.

Step 4:

Calculate the difference between the two numbers to determine the change in cost.

Perform

Step 2:

Previous Sell Rate = 1.0218 × (1 + 0.015) = 1.0371 per C\$

Previous Sell Rate $= \frac{1}{1.0371} = 0.9642$ per US\$

If the Canadian dollar has appreciated, it buys more US dollars per C\$. Therefore,

New Sell Rate = (1.0218 + 0.0178) × (1 + 0.015) = 1.0552 per C\$

New Sell Rate $= \frac{1}{1.0552} = 0.9477$ per US\$

Step 3:

Previous C\$ = 0.9642 × \$725,000 = \$699,045

New C\$ = 0.9477 × \$725,000 = \$687,082.50

Step 4:

Change in Cost = New C\$ − Previous C\$ = \$687,082.50 − \$699,045 = −\$11,962.50

The manufacturer has its input costs decrease by \$11,962.50 since C\$1 now purchases more US\$.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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7.4: Invoicing - Terms of Payment and Cash Discounts

What a way to start your Monday morning! With dread you face a great stack of envelopes in your in-basket. The first envelope contains an invoice from your major supplier indicating an outstanding balance of $3,600 with terms of 2/10, 1/20, net 30. In the second envelope, an office supplies bill totals $500 with terms of 3/15, net 45 EOM. The third envelope holds yet another invoice from your transportation company indicating an outstanding balance of $21,000 with terms of 2/15, 1/25, net 60 ROG. It also indicates that you have an overdue balance of $4,000 subject to a 3% monthly penalty. Before opening any more envelopes that probably contain still more bills, you pour yourself another cup of coffee and settle down to figuring out how much you need to pay and when.

In the world of business, most purchases are not paid for in cash. Instead, businesses tend to work through an invoicing system in which they send out bills to their clients on a regular basis. In accounting, this means that the purchase is placed into accounts receivables until such time as a cheque arrives and the purchase is converted to cash. Invoices provide detailed transaction information, listing the amount owed and also indicating the terms on which payment is expected. Companies may offer what are known as cash discounts as incentives for early invoice payment. The rationale for these discounts is simple—a sale is not a sale until you have the cash in hand. The longer an invoice remains in accounts receivable, the less likely that it will be paid. Thus, it might turn into bad debt, which the creditor cannot collect at all.

Invoicing is less common in consumer purchases because of the sheer volume of transactions involved and the higher risk of nonpayment. Imagine purchasing items at Walmart and receiving an invoice to pay your bill next month instead of paying cash. It is hard to fathom the number of invoices Walmart would have to distribute monthly. How much would it cost to collect those debts? How many of those invoices would go unpaid? That is why consumer purchases typically do not involve invoicing.

Invoicing does commonly occur at a consumer level on credit card transactions along with many services where the business may not be able to assess the exact amount of the bill at the time of the transaction or until the service is delivered. Two examples illustrate this point:

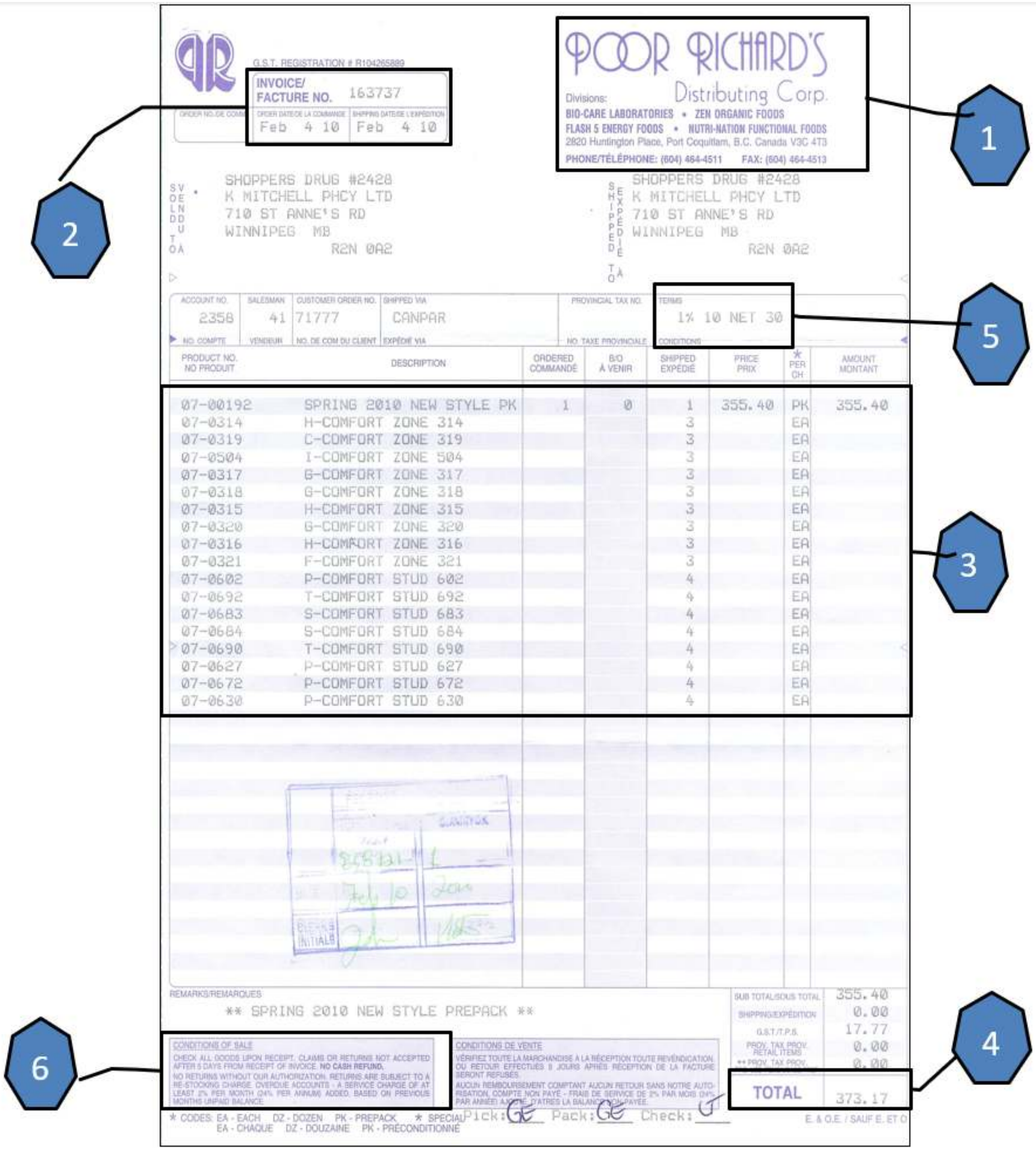
1. Think of your MasterCard bill. You are able to make purchases, say, from March 9 to April 8. Then on April 9 the company sends out a statement saying you have until April 29 to pay your bill. If you do not, interest and late payment penalties are involved.
2. You have a dental visit for a regular cleaning. Before charging you, the dentist's office needs to determine how much your insurance will cover. It may not find out the answer for a day or two, so it sends you an invoice at a later date once it hears from the insurance company. The invoice terms indicate that payment is due upon receipt and you will incur late penalties if payment is not forthcoming.

This section explores the most common aspects of invoicing, including terms of payments and cash discounts. You will learn to calculate the amount required to pay invoices and how to reduce the outstanding balance of an invoice if a partial payment is received. Additionally, the calculation of late payments and penalties is introduced.

Invoice Terms and Invoice Dating

You must know how to read a business invoice. The figure below[1] provides a sample invoice and highlights the following areas:

1. **Invoicing Company**. The invoice must identify who is sending and issuing the invoice.
2. **Invoice Date**. The date on which the invoice was printed, along with the invoice tracking number (in this case, the order date and shipping date are equivalent to the invoice date). When an invoice is paid, the cheque must reference the invoice number so that the invoicing company can identify which invoice to credit the payment to.
3. **Transaction Details**. The details of the transaction might include number of units, unit prices, and any discounts for which the item is eligible.
4. **Invoice Total**. The total amount owing is indicated, including any taxes or additional charges.
5. **Terms of Payment**. The terms of payment include any cash discounts and due dates. The date of commencement (as discussed below) is determined from this part of the invoice.
6. **Late Penalty**. A penalty, if any, for late payments is indicated on the invoice. Whether or not a company enforces these late penalties is up to the invoice-issuing organization.



G.S.T. REGISTRATION # R104265889

INVOICE/ FACTURE NO. 163737

ORDER NO./DE COMM	ORDER DATE/DE LA COMMANDE	SHIPPING DATE/DE L'EXPÉDITION
	Feb 4 10	Feb 4 10

POOR RICHARD'S
Distributing Corp.
Divisions:
BIO-CARE LABORATORIES • ZEN ORGANIC FOODS
FLASH 5 ENERGY FOODS • NUTRI-NATION FUNCTIONAL FOODS
2820 Huntington Place, Port Coquitlam, B.C. Canada V3C 4T3
PHONE/TÉLÉPHONE: (604) 464-4511 FAX: (604) 464-4513

SOLD TO / VENDU À
SHOPPERS DRUG #2428
K MITCHELL PHCY LTD
710 ST ANNE'S RD
WINNIPEG MB
R2N 0A2

SHIPPED TO / EXPÉDIÉ À
SHOPPERS DRUG #2428
K MITCHELL PHCY LTD
710 ST ANNE'S RD
WINNIPEG MB
R2N 0A2

ACCOUNT NO. / NO. COMPTE	SALESMAN / VENDEUR	CUSTOMER ORDER NO. / NO. DE COM DU CLIENT	SHIPPED VIA / EXPÉDIÉ VIA	PROVINCIAL TAX NO. / NO. TAXE PROVINCIALE	TERMS / CONDITIONS
2358	41	71777	CANPAR		1% 10 NET 30

PRODUCT NO. NO PRODUIT	DESCRIPTION	ORDERED COMMANDÉ	B/O À VENIR	SHIPPED EXPÉDIÉ	PRICE PRIX	* PER CH	AMOUNT MONTANT
07-00192	SPRING 2010 NEW STYLE PK	1	0	1	355.40	PK	355.40
07-0314	H-COMFORT ZONE 314			3		EA	
07-0319	C-COMFORT ZONE 319			3		EA	
07-0504	I-COMFORT ZONE 504			3		EA	
07-0317	G-COMFORT ZONE 317			3		EA	
07-0318	G-COMFORT ZONE 318			3		EA	
07-0315	H-COMFORT ZONE 315			3		EA	
07-0320	G-COMFORT ZONE 320			3		EA	
07-0316	H-COMFORT ZONE 316			3		EA	
07-0321	F-COMFORT ZONE 321			3		EA	
07-0602	P-COMFORT STUD 602			4		EA	
07-0692	T-COMFORT STUD 692			4		EA	
07-0683	S-COMFORT STUD 683			4		EA	
07-0684	S-COMFORT STUD 684			4		EA	
07-0690	T-COMFORT STUD 690			4		EA	
07-0627	P-COMFORT STUD 627			4		EA	
07-0672	P-COMFORT STUD 672			4		EA	
07-0630	P-COMFORT STUD 630			4		EA	

CLERKS INITIALS

Feb 10 2010

REMARKS/REMARQUES

** SPRING 2010 NEW STYLE PREPACK **

SUB TOTAL/SOUS TOTAL	355.40
SHIPPING/EXPÉDITION	0.00
G.S.T./T.P.S.	17.77
PROV. TAX PROV. RETAIL ITEMS	0.00
** PROV. TAX PROV.	0.00
TOTAL	373.17

CONDITIONS OF SALE

CHECK ALL GOODS UPON RECEIPT. CLAIMS OR RETURNS NOT ACCEPTED AFTER 5 DAYS FROM RECEIPT OF INVOICE. **NO CASH REFUND.**

NO RETURNS WITHOUT OUR AUTHORIZATION. RETURNS ARE SUBJECT TO A RE-STOCKING CHARGE. OVERDUE ACCOUNTS - A SERVICE CHARGE OF AT LEAST 2% PER MONTH (24% PER ANNUM) ADDED. BASED ON PREVIOUS MONTHS UNPAID BALANCE.

CONDITIONS DE VENTE

VÉRIFIEZ TOUTE LA MARCHANDISE À LA RÉCEPTION TOUTE REVENDICATION OU RETOUR EFFECTUÉS 5 JOURS APRÈS RÉCEPTION DE LA FACTURE SERONT REFUSÉS.

AUCUN REMBOURSEMENT COMPTANT AUCUN RETOUR SANS NOTRE AUTORISATION. COMPTE NON PAYÉ - FRAIS DE SERVICE DE 2% PAR MOIS (24% PAR ANNÉE) AJOUTÉ D'ATRES LA BALANCE NON PAYÉE.

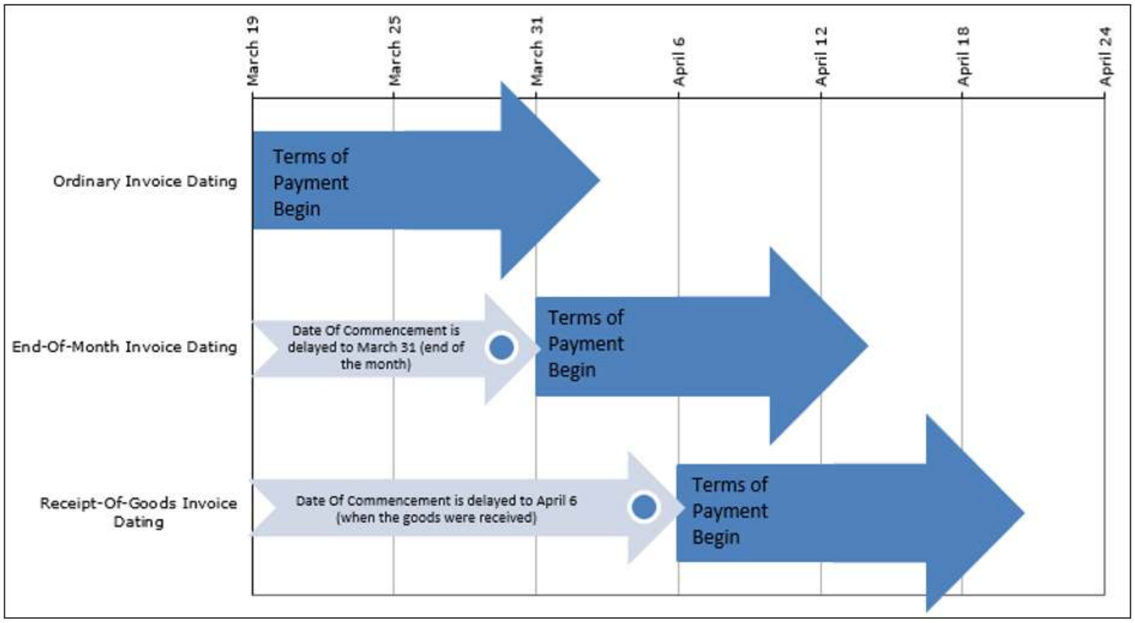
Pick: GE Pack: GE Check:

* CODES: EA - EACH DZ - DOZEN PK - PREPACK * SPECIAL
EA - CHAQUE DZ - DOUZAINE PK - PRÉCONDITIONNÉ

E. & O.E. / SAUF E. ET O

Three Dates of Commencement

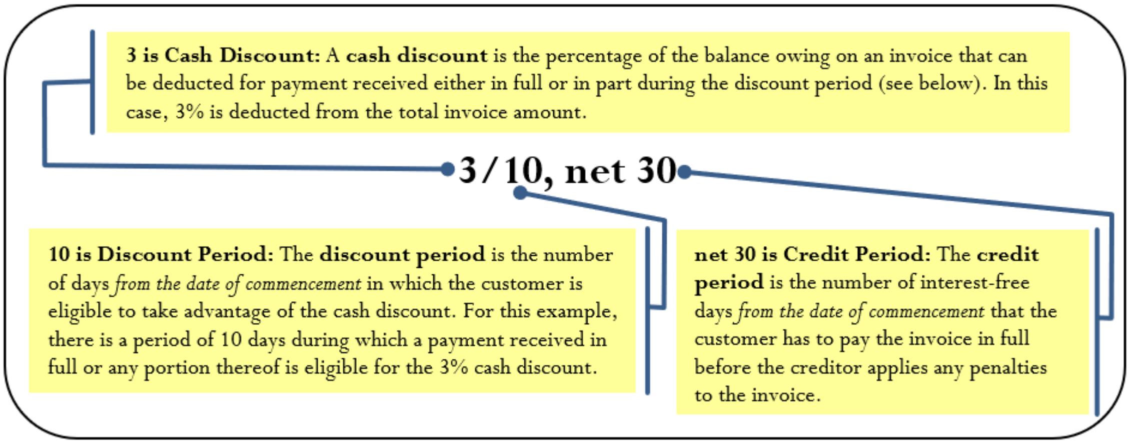
All invoice terms are affected by what is known as the date of commencement, which is the first day from which all due dates stated on the invoice will stem. The date of commencement is determined in one of three ways as illustrated below.


March 19
March 25
March 31
April 6
April 12
April 18
April 24
Ordinary Invoice Dating
Terms of Payment Begin
End-Of-Month Invoice Dating
Date Of Commencement is delayed to March 31 (end of the month)
Terms of Payment Begin
Receipt-Of-Goods Invoice Dating
Date Of Commencement is delayed to April 6 (when the goods were received)
Terms of Payment Begin

1. **Ordinary Invoice Dating**. In ordinary invoice dating, or just invoice dating for short, the date of commencement is the same date as the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on March 19. This is the default manner in which most companies issue their invoices, so if an invoice does not specify any other date of commencement you can safely assume it is using ordinary dating.
2. **End-of-Month Invoice Dating**. End-of-month invoice dating applies when the terms of payment include the wording "end-of-month" or the abbreviation "EOM" appears after the terms of payment. In end-of-month dating, the date of commencement is the last day of the same month as indicated by the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on the last day of March, or March 31. Many companies use this method of dating to simplify and standardize all of their due dates, in that if the date of commencement is the same for all invoices, then any terms of payment will also share the same dates.
3. **Receipt-Of-Goods Invoice Dating**. Receipt-of-goods invoice dating applies when the terms of payment include the wording "receipt-of-goods" or the abbreviation "ROG" appears after the terms of payment. In receipt-of-goods dating, the date of commencement is the day on which the customer physically receives the goods. Therefore, if the invoice is printed on March 19 but the goods are not physically received until April 6, then all terms of payment commence on April 6. Companies with long shipping times involved in product distribution or long lead times in production commonly use this method of dating.

Terms of Payment

The most common format for the terms of paying an invoice is shown on the next page and illustrated by an example.



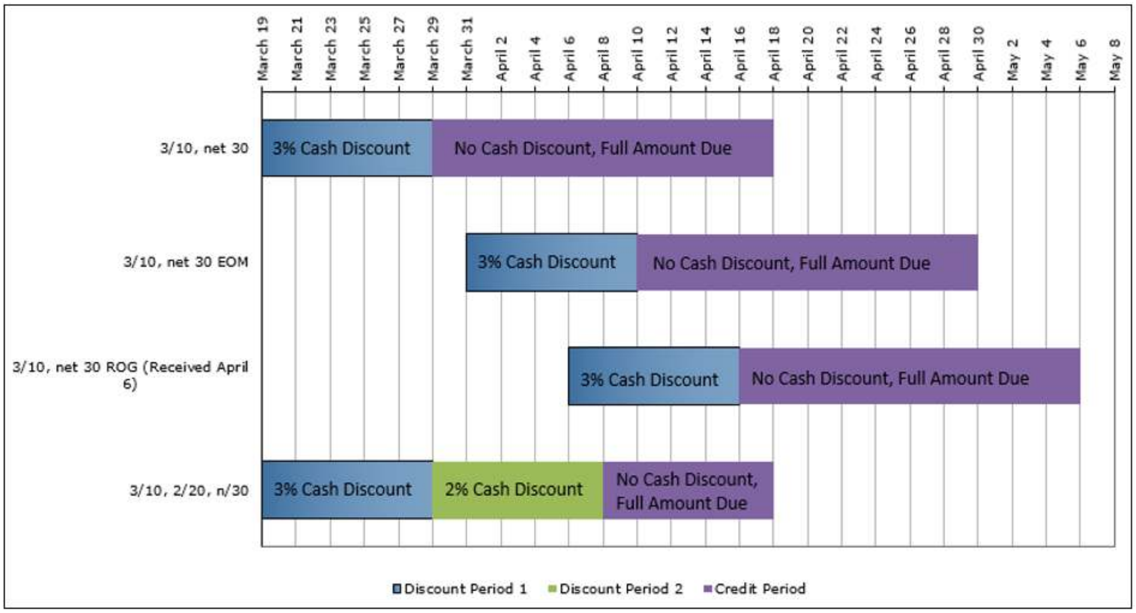
3 is Cash Discount: A **cash discount** is the percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period (see below). In this case, 3% is deducted from the total invoice amount.

3/10, net 30

10 is Discount Period: The **discount period** is the number of days *from the date of commencement* in which the customer is eligible to take advantage of the cash discount. For this example, there is a period of 10 days during which a payment received in full or any portion thereof is eligible for the 3% cash discount.

net 30 is Credit Period: The **credit period** is the number of interest-free days *from the date of commencement* that the customer has to pay the invoice in full before the creditor applies any penalties to the invoice.

The figure below plots the invoice term of "3/10, net 30" along a time diagram to illustrate the terms of payment. This term is combined with various dates of commencement using the examples from the "Date of Commencement" section.



Note the following observations:

- In all four scenarios, the total length of all bars is the same, since the credit period is 30 days.
- Although all four invoices have the same invoice date of March 19, the date of commencement is modified in the ROG and EOM scenarios, which shift the discount and credit periods into the future. In these cases, any payment before the end of the discount period qualifies for the discount. For example, in the EOM scenario a company could pay its bill early on March 27 and qualify for the 3% cash discount.

You may see invoice terms displayed in several other common formats:

1. 3/10, n/30: In this case, the word "net" has been abbreviated to "n/". This is illustrated in the fourth scenario in the figure.
2. 3/10, 2/20, n/30. In this case, multiple cash discounts are being offered, meaning 3% within 10 days from the date of commencement and 2% from the 11th to the 20th day from the date of commencement. This is also illustrated in the fourth scenario of the figure.
3. n/30. In this case, no cash discount is offered and only the credit period is identified.

Rules for Invoice Terms

Some common business practices are implemented across most industries. Note that you must always check with the invoicing organization to ensure it is applying these practices.

1. **No Net Figure**. If no net figure is stated in the terms of payment, you should assume that the credit period is 20 days after the last discount period. If there are no discount periods, then the credit period is 20 days from the date of commencement. For example, "3/10, 2/15" means that the credit period ends 20 days after the second discount period of 15 days. Hence, the credit period is 35 days after the date of commencement.
2. **No Cash Discount**. Not every invoice receives a cash discount. If no terms indicate a cash discount, then the invoicing company seeks full payment only. For example, "n/30" means that no cash discount applies and the credit period ends 30 days from the date of commencement.
3. **Nonbusiness Days**. Most businesses operate Monday through Friday and are closed on weekends and holidays. As such, any date falling on a nonworking day is moved to the next business day. For example, if an invoice is ordinary dated December 21 and lists a cash discount of 2/10, the end of the discount period falls on January 1. Since this is New Year's Day, the discount period extends to January 2. In this textbook, you should apply this practice to any of the five known statutory holidays discussed in Chapter 4 (New Year's Day, Good Friday or Easter Monday in Quebec only, Canada Day, Labour Day, and Christmas Day).

Three Types of Payments

When businesses pay invoices, three situations can occur:

1. **Full Payment**. A full payment means that the company wants to pay its invoice in full and reduce its balance owing to zero dollars. This is the most common practice.
2. **Partial Payment**. A partial payment means that the company wants to lower its balance owing but will not reduce that balance to zero. A company will generally employ this method when it wants to either lower its accounts payable or demonstrate good faith in paying its invoices. Partial payment may also occur if the company wants to take advantage of a cash discount but lacks the funds to clear the invoice in its entirety.
3. **Late Payment**. A late payment occurs when the company pays its invoice either in full or partially after the credit period has elapsed. Late payments occur for a variety of reasons, but the most common are either insufficient funds to pay the invoice or a simple administrative oversight.

The rest of this section shows you how to handle each of these types of payments mathematically, and it also explores the implications of cash discounts.

? Exercise 7.4.1: Give It Some Thought

In each of the following cases, determine which term of payment results in the longest credit period extending from the invoice date.

a. 3/15, n/45 or 2/10, 1/20
b. 3/15, n/45 or 2/10, n/35 ROG, or 2/20 EOM on an invoice dated September 22 and goods received on September 29

Answer

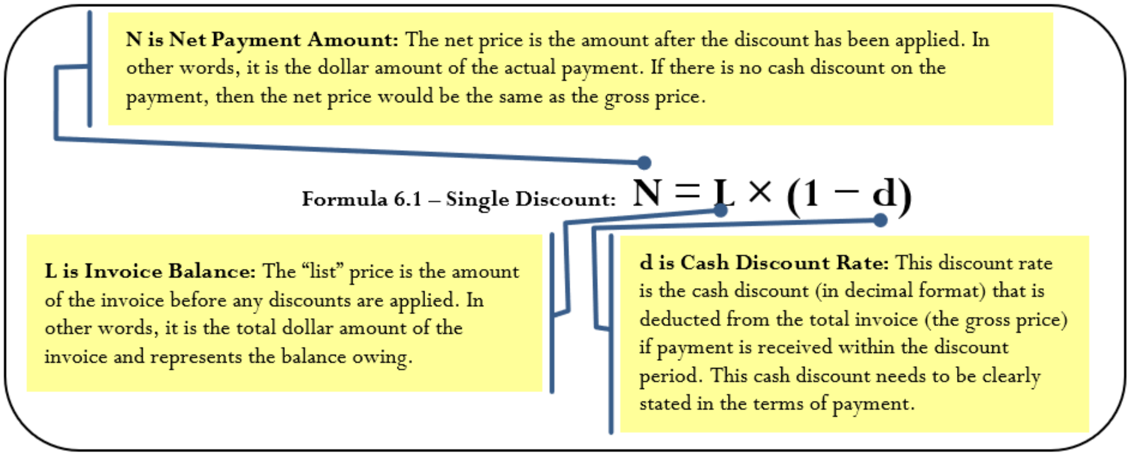
a. 30 days or 35 days
b. 45 days or 40 days
c. 60 days or 54 days (24 days left in January plus 30 more)
d. 45 days or 42 days (7 days until received plus 35) or 48 days (8 left in month plus 40 more)

Full Payments

In the opening scenario in this section, the first invoice on your desk was for $3,600 with terms of 2/10, 1/20, net 30. Suppose that invoice was dated March 19. If you wanted to take advantage of the 2% cash discount, when is the last day that payment could be received, and in what amount would you need to write the cheque? In this section, we will look at how to clear an invoice in its entirety.

The Formula

The good news is that a cash discount is just another type of discount similar to what you encountered in Section 6.1, and you do the calculations using the exact same formula. Apply Formula 6.1 on single discounts, which is reprinted below.



N is Net Payment Amount: The net price is the amount after the discount has been applied. In other words, it is the dollar amount of the actual payment. If there is no cash discount on the payment, then the net price would be the same as the gross price.

Formula 6.1 – Single Discount: $\mathbf{N = L \times (1 - d)}$

L is Invoice Balance: The "list" price is the amount of the invoice before any discounts are applied. In other words, it is the total dollar amount of the invoice and represents the balance owing.

d is Cash Discount Rate: This discount rate is the cash discount (in decimal format) that is deducted from the total invoice (the gross price) if payment is received within the discount period. This cash discount needs to be clearly stated in the terms of payment.

How It Works

Follow these steps when working with payments and invoices:

Step 1: Look for and identify key information such as the invoice date, date of commencement modifiers such as EOM or ROG, terms of payment, late penalty, and the invoice amount.

Step 2: Draw a timeline similar to the figure on the next page. On this timeline, chronologically arrange from left to right the invoice date and amount, the date of commencement, the end of any discount (if any) or credit periods, cash discounts (if any) that are being offered, and penalties (if any).

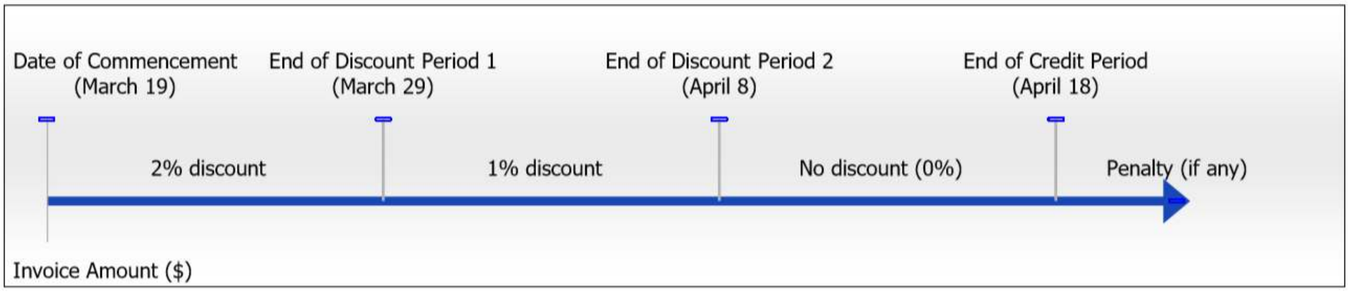
Step 3: Determine when payments are being made and in what amount. Locate them on the timeline.

Step 4: Apply the correct calculations for the payment, depending on whether the payment is a full payment, partial payment, or late payment.

Let's continue with the example of the first invoice on your desk. You mail in a cheque to be received by March 29. What amount is the cheque?

Step 1: The invoice amount is L = \$3,600, invoice date is March 19, and terms of payment are 2/10, 1/20, net 30.

Step 2: The figure on the next page displays the invoice timeline.


Date of Commencement (March 19)
End of Discount Period 1 (March 29)
End of Discount Period 2 (April 8)
End of Credit Period (April 18)
2% discount
1% discount
No discount (0%)
Penalty (if any)
Invoice Amount ($)

Step 3: Note on the timeline that a payment on March 29 is the last day of the 2% discount period.

Step 4: According to Formula 6.1, the amount to pay is $N = \$3,600 \times (1 - 0.02) = \$3,600 \times .098 = \$3,528$. A cheque for \$3,528 pays your invoice in full. By taking advantage of the cash discount, you reduce your payment by \$72.

Things To Watch Out For

If there is one area of invoicing that causes the most confusion, it is assigning information to the net price and invoice amount variables. These are commonly assigned backwards. Remember these rules so that you always get the correct answer:

1. The invoice amount or any reference to the balance owing on an invoice is always a list price.

$$\text{List Price} = \text{Invoice Amount}$$

2. The payment of an invoice, whether in full, partial, or late, is always a net price.

$$\text{Net Price } = \text{ Payment Amount}$$

Paths To Success

Who cares about 1% or 2% discounts when paying bills? While these percentages may not sound like a lot, remember that these discounts occur over a very short time frame. For example, assume you just received an invoice for $102.04 with terms of 2/10, n/30. When taking advantage of the 2% cash discount, you must pay the bill 20 days early, resulting in a $100 payment. That is a $2.04 savings over the course of 20 days. To understand the significance of that discount, imagine that you had $100 sitting in a savings account at your bank. Your savings account must have a balance of $102.04 twenty days later. This requires your savings account to earn an annual interest rate of 44.56%! Therefore, a 20-day 2% discount is the same thing as earning interest at 44.56%. Outrageously good!

Exercise 7.4.2: Give It Some Thought

In each of the following cases, determine the cash discount or penalty for which the payment qualifies.

Invoice Date	Terms of Payment	Payment Received on
1. April 14	2/10, n/30	April 24
2. July 7	3/10, 2/20, n/30 EOM, 1% per month penalty	August 12
3. November 12 (goods received November 28)	2/20 ROG, 2% per month penalty	December 29
4. February 27 (non–leap year)	4/10, 2/15, 1/25 EOM	March 25

Answer

a. 2% discount
b. 2% discount
c. no discount (0%) or penalty
d. 1% discount

Example 7.4.1: Paying Your Bills in Full

You receive an invoice for $35,545.50 dated August 14. The terms of payment are listed as 3/10, 2/20, net 45 EOM. The accounting department is considering paying this debt on one of three days. Determine the full payment required if payment is received by the invoicing company on each of the following dates:

a. September 3
b. September 19
c. September 30

Solution

In each case, calculate the full payment (N) after applying any cash discount.

What You Already Know

Step 1:

The invoice amount, terms of payment, and any payments are known:

L = $35,545.50

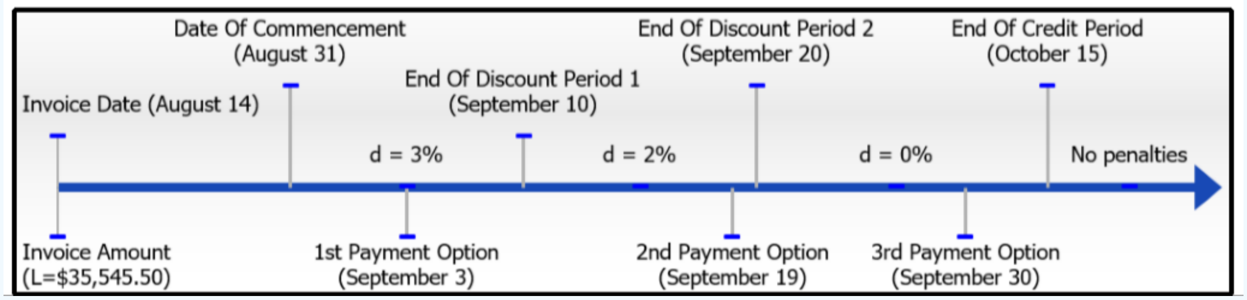
Invoice date = August 14

Terms of payment = 3/10, 2/20, net 45 EOM

Three payment date options are September 3, September 19, and September 30.

Steps 2 & 3:

The figure below illustrates the timeline for the invoice and identification of payments.


Date Of Commencement (August 31)
End Of Discount Period 2 (September 20)
End Of Credit Period (October 15)
End Of Discount Period 1 (September 10)
Invoice Date (August 14)
d = 3%
d = 2%
d = 0%
No penalties
Invoice Amount (L=$35,545.50)
1st Payment Option (September 3)
2nd Payment Option (September 19)
3rd Payment Option (September 30)

How You Will Get There

Step 4:

Calculate the payment required by applying Formula 6.1. As noted on the timeline:

a. The first payment option qualifies for $d = 3\%$.
b. The second payment option qualifies for $d = 2\%$.
c. The third payment option qualifies for $d = 0\%$ (no cash discount).

Perform

Step 4:

a. $N = \$35{,}545.50 \times (1 - 0.03) = \$35{,}545.50 \times 0.97 = \$34{,}479.14$
b. $N = \$35{,}545.50 \times (1 - 0.02) = \$35{,}545.50 \times 0.98 = \$34{,}834.59$
c. $N = \$35{,}545.50 \times (1 - 0) = \$35{,}545.50 \times 1 = \$35{,}545.50$

(Note that since there is no discount in part (c), you could have concluded that $N = L$)

If the invoicing company receives payment on September 3, a 3% discount is allowed and $34,479.14 clears the invoice. If the payment is received on September 19, a 2% discount is allowed and $34,834.59 pays it off. Finally, if the payment arrives on September 30, there is no discount so the full invoice amount of $35,545.50 is due.

Partial Payments

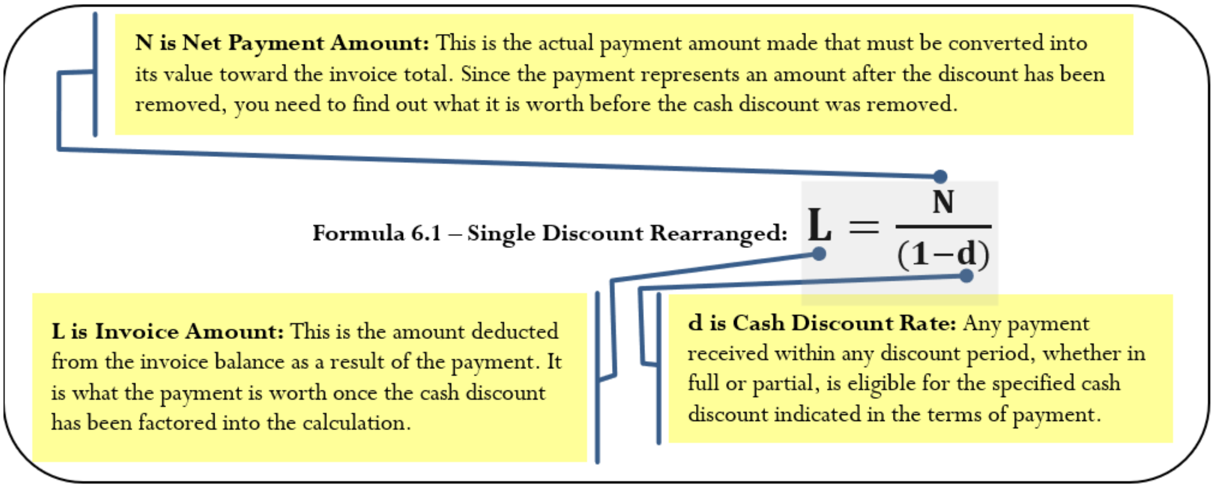
In the section opener, your third invoice is for $21,000 with terms of 2/15, 1/25, net 60 ROG. After you paid the $3,528 to clear the first invoice, you realize that your company has insufficient funds to take full advantage of the 2% cash discount being offered by the transportation company. Not wanting to lose out entirely, you decide to submit a partial payment of $10,000 before the first discount period elapses. What is the balance remaining on the invoice?

Unless you pay attention to invoicing concepts, it is easy to get confused. Perhaps you think that $10,000 should be removed from the $21,000 invoice total, thereby leaving a balance owing of $11,000. Or maybe you think the payment should receive the discount of 2%, which would be $200. Are you credited with $9,800 off of your balance? Or maybe $10,200 off of your balance? In all of these scenarios, you would be committing a serious mistake and miscalculating your balance owing. Let’s look at the correct way of handling this payment.

The Formula

Recall that invoice amounts are gross amounts (amounts before discounts), or G, and that payment amounts are net amounts (amounts after discounts), or N. Also recall that an algebraic equation requires all terms to be expressed in the same unit.

In the case of invoice payments, you can think of the balance owing as being in the unit of "pre-discount” and any payment being in the unit of "post-discount." Therefore, a payment cannot be directly deducted from the invoice balance since it is in the wrong unit. You must convert the payment from a “post-discount" amount into a "pre-discount" amount using a rearranged version of Formula 6.1. You can then deduct it from the invoice total to calculate any balance remaining.



N is Net Payment Amount: This is the actual payment amount made that must be converted into its value toward the invoice total. Since the payment represents an amount after the discount has been removed, you need to find out what it is worth before the cash discount was removed.

Formula 6.1 – Single Discount Rearranged: $\mathbf{L} = \dfrac{\mathbf{N}}{(\mathbf{1-d})}$

L is Invoice Amount: This is the amount deducted from the invoice balance as a result of the payment. It is what the payment is worth once the cash discount has been factored into the calculation.

d is Cash Discount Rate: Any payment received within any discount period, whether in full or partial, is eligible for the specified cash discount indicated in the terms of payment.

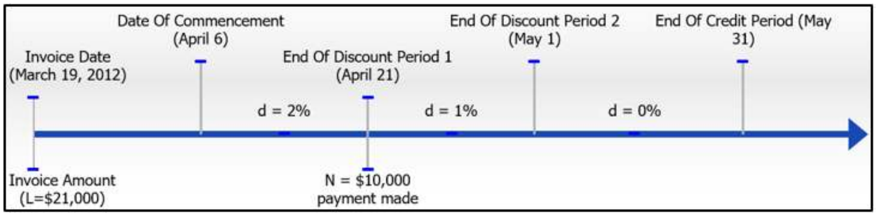
How It Works

Follow the same invoice payment steps even when working with partial payments. Commonly, once you calculate the gross amount of the payment in step 4 you will also have to calculate the new invoice balance by deducting the gross payment amount.

Let's continue working with the $21,000 invoice with terms of 2/15, 1/25, net 60 ROG. Assume the invoice is dated March 19 and the goods are received on April 6. If a $10,000 payment is made on April 21, what balance remains on the invoice?

Step 1: The invoice amount is $21,000, the invoice date is March 19, goods are received on April 6, and the terms of payment are 2/15, 1/25, net 60. A payment of $10,000 is made on April 21.

Step 2: The figure below shows the invoice timeline.


Date Of Commencement (April 6)
End Of Discount Period 2 (May 1)
End Of Credit Period (May 31)
Invoice Date (March 19, 2012)
End Of Discount Period 1 (April 21)
d = 2%
d = 1%
d = 0%
Invoice Amount (L=$21,000)
N = $10,000 payment made

Step 3: The payment of $10,000 on April 21 falls at the end of the first discount period and qualifies for a 2% discount.

Step 4: To credit the invoice, apply the rearranged Formula 6.1:

$$L = \frac{\$10,000}{(100\% - 2\%)} = \frac{\$10,000}{98\%} = \frac{\$10,000}{0.98} = \$10,204.08$$

This means that before any cash discounts, the payment is worth $10,204.08 toward the invoice total. The balance remaining is $\$21,000.00 - \$10,204.08 = \$10,795.92$.

Important Notes

In the case where a payment falls within a cash discount period, the amount credited toward an invoice total is always larger than the actual payment amount. If the payment does not fall within any discount period, then the amount credited toward an invoice total is equal to the actual payment amount (since there is no cash discount).

To help you understand why a partial payment works in this manner, assume an invoice is received in the amount of $103.09 and the customer pays the invoice in full during a 3% cash discount period. What amount is paid? The answer is $N = \$103.09(100\% - 3\%) = \$100$. Therefore, any payment of $100 made during a 3% cash discount period is always equivalent to a pre-discount invoice credit of $103.09. If the balance owing is more than $103.09, that does not change the fact that the $100 payment during the discount period is worth $103.09 toward the invoice balance.

? Exercise 7.4.3: Give It Some Thought

In each of the following situations, determine for the partial payment whether you would credit the invoice for an amount that is larger than, equal to, or less than the partial payment amount.

a. An invoice dated April 7 with terms 4/20, 2/20, n/60 EOM. The goods are received on April 9 and a partial payment is made on May 21.
b. An invoice dated July 26 with terms 3/30, n/45 ROG. The goods are received on August 2 and a partial payment is made on September 3.
c. An invoice dated January 3 with terms 2½/10, 1/20. The goods are received on January 10 and a partial payment is made on January 24.

Answer

a. Since the payment falls within a discount period, the credited amount is larger than the payment.
b. Since the payment does not fall within a discount period, the credited amount is equal to the payment.
c. Since the payment falls within a discount period, the credited amount is larger than the payment.

✓ Example 7.4.2: Applying Your Partial Payments toward Your Balance Owing

Heri just received an invoice from his supplier, R&B Foods. The invoice totalling $68,435.27 is dated June 5 with terms of 2½/10, 1/25, n/45. Heri sent two partial payments in the amounts of $20,000 and $30,000 that R&B Foods received on June 15 and June 29, respectively. He wants to clear his invoice by making a final payment to be received by R&B Foods on July 18. What is the amount of the final payment?

Solution

Determine the amount of the final payment (N) after the first two partial payments are credited toward the invoice total. To credit the two partial payments you must find the list amount before any cash discounts (L) such that the payment can be deducted from the invoice total.

What You Already Know

Step 1:

The invoice amount, terms of payment, and payment amounts are known:

L = $68,435.27

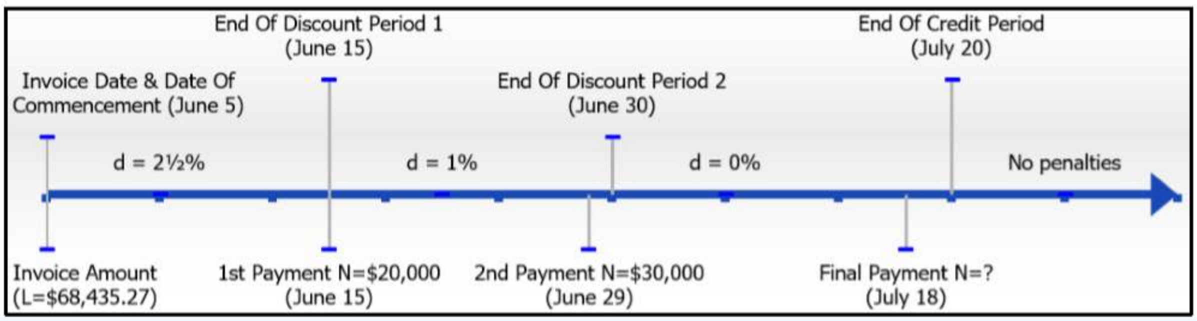
Invoice date = June 5

Terms of payment = 2½/10, 1/25, n/45

N_1 = $20,000 on June 15, N_2 = $30,000 on June 29, N_3 = ? on July 18

Steps 2 & 3:

The figure illustrates the timeline for the invoice and identifies the payments.


End Of Discount Period 1
(June 15)
End Of Credit Period
(July 20)
Invoice Date & Date Of
Commencement (June 5)
End Of Discount Period 2
(June 30)
d = 2½%
d = 1%
d = 0%
No penalties
Invoice Amount
(L=$68,435.27)
1st Payment N=$20,000
(June 15)
2nd Payment N=$30,000
(June 29)
Final Payment N=?
(July 18)

How You Will Get There

Step 4:

The first two payments are partial. Apply the rearranged Formula 6.1. As noted on the timeline:

a. The first partial payment qualifies for d = 2½%.
b. The second partial payment qualifies for d = 1%.
c. The last payment is a full payment, where d = 0% (no cash discount).

Perform

Step 4:

a. $L=\frac{\$20,000}{(1-0.025)}=\frac{\$20,000}{0.975}=\$20,512.82, \text{Balance Owing }=\$68,435.27-\$20,512.82=\$47,922.45$
b. $L=\frac{\$30,000}{(1-0.01)}=\frac{\$30,000}{0.99}=\$30,303.03, \text{Balance Owing }=\$47,922.45-\$30,303.03=\$17,619.42$
c. Since there is no cash discount, $N_3=L$. The balance owing is L = \$17,619.42. Therefore, the payment is the same number and N_3 = \$17,619.42.

The first two payments receive a credit of \$20,512.82 and \$30,303.03 toward the invoice total, respectively. This leaves a balance on the invoice of \$17,619.42. Since the final payment is made during the credit period, but not within any cash discount period, the final payment is the exact amount of the balance owing and equals \$17,619.42.

Late Payment Penalties

Among all of the invoices that you were paying in the section opener, you realize upon opening that third envelope that you somehow missed an invoice last month and failed to pay the \$4,000 owing, which is now overdue. In looking at the invoice, you notice that it states on the bottom that late payments are subject to a 3% per month penalty. You urgently want to pay this invoice to maintain good relations with your supplier but wonder in what amount you must issue the cheque?

Why Do Late Penalties Exist?

Suppliers are doing their business customers a favor when they use invoicing to seek payment. In essence, suppliers provide the products to the customer but are not paid for those products up front. This means customers are granted a period during which they "borrow" the products for free without any associated interest costs that are usually tied to borrowing. If the invoice is not paid in full by the time the credit period elapses, then the supplier starts treating any remaining balance like a loan and charges interest, which is called a late payment penalty.

The Formula

When a payment arrives after the credit period has expired, you need to adjust the balance of the invoice upward to account for the penalty. Continue to use Formula 6.1 on Single Discounts for this calculation. In this unique case, the discount rate represents a penalty. Instead of deducting money from the balance, you add a penalty to the balance. As a result, the late penalty is a negative discount. For example, if the penalty is 3% then d = −3%.

How It Works

Follow the same steps to solve the invoice payment once again when working with late payments. As an example, work with the \$4,000 overdue invoice that is subject to a 3% per month late penalty. If the invoice is paid within the first month of being overdue, what payment is made?

Step 1: The invoice balance is L = \$4,000, and the penalty percent is d = −0.03.

Steps 2 & 3: Since the focus is strictly on the late payment calculation, you do not need a timeline for the discount or credit periods. The late payment occurs within the first month past the expiry of the credit period.

Step 4: To determine the new invoice balance, apply Formula 6.1 to calculate $N=\$4,000\times(1-(-0.03))=\$4,000\times1.03=\$4,120$. Therefore, to clear the invoice you must issue a cheque in the amount of \$4,120. This covers the \$4,000 owing along with a late penalty of \$120.

Important Notes

Not all invoices have penalties, nor are they always enforced by the supplier. There are many reasons for this:

1. Most businesses pay their invoices in a timely manner, which minimizes the need to institute penalties.
2. Waiving a penalty maintains good customer relations and reinforces a positive, cooperative business partnership.

3. Strict application of late penalties results in difficult situations. For example, if a payment was dropped in the mail on August 13 from your best customer and it arrives one day after the credit period expires on August 15, do you penalize that customer? Is it worth the hassle or the risk of upsetting the customer?
4. The application of late penalties generally involves smaller sums of money that incur many administrative costs. The financial gains from applying penalties may be completely wiped out by the administrative expenses incurred.
5. Businesses have other means at their disposal for dealing with delinquent accounts. For example, if a customer regularly pays its invoices in an untimely manner, the supplier may just to decide to withdraw the privilege of invoicing and have the customer always pay up front instead.

In this textbook, all penalties are strictly and rigidly applied. If a payment is late, the invoice balance has the appropriate late penalty applied.

Things To Watch Out For

When a late penalty is involved, you must resist the temptation to adjust the payment by the penalty percentage. The discussion and calculations in this section focus on adjusting the outstanding invoice balance to figure out the payment required. For example, assume there is a $500 outstanding balance subject to a 2% penalty.

- This means that with the penalty, the invoice total is $510, which determines that a payment of $510 is required.
- If the customer makes a $200 partial payment on this late debt, you cannot apply the partial payment procedure, which would give a credit of $204.08 for the payment, resulting in a balance owing of $295.92. If you make this mistake, you in fact reward the customer for being tardy!
- Nor can you deduct the 2% penalty from the payment and give credit for $200(1 – 0.02) = $196 (resulting in a balance owing of $304) because the 2% penalty applies to the whole invoice amount and not just the partial payment.
- Instead, the $200 payment must be deducted directly from the penalty-adjusted balance of $510. In this case, the customer still owes $310 to clear the invoice.

Paths To Success

An alternative method for applying a late penalty is to treat the penalty like a positive percent change (see Section 3.1). In this case, the penalty is how much you want to increase the balance owing. Thus, if an invoice for $500 is subject to a 2% penalty, the Δ% = 2% and Old = $500. The goal is to look for the New balance in Formula 3.1:

$$\Delta\% = \frac{\text{New -Old}}{\text{old}} \times 100 \quad 2\% = \frac{\text{New } - \$500}{\$500} \times 100 \quad \text{New } \$ = \$10$$

✓ Example $7.4.3$: Whoops! I Missed the Due Date!

You received an invoice dated December 17 in the amount of $53,455.55 with terms of 4/15, 2/30, n/60 ROG, 2.75% penalty per month for late payments. The merchandise is received on January 24. You sent in a partial payment of $40,000 on January 31 with intention to pay the remaining balance before the credit period expired. However, you forgot about the invoice and realized your mistake on March 30, when you submit payment in full for the invoice. What amount is the final payment? (Note: Assume this is not a leap year).

Solution

The final payment must cover the invoice balance owing on any remaining balance after the partial payment is credited and the outstanding balance is penalized as per the late penalty.

What You Already Know

Step 1:

The invoice amount, terms of payment, and payment information are known:

L = $53,455.55, Invoice date = December 17

Goods received = January 24

Terms of payment = 4/15, 2/30, n/60 ROG

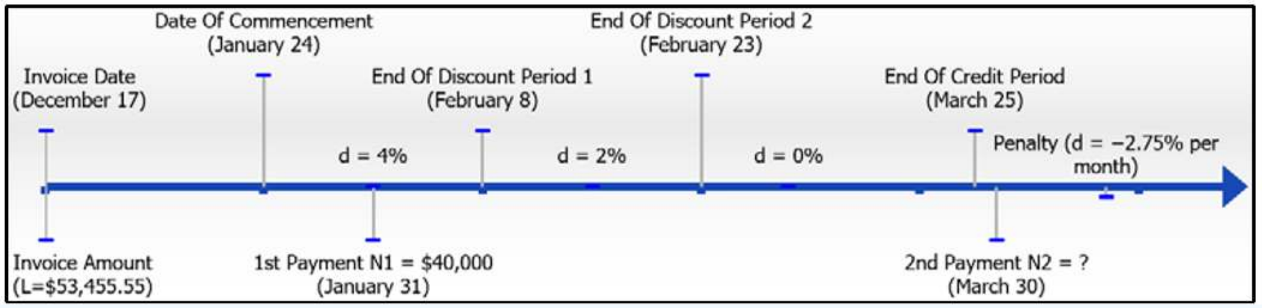
2.75% penalty per month for late payments

N_1 = \$40,000 on January 31

N_2 = ? on March 30

Steps 2 & 3:

The figure on the next page illustrates the timeline for the invoice and identifies the payments.


Date Of Commencement
(January 24)
End Of Discount Period 2
(February 23)
Invoice Date
(December 17)
End Of Discount Period 1
(February 8)
End Of Credit Period
(March 25)
d = 4%
d = 2%
d = 0%
Penalty (d = −2.75% per month)
Invoice Amount
(L=$53,455.55)
1st Payment N1 = $40,000
(January 31)
2nd Payment N2 = ?
(March 30)

How You Will Get There

Step 4 (Partial Payment):

Credit the partial payment during the discount period (d = 4%) by applying the rearranged Formula 6.1.

Step 4 (New Invoice Balance):

Deduct the partial payment from the invoice total.

Step 4 (Late Payment):

The second payment occurs after the credit period elapses. Therefore the outstanding invoice balance is penalized by d = −0.0275 as per the policy. Apply Formula 6.1. This calculates the outstanding balance including the penalty, which equals the final payment.

Perform

Step 4 (Partial Payment):

$$L=\frac{\$40,000}{(1-0.04)}=\frac{\$40,000}{0.96}=\$41,666.67$$

Step 4 (New Invoice Balance):

$$\text{Balance Owing }=\$53,455.55-\$41,666.67=\$11,788.88$$

Step 4 (Late Payment):

$$\begin{aligned}N&=\$11,788.88\times(1-(-0.0275))\\N&=\$11,788.88\times 1.0275\\&=\$12,113.07\end{aligned}$$

The first payment resulted in a \$41,666.67 deduction from the invoice. The overdue remaining balance of \$11,788.88 has a penalty of \$324.19 added to it, resulting in a final clearing payment of \$12,113.07.

References

1. Invoice provided by and courtesy of Kerry Mitchell Pharmacy Ltd. And Shoppers Drug Mart

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7.S: Summary

Key Concepts Summary

7.1: Sales Taxes (Everybody Wants a Piece of My Pie)

- The three sales taxes, current rates, and how to calculate prices including taxes
- How businesses complete GST/HST remittances

7.2: Property Taxes (I Owe, I Owe)

- How municipalities levy mill rates and tax rates against real estate owners

7.3: Exchange Rates and Currency Exchange (It Is a Global World)

- Converting currencies through exchange rates
- The rise and decline of exchange rates: currency appreciation and depreciation

7.4: Invoicing: Terms of Payment and Cash Discounts (Make Sure You Bill Them)

- How invoicing works: understanding invoice terms and invoice dating
- The calculations involved when a full payment amount is made
- The calculations involved when a partial payment amount is made
- The calculations involved when a late payment amount is made

The Language of Business Mathematics

assessed value

The portion of the market value of a property that is subject to municipal taxes.

buy rate

The rate at which a foreign currency is bought; it will always be lower than the mid-rate in terms of money per unit of foreign currency.

cash discount

A percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period.

credit period

The number of interest-free days from the date of commencement before full payment of the invoice is required.

currency appreciation

When one currency strengthens relative to another currency, resulting in an ability to purchase more of that other currency than it could previously.

currency depreciation

When one currency weakens relative to another currency, resulting in an ability to purchase less of that other currency than it could previously.

date of commencement

The first day from which the invoice terms extend forward in time and from which all due dates are established.

discount period

The number of days from the date of commencement for which a cash discount is offered.

end-of-month invoice dating

A term of payment where the date of commencement is the last day of the same month as indicated by the invoice date.

exchange rate

The number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa.

GST

The goods and services tax is a Canadian federal sales tax on most products purchased by businesses and consumers alike.

HST

The harmonized sales tax is a tax that combines the GST and PST into a single tax.

market value

A snapshot of the estimated selling price of a property at some specified point in time.

mid-rate

An exchange rate that does not involve or provide for any charges for currency conversion.

mill rate

A tax per $1,000 of assessed value to determine property taxes.

ordinary invoice dating

A term of payment where the date of commencement is the same date as the invoice date.

property taxes

Annual taxes paid by the owners of real estate to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services.

PST

The provincial sales tax is a consumer sales tax administered by Canadian provinces and territories.

receipt-of-goods invoice dating

A term of payment where the date of commencement is the day on which the customer physically receives the goods.

sales tax

A percent fee levied by a government on the supply of products.

sell rate

The rate at which a foreign currency is sold. It is always higher than the mid-rate in terms of money per unit of foreign currency.

tax policy

A municipality-based percentage of the market value of a property that is used to convert the market value into an assessed value.

tax rate

A tax per $100 of assessed value to determine property taxes.

tax remittance

The fulfillment of a tax obligation.

The Formulas You Need to Know

Symbols Used

AV = assessed value of a property

Current Currency = currency you are converting from

Desired Currency = currency you are converting to

Exchange Rate = per-unit conversion rate for current currency

Property Tax = property tax amount

PTR= property tax rate, usually set on a per \$100 (tax rate) or per \$1,000 (mill rate) basis

Rate = sales tax

Remit = tax remittance

S = selling price

S_{tax} = selling price including taxes

Tax Collected = total tax collected through sales

Tax Paid = total tax paid through purchases

Formulas Introduced

Formula 7.1 Selling Price Including Tax: $S_{tax} = S + (S \times \text{ Rate })$

Formula 7.2 GST/HST Remittance: $\text{Remit } = \text{ Tax Collected } - \text{ Tax Paid}$

Formula 7.3 Property Taxes: $\text{ Property Tax } = \Sigma(AV \times PTR)$

Formula 7.4 Currency Exchange: $\text{Desired Currency } = \text{ Exchange Rate } \times \text{ Current Currency}$

Technology

Calculator

The Percent Change function is used in this chapter. See the end of Chapter 3 for a full discussion on this calculator function.

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7.CS: Case Study - Completing Summary Financial Reports

The Situation

An accountant at Lightning Wholesale needs to summarize the GST remittances, costs of goods sold, and sales for 2013 by quarter. The table below presents the purchases (in US$) that it made from its American supplier, average monthly supplier cash discounts, total monthly sales (in C$), and the average monthly exchange rate appreciation or depreciation.

The Data

Quarter	Month	2013 Purchases in US$ Before Discounts	Average Cash Discount	2013 Sales in C$	C$ Appreciation (+) or Depreciation (−) Against US$
I	January	$2,021	2.00%	$1,798	Start 0.9423
	February	$2,200	3.00%	$2,407	1.00%
	March	$2,531	2.50%	$2,568	–0.50%
II	April	$2,623	1.50%	$2,985	0.35%
	May	$2,676	1.00%	$3,114	–1.50%
	June	$2,805	2.00%	$3,242	1.75%
III	July	$3,309	2.25%	$3,306	1.00%
	August	$4,146	2.00%	$3,852	0.50%
	September	$6,196	3.00%	$4,815	–1.40%
IV	October	$10,981	3.00%	$7,222	–0.30%
	November	$8,332	3.25%	$12,839	0.90%
	December	$1,520	1.75%	$9,630	–0.75%

(all figures are in thousands of dollars)

Important Information

- The imported products require GST to be paid on the equivalent Canadian value of the imported goods before any discounts.
- All sales are subject to GST.
- Lightning Wholesale’s suppliers offer it various cash discounts, which the accounting department always takes advantage of. These discounts have not been deducted from the purchase amounts listed in the table.

Your Tasks

1. Calculate the exchange rates for each month throughout the 2013 operating year. Round all exchange rates to four decimals before calculating the next exchange rate.
2. Compute quarterly GST remittances.
    a. Convert monthly purchases into Canadian funds and determine the monthly GST paid to the CBSA (Canada Border Services Agency).
    b. Assess the monthly GST collected on all sales.
    c. Complete a quarterly GST remittance table summarizing GST paid, GST collected, and the net amount refunded or remitted in 2013.
3. Create a quarterly cost of goods sold and revenue table.
    a. Apply the average monthly cash discount to the Canadian monthly purchases amounts. Round all answers to the nearest thousands of dollars.
    b. Create a quarterly table summarizing costs of goods sold and sales with yearly totals of each.

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7.E: Accounting Applications (Exercises)

7.1: Sales Taxes

Mechanics

You are purchasing a new BlackBerry at the MSRP of $649.99. Calculate the price including taxes in the following provinces or territories:

1. Northwest Territories
2. New Brunswick
3. Nova Scotia
4. British Columbia

The Brick is advertising a new Serta mattress nationally for a price of $899.99 including taxes. What is the price before taxes and the sales tax amounts in each of the following provinces?

5. Ontario
6. Saskatchewan
7. Audiophonic Electronics is calculating its HST remittance in Prince Edward Island. For each of the following months, calculate the HST remittance or refund on these HST-eligible amounts.

Month	Purchases	Sales
January	$48,693	$94,288
February	$71,997	$53,639

8. Airwaves Mobility is calculating its GST remittance in Alberta. For each of the following quarters, calculate the GST remittance or refund on these GST-eligible amounts.

Quarter	Purchases	Sales
Winter	$123,698	$267,122
Spring	$179,410	$158,905
Summer	$216,045	$412,111
Fall	$198,836	$175,003

Applications

9. Elena lives in Nova Scotia and has relatives in Alberta, Saskatchewan, and Quebec. She gets together with them often. She wants to purchase a new aerobic trainer and would like to pay the lowest price. If a family member buys the item, Elena can pick it up at one of their regular family gatherings. The price of the trainer for each province is listed below:

Province	Regular Selling Price before Taxes
Nova Scotia	$1,229.50
Alberta	$1,329.95
Saskatchewan	$1,274.25
Quebec	$1,219.75

   a. Where should Elena have the aerobic trainer purchased and how much would she pay?
   b. How much money would she save from her most expensive option?

10. Mary Lou just purchased a new digital camera in Nunavut for $556.49 including taxes. What was the price of the camera before taxes? What amount of sales tax is paid?

11. Marley is at Peoples Jewellers in New Brunswick wanting to purchase an engagement ring for his girlfriend. The price of the ring is $2,699.95. If the credit limit on his credit card is $3,000, will he be able to purchase the ring on his credit card? If not, what is the minimum amount of cash that he must put down to use his credit card?
12. In the IKEA store in Vancouver, British Columbia, you are considering the purchase of a set of kitchen cabinets priced at $3,997.59. Calculate the amount of GST and PST you must pay for the cabinets, along with the total price including taxes.
13. A company in Saskatchewan recorded the following GST-eligible purchases and sales throughout the year. Determine the GST remittance or refund per quarter.

Quarter	Purchases	Sales
1st	$2,164,700	$2,522,000
2nd	$1,571,300	$2,278,700
3rd	$1,816,100	$1,654,000
4th	$2,395,900	$1,911,700

14. A manufacturer in Nova Scotia recorded the following HST-eligible purchases and sales in its first three months of its fiscal year. Determine the HST remittance or refund per month.

Month	Purchases	Sales
March	$20,209	$26,550
April	$28,861	$20,480
May	$22,649	$42,340

Challenge, Critical Thinking, & Other Applications

15. If the selling price of an item is 6% higher in Yukon than in Ontario, will the price including taxes be higher in Yukon or Ontario? What percentage more?
16. Colin just travelled across the country on a road trip. He bought some skis in Alberta for $879.95 plus tax, a boombox in British Columbia for $145.58 including taxes, a Niagara Falls souvenir in Ontario for $99.97 plus tax, and some maple syrup in Quebec for $45.14 including tax. Overall, how much GST, PST, and HST did Colin pay on his trip?
17. Cisco Enterprises in Ontario purchased the following in a single month:
    - 16,000 units of network routers at $79.25 each, priced at $97.97 each
    - 12,000 units of wireless LAN adapters at $129.95 each, priced at $189.55 each
    - 13,500 units of computer boards at $229.15 each, priced at $369.50 each.

    Assuming that all units purchased are sold during the same month and that all purchases and sales are taxable, calculate the tax remittance or refund for the month.
18. In Quebec, the PST used to be calculated on the price including GST. When the PST was calculated in this manner, what PST rate did Quebec set to arrive at the same price including taxes?
19. For each of the following situations, compute the selling price of the product before taxes in the other province/territory that would result in the same selling price including taxes as the item listed.

Price before Tax	Sold In	Find Equivalent Price before Tax in This Province
a. $363.75	British Columbia	Prince Edward Island
b. $1,795.00	Alberta	Manitoba
c. $19,995.95	Saskatchewan	Ontario
d. $4,819.35	New Brunswick	Quebec

20. A company made the following taxable transactions in a single month. Compute the GST remittance on its operations assuming all sales and purchases are eligible for GST.

Transaction Type	Unit Price	Quantity Involved
Purchase	$168.70	5,430
Sale	$130.00	4,000
Sale	$148.39	3,600
Purchase	$93.47	2,950
Purchase	$24.23	3,325
Purchase	$121.20	2,770
Sale	$188.88	6,250

7.2: Property Taxes

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

Market Value	Tax Policy	Assessed Value	Rate	Type of Rate	Property Tax
1. $320,000	55%	?	26.8145	Mill	?
2. ?	85%	$136,000	1.984561	Tax	?
3. $500,000	?	?	9.1652	Mill	$3,666.08
4. ?	50%	$650,000	?	Tax	$4,392.91

Applications

5. A house with an assessed value of $375,000 is subject to a tax rate of 1.397645. What is the property tax?
6. If a commercial railway property has a property tax bill of $166,950 and the mill rate is 18.5500, what is the assessed value of the property?
7. A house in Calgary has a market value of $450,000. The tax policy is 100%. The property is subject to a 2.6402 mill rate from the City of Calgary and a 2.3599 mill rate from the province of Alberta. What are the total property taxes?
8. A residential property in Regina has a market value of $210,000. The Saskatchewan tax policy is 70%. The property is subject to three mill rates: 13.4420 in municipal taxes, 1.4967 in library taxes, and 10.0800 in school taxes. What amount of tax is collected for each, and what are the total property taxes?
9. A municipality needs to increase its operating budget. Currently, the assessed value of all properties in its municipality total $1.3555 billion and the tax rate is set at 0.976513. If the municipality needs an additional $1.8 million next year, what tax rate should it set assuming the assessed values remain constant?
10. A municipality set its new mill rate to 10.2967, which increased its total operating budget by $10 million on a constant assessed value of $7.67 billion. What was last year's mill rate?

Challenge, Critical Thinking, & Other Applications

11. A school board is determining the mill rate to set for next year. The assessed property values for next year total $5.782035 billion, representing an increase of 5% over the current year. If the school board needs an additional $5.4 million in funding next year, by what amount should it change its current year mill rate of 11.9985?
12. In the current year, the market value of properties totals $6.896 billion. The current tax policy is 85% and the current mill rate is 15.6712. If the municipality requires an additional $2 million in its operating budget next year, market values increase by 4%, and the tax policy changes to 90%, what mill rate should it set for next year?
13. A $600,000 market value property is assessed with a tax policy of 75% and subject to two mill rates. If the total property taxes are $6,766.67 and the second mill rate is half of the first tax rate, calculate each mill rate.

14. Two properties in different provinces pay the same property taxes of $2,840. One province uses a mill rate of 24.6119 with a 60% tax policy, while the other province uses a tax rate of 1.977442 with an 80% tax policy. Compute the market values for each of these properties.
15. A water utility funded through property taxes requires $900 million annually to operate. It has forecasted increases in its operating costs of 7% and 3.5% over the next two years. Currently, properties in its area have a market value of $234.85 billion, with projected annual increases of 3% and 5% over the next two years. The provincial government has tabled a bill that might change the tax policy from 70% to 75% effective next year, but it is unclear if the bill will pass at this point. For planning purposes, the utility wants to forecast its new mill rates for the next two years under either tax policy. Perform the necessary calculations for the utility.

7.3: Exchange Rates and Currency Exchange

For questions 1–6, use the mid-rates in the cross-rate table below to convert the current currency to the desired currency.

	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU$
Canadian Dollar (C$)	\	0.9787	1.4012	0.0122	0.0823	1.0360
US Dollar (US$)	1.0218	\	1.4317	0.0125	0.0841	1.0585
Euro (€)	0.7137	0.6985	\	0.0087	0.0588	0.7394
Japanese Yen (¥)	82.0233	80.2765	114.9287	\	6.7540	84.9747
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481	\	12.5814
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	\

Mechanics

Current Currency	Desired Currency
1. C$68,000	US$
2. ¥15,000,000	€
3. AU$3,000	MXN$
4. US$180,000	AU$
5. €230,500	C$
6. MXN$1,300,000	¥

Applications

7. If the exchange rate is ¥97.3422 per C$, what is the exchange rate for C$ per ¥?
8. Procter & Gamble just received payment for a large export of Tide in the amount of 275,000 Denmark kroner (DKK). If the exchange mid-rate is C$0.1992 per DKK and the bank charges 3% on its buy rates, how many Canadian dollars will Procter & Gamble receive?
9. The exchange rate per US$ is C$0.9863. If the Canadian dollar depreciates by $0.0421 per US$, how many more or less US$ is C$12,500 able to purchase?
10. Jack is heading home to visit his family in Great Britain and decided to stop at the airport kiosk to convert his money. He needs to convert C$5,000 to British pounds (£). The exchange mid-rate per C$ is £0.5841. The kiosk charges a commission of 4.5% on the conversion, plus a flat fee of £5.00.
    a. How many pounds will Jack have?
    b. What is the percentage cost of his transaction?

11. Yarianni is heading on a vacation. She converts her C$4,000 into Chinese yuan renminbi (CNY) at a sell rate of CNY6.3802 per C$. While in China, she spends CNY14,000 of her money. At the airport, she converts her remaining money into Indian rupees (INR) at a sell rate of INR6.7803 per CNY. In India, she spends INR50,000. When she returns home, she converts her INR back to C$ at a buy rate of C$0.0231 per INR. How many Canadian dollars did she return with? Note that all currencies involved have two decimals.
12. Elena is an international investor. Four years ago she purchased 2,700 shares of a US firm at a price of US$23.11 per share when the exchange rate was US$0.7536 per C$. Today, she sold those shares at a price of US$19.87 per share when the exchange rate was US$1.0063 per C$. In Canadian dollars, determine how much money Elena earned or lost on her investment.
13. International Traders regularly imports products from Hong Kong. If the exchange rate of C$ per Hong Kong dollar (HKD) is 0.1378 and the Canadian dollar appreciates by HKD0.0128, by what amount would the cost of a HKD1,000,000 purchase increase or decrease in Canadian dollars for International Traders?
14. Brian needs to purchase some Brazilian reals (BRL). He takes C$7,500 to the bank and leaves the bank with BRL12,753.20. If the exchange mid-rate per C$ is 1.7621, determine the sell rate commission percentage (rounded to two decimals) charged by the bank.

Challenge, Critical Thinking, & Other Applications

15. Fernando could purchase a 55" Samsung HDTV in Winnipeg, Manitoba, for $2,999.99 plus taxes. Alternatively, he could head across the border on Black Friday and shop in Grand Forks, North Dakota, where the same product is selling for US$2,499.99 (plus 5% state sales tax and 1.75% local sales tax) at Best Buy. He estimates he would incur $65 in gasoline and vehicle wear and tear, $130 in accommodations, and $25 in food (all money in US$). He would make all purchases on his credit card, which uses the mid-rate plus 2.5%. When returning across the border, he would have to pay in Canadian dollars the 5% GST on the Canadian value of the HDTV not including taxes. Once home, he can then have the North Dakota government refund all taxes paid on the HDTV through their Canadian sales tax rebate program. For all currency exchanges, assume a mid-rate of US$0.9222 per C$. Which alternative is Fernando's better choice and by how much?
16. The current mid-rate is C$1.5832 per €. Scotiabank has a sell rate of C$1.6196 per € while an airport kiosk has a sell rate of C$1.6544 per € plus a service charge of C$4.75. You need to purchase €800.
    a. Calculate the fee percentages charged by each financial institution. Round your answers to one decimal.
    b. Rounded to two decimals, what percentage more than Scotiabank is the airport kiosk charging on your purchase?
17. Henri and Fran have retired and are considering two options for a two-month vacation in Europe. Their local Lethbridge travel agent is offering them an all-inclusive package deal at C$7,975 per person. Alternatively, they can book their own flights for C$1,425 per person, stay in Britain at a small apartment averaging £65 per night for 30 days, and then in France for €70 average per night for 30 days. Estimated groceries cost a total of £250 in Britain and €400 in France. They will need to purchase a Eurail pass for €986 each while they are there. The exchange rates are €0.6808 per C$ and £0.5062 per C$. Which alternative is their cheapest option and by how much in Canadian dollars?

7.4: Invoicing - Terms of Payment and Cash Discounts

Round all money to two decimals and percentages to four decimals for each of the following exercises. In all questions, assume that the dates indicated are the dates upon which the payment is received.

Mechanics

For questions 1–3, calculate the full payment required on the payment date that reduces the balance on the invoice to zero. Assume this is not a leap year.

Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date	Date of Full Payment
1. $136,294.57	January 14	2/10, n/30	January 10	January 22
2. $98,482.75	September 28	3/10, 2/20, 1/30, n/50 EOM	October 3	October 19
3. $48,190.38	February 21	4/15, 3/40, n/60 ROG	February 27	April 3

For questions 4–6, calculate the remaining invoice balance after crediting the invoice for the partial payments indicated.

Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date	Partial Payments and Dates
4. $57,775.00	June 16	3/15, 1½/25	June 12	$20,000 on July 2; $15,000 on July 10
5. $1,200,310.75	August 25	2½/10, 1/20, n/30 ROG	September 2	$500,000 on September 4; $250,000 on September 14
6. $17,481.68	December 3	4/10, 1/30, n/60 EOM	December 20	$10,000 on January 2

For questions 7–8, calculate the final payment required on the payment date that reduces the balance of the invoice to zero.

Invoice Amount	Invoice Date	Invoice Terms	Late Payment Penalty	Receipt of Goods Date	Partial Payments and Dates	Final Payment Date
7. $23,694.50	April 13	3/15, 2/25, 1/40, n/60	1½% per month	April 18	$10,000 on April 30	June 15
8. $332,053.85	October 4	4/10 EOM	2% per month	October 30	$100,000 on November 5	December 15

Applications

9. Hudson's Bay received an invoice dated July 13 from Nygard International in the amount of $206,731.75. The terms are 2/10, 1/20, n/30. If Nygard receives full payment on August 1, what amount is paid?
10. Time Bomb Traders Inc. in Burnaby just received an invoice dated February 17 (of a leap year) from UrbanEars headphones in the amount of $36,448.50 with terms of 1½/15, ½/30, n/45 ROG. The items on the invoice are received on March 3. What amount is the full payment if UrbanEars receives it on March 18?
11. Family Foods received an invoice dated July 2 from Kraft Canada in the amount of $13,002.96 with terms of 2/15, 1/30, n/45 EOM and a late penalty of 2% per month. What amount is paid in full if Kraft Canada receives a cheque from Family Foods on August 17?
12. An invoice dated November 6 in the amount of $38,993.65 with terms of 2½/10, 1/15 ROG, 2% penalty per month is received by Cargill Limited from Agricore United. The wheat shipment is received on December 2. Cargill Limited made a partial payment of $15,000 on December 10. What amount should Cargill pay to clear its invoice on December 13?
13. Yamaha Music received two invoices from the same vendor. The first is for $1,260 dated March 9 with terms 3/10, net 30. The second is for $2,450 dated March 12 with terms 2/10, 1/20, net 30. If a payment of $1,000 is made on March 19, what payment amount on March 31 settles both invoices? (Note: Payments are applied to the earlier invoice first.)
14. An invoice is dated July 26 for $5,345.50 with terms of 3¼/10, net 45, 2½% per month penalty. If the invoice is paid on September 10, what payment amount is required to pay the entire balance owing?
15. Mohawk College received an invoice dated August 20 from Office Depot for office supplies totaling $10,235.97 with terms of 3¾/15, 1½/30, *n*/45, EOM, 2¾% per month penalty. The college made three payments of $2,000 dated September 4, September 29, and October 10. What payment on October 25 settles the invoice?

Challenge, Critical Thinking, & Other Applications

16. An invoice for $100,000 dated February 2 (of a non–leap year) with terms of 4/10, 3/20, 2/30, 1/40, net 60, ROG, 1¾% per month penalty. The merchandise is received on February 16. If four equal payments are made on February 20, March 17, April 1, and April 20 resulting in full payment of the invoice, calculate the amount of each payment.
17. An accounting department receives the following invoices from the same vendor and makes the indicated payments. The vendor always applies payments to the earliest invoices first and its late payment policy stands at 1½% per month for any outstanding balance.

Invoice #	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date
3866	$47,690.11	June 18	2/10, EOM	July 3

Invoice #	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date
2928	$123,691.82	June 26	2½/15, 1/25	July 6
4133	$96,004.21	June 30	2/20, ROG	July 7
6767	$16,927.50	July 10	2/10, 1/20	July 11

Payment Received by Vendor on	Payment Amount
July 2	$40,000
July 11	$75,000
July 21	$100,000
July 30	$25,000

Calculate the amount of the final payment on August 17 that reduces the total balance owing to zero.

18. John's Home Hardware was invoiced on May 29 for the following non-taxable items with terms of 2/20, 1/40, 2% per month penalty.

Item	Quantity	Unit Price
2"x 4"x 8' framing studs	13,000	$0.96
4"x 4" x 8' fence posts	2,400	$2.87
2"x 8"x 16' wood planks	480	$8.47
1"x 6"x 5' fence boards	8,000	$1.22

If a partial payment of $20,000 is made on July 7, what payment amount on August 30 reduces the balance owing to zero? What is the total dollar amount of the late penalty?

19. Your company receives a $138,175.00 invoice dated April 12 with terms of 4/10, 3/20, 2/30, 1/40, n/60. Due to the large amount, the accounting department proposes two different ways to pay this invoice, depending on company income and financing alternatives. These plans are listed in the table below.

Payment Date	Plan #1	Plan #2
April 21	$35,000	$50,000
May 2	$20,000	$10,000
May 10	$25,000	$50,000
May 13	$30,000	$20,000
June 11	Balance owing	Balance owing

a. Which alternative do you recommend?
b. If your recommendation is followed, how much money is saved over the other option?

20. You receive an invoice dated July 29 with terms of 3/15, 1¾/30, n/50 EOM, 2% penalty per month. If the payments in the following table are made and you pay the invoice in full, what is the total invoice amount?

Payment Date	Payment Amount
August 14	$67,000.00
August 28	$83,000.00

Payment Date	Payment Amount
September 18	$15,000.00
September 30	$83,297.16

Review Exercises

Mechanics

1. Suppose you own property with a market value of $412,000 in a municipality that has a 60% tax policy. What are your total property taxes if the mill rate is set at 15.4367?
2. Convert 140,000 Indian rupees (INR) to Belgian francs (BEF) if the exchange rate is BEF0.6590 per INR.
3. Calculate the price including taxes for a product that has a regular selling price of $995.95 in:
   a. Ontario
   b. Alberta
   c. Saskatchewan
   d. Prince Edward Island
4. An invoice for $15,000.00 dated February 16 of a leap year has terms of 3/15, 2/30. What payment in full is required if the payment is received on March 3?
5. The exchange rate is US$0.9063 per Canadian dollar. If over the course of a year the Canadian dollar appreciates by $0.0644 per C$, how many more or less US dollars are purchased with C$1,750?
6. A company recorded the following purchases and sales throughout the year. Assume all amounts are taxable.

Quarter	Purchases	Sales
1st	$2,138,450.25	$1,444,297.93
2nd	$3,496,821.15	$4,989,201.01
3rd	$1,265,733.49	$6,422,604.19
4th	$2,725,706.28	$2,003,125.50

a. Calculate the GST remittance or refund per quarter.
b. Calculate the HST remittance or refund per quarter in Newfoundland and Labrador.

7. The assessed values in a municipality total $4.986 billion. If the city council needs an operating budget of $100 million, what mill rate should it set for its constituent property owners?

Applications

8. An invoice for $37,650 is dated June 24 with terms of 2/10, 1/30, ROG. Merchandise is received on June 29. If a payment of $20,000 is made on July 5 and another payment of $10,000 is made on July 26, what is the balance remaining on the invoice? Assume all payments are received on the dates indicated.
9. If the price including taxes for an item is $1,968.35 in Quebec, what is the dollar amount of each of the PST and GST on the item?
10. Obiwan is going on vacation to Slovakia and stops at a currency shop in his local airport. He needs to convert C$7,500 to the Slovakian koruna (SKK). The exchange mid-rate per C$ is 21.4624. The shop charges a commission of 6.5% on the conversion, plus a flat fee of C$15.75. How many koruna will Obiwan have?
11. A town in Saskatchewan has residential property values totalling $968 million in market value. The councillors have set the tax policy at 75% of market value. The following is a list of projected municipal expenditures in the coming year.

Sewers & water	$5,250,000
Public education	$13,500,000
City services (fire, police, ambulance)	$1,115,000
Roads & maintenance	$3,300,000

Other	$2,778,000

If the council expects to raise 85% of these expenditures from residential property taxes, what mill rate must it set?

12. What amount must be remitted if invoices dated January 25 for $1,700, February 1 for $1,800, and February 15 for $900, all with terms of 3/25, n/50, EOM are paid with the same cheque on March 12? Assume it is a non–leap year.
13. Many Canadians complain about the high cost of automobile gasoline. In the United States, regular grade gasoline costs approximately $3.652 per gallon (as of June 2011).
    a. What is the equivalent Canadian cost per litre (rounded to three decimals) if the exchange rate is US$1.0152 per C$? A US gallon is equivalent to 3.7854 litres.
    b. At the same time, Canadians were paying $1.239 per litre. What percentage more (rounded to two decimals in percent format) than Americans do Canadians pay?
14. An invoice for $13,398.25 dated August 27 with terms of 5/10, 2/25, n/45 was partially paid on September 7. If the invoice balance was reduced to $2,598.25, what was the amount of the payment? Assume the payment was received on the date indicated.

Challenge, Critical Thinking, & Other Applications

15. In Saint-Lazare, Quebec, the average market value of a residential home in 2010 rose to $308,000 from its previous 2009 value of $262,000. The residential property tax rate was lowered from 2009 to 2010 by 9.8918% to 0.583. In Saint-Lazare, the tax policy is to tax 100% of the market value. By what percentage did the residential property tax bill go up or down in 2010 for a Saint-Lazare residential homeowner?
16. What percentage of a tax-inclusive price is each of the GST and PST in the following provinces?
    a. Quebec
    b. Manitoba
17. It is well known that airport kiosks are much more expensive for currency exchange than visiting a bank. The current mid-rate is $2.4226 Brazilian reals (BRL) per euro. The Banco do Brasil has a sell rate of BRL$2.4710 per euro, while a kiosk at Salvador airport has a sell rate of BRL$2.5473 per euro plus a flat service charge of BRL$3.00. A traveler through the airport wants to convert BRL$2,000 to euros. Assuming the traveler has the time, would it be worth the BRL$15.00 return taxi ride to the nearest bank to convert the currency? How many euros more would the best option present?
18. An invoice for $147,477.77 dated July 26 with terms of 5/15, 3½/25, 2/40, EOM was partially paid with two equal payments on August 10 and September 4, reducing the balance owing to $76,579.81. What amount was paid on each of the two dates? Assume the payment was received on the date indicated.
19. An American tourist is traveling across Canada and makes the following purchases on her credit card (all purchases are fully taxable):

Total Purchase Amount before Taxes	Province/Territory Purchased
$319.95	British Columbia
$114.75	Alberta
$82.45	Northwest Territories
$183.85	Manitoba
$59.49	Quebec
$123.99	Prince Edward Island

On the first purchase, the credit card company used an exchange rate of C$1.0412 per US$. Each subsequent purchase used an exchange rate that appreciated (for the US dollar) by 0.2% each time. What total amount in US dollars was charged to the traveller's credit card?

20. A Canadian manufacturer imports 2,000 units of product worth US$262.25 each from an American supplier. These imports are subject to GST on the equivalent Canadian value of the product. All of the product is then sold for C$419.50 each to another

Canadian distributor. The sale is subject to GST. Assume an exchange rate of US$0.9345 per C$. If the company needs to submit a GST remittance on this transaction, what amount is remitted or refunded?

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CHAPTER OVERVIEW

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8.0: Introduction

If you have a lot of surplus cash, it would hardly make sense to hide it under your mattress or bury it in the backyard of your townhouse. As any investment specialist will tell you, you want your money to work for you instead of just sitting idle. So you can invest and be rewarded by earning interest.

Do you know if your account at the Royal Bank of Canada (RBC) has paid you the right amount of interest? You might assume that because everything at a bank is fully automated it is guaranteed to be correct. However, ultimately all automation at the Royal Bank relies on input by an employee. What if your interest rate is keyed in wrong? Maybe the computer programmer erred in a line of computer code and the amount of interest is miscalculated. What if RBC had a computer glitch? These pitfalls highlight the importance of always checking your account statements.

In business, it would be rare for an organization to borrow or lend money for free. That is not to say that businesses are ruthless with each other; in the previous chapter, you saw that most business transactions are completed through interest free credit transactions involving invoicing. However, this generosity does not extend indefinitely. After the credit period elapses, the business essentially treats an unpaid invoice as a loan and starts charging an interest penalty.

When businesses need to borrow money, promissory notes, demand loans, and commercial papers are just three of the possibilities discussed later in this chapter.

Even governments need to borrow money. Have you ever thought about how the government of Canada or your provincial government goes about doing that? While a government has many methods at its disposal, a common method is to use treasury bills—in essence, borrowing the money from investors.

On a personal level, almost all people invest and borrow money at some time. This chapter examines consumer lines of credit and student loans, both of which carry simple interest rates. On the earnings side, it is common for people to have savings accounts that earn low rates of interest. Other short-term investments include treasury bills, commercial papers, and short-term guaranteed investment certificates (GICs).

The bottom line is that money does not come free. In this chapter, you will explore the concept of simple interest, learn how to calculate it, and then apply simple interest to various financial tools.

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8.1: Principal, Rate, Time

Your investments may be at risk if stock and bond markets slump, as a story in the Globe and Mail predicts. You wonder if you should shift your money into relatively secure short-term investments until the market booms again. You consider your high-interest savings account, but realize that only the first $60,000 of your savings account is insured. Perhaps you should put some of that money into treasury bills instead.

Looking ahead, what income will you live on once you are no longer working? As your career develops, you need to save money to fund your lifestyle in retirement. Some day you will have $100,000 or more that you must invest and reinvest to reach your financial retirement goals.

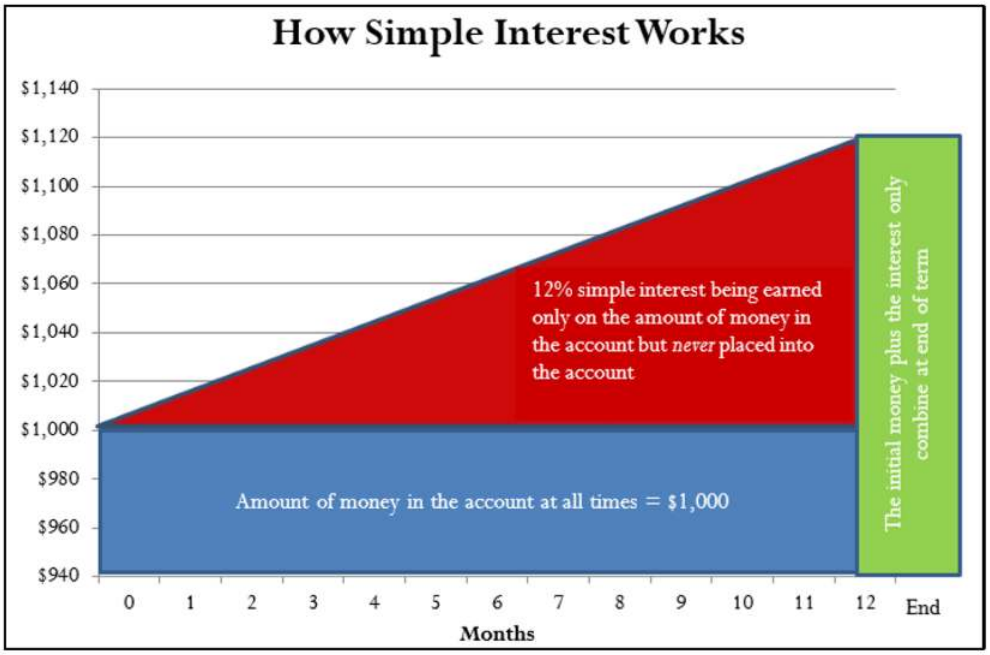
To make such decisions, you must first understand how to calculate simple interest. Second, you need to understand the characteristics of the various financial options that use simple interest. Armed with this knowledge, you can make smart financial decisions!

The world of finance calculates interest in two different ways:

1. Simple Interest. A simple interest system primarily applies to short-term financial transactions, with a time frame of less than one year. In this system, which is explored in this chapter, interest accrues but does not compound.
2. Compound Interest. A compound interest system primarily applies to long-term financial transactions, with a time frame of one year or more. In this system, which the next chapter explores, interest accrues and compounds upon previously earned interest.

Simple Interest

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction. When the term of the transaction ends, you add the amount of the simple interest to the initial amount. Therefore, throughout the entire transaction the amount of money placed into the account remains unchanged until the term expires. It is only on this date that the amount of money increases. Thus, an investor has more money or a borrower owes more money at the end.



The figure illustrates the concept of simple interest. In this example, assume $1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals $1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the $120 of interest and the initial $1,000 are then combined to total $1,120.

A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

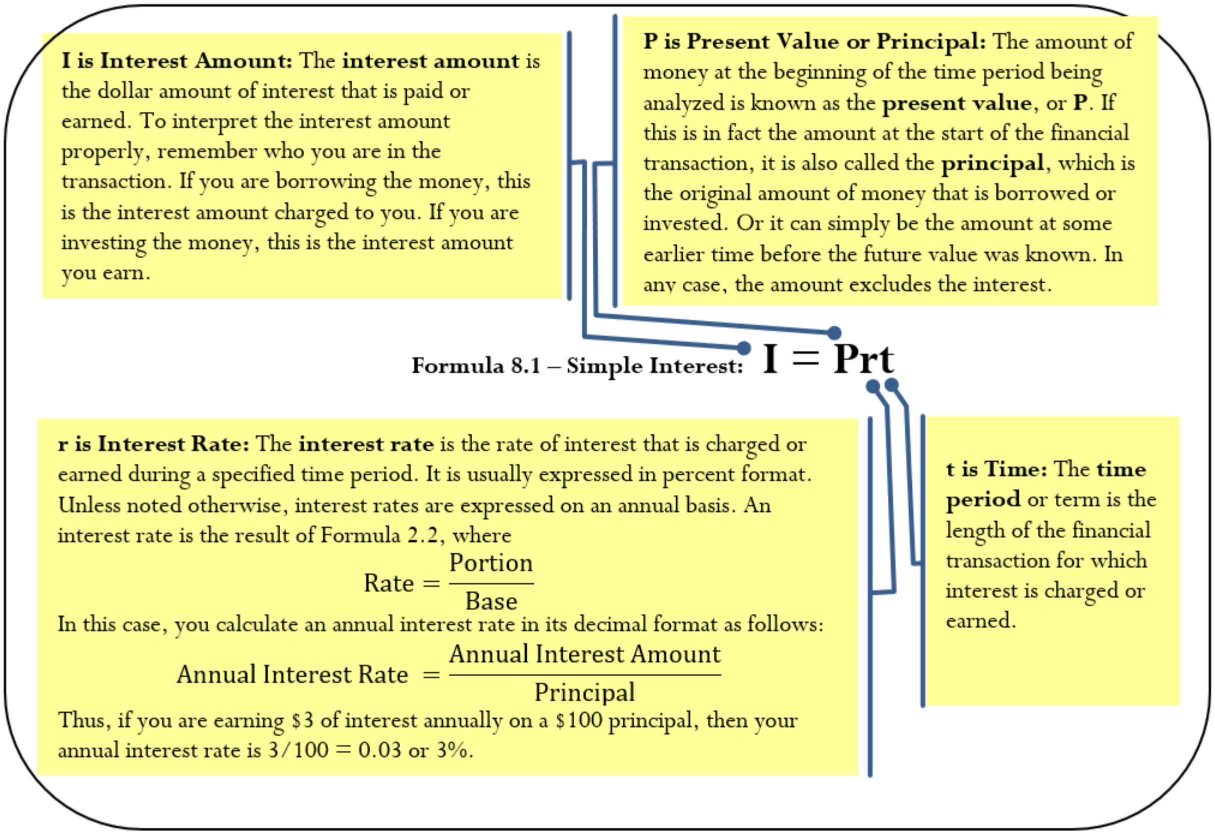
- If you take out a personal loan from a bank, the bank gives you the money and you receive the money. In this situation, the bank earns the simple interest and you are being charged simple interest on your loan. In the figure, this means you must pay back not only the $1,000 you borrowed initially but an additional $120 in interest.
- If you place your money into an investment account at the bank, you have given the money and the bank has received the money. In this situation, you earn the simple interest on your money and the bank pays simple interest to your investment account. In the figure, this means the bank must give you back your initial $1,000 at the end plus an additional $120 of interest earned.

The Formula

The best way to understand how simple interest is calculated is to think of the following relationship:

Amount of simple interest = *How much at What simple interest rate* for *How long*

Notice that the key variables are the amount, the simple interest rate, and time. Formula 8.1 combines these elements into a formula for simple interest.



I is Interest Amount: The **interest amount** is the dollar amount of interest that is paid or earned. To interpret the interest amount properly, remember who you are in the transaction. If you are borrowing the money, this is the interest amount charged to you. If you are investing the money, this is the interest amount you earn.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed is known as the **present value**, or **P**. If this is in fact the amount at the start of the financial transaction, it is also called the **principal**, which is the original amount of money that is borrowed or invested. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.1 – Simple Interest: $\mathbf{I = Prt}$

r is Interest Rate: The **interest rate** is the rate of interest that is charged or earned during a specified time period. It is usually expressed in percent format. Unless noted otherwise, interest rates are expressed on an annual basis. An interest rate is the result of Formula 2.2, where

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

In this case, you calculate an annual interest rate in its decimal format as follows:

$$\text{Annual Interest Rate} = \frac{\text{Annual Interest Amount}}{\text{Principal}}$$

Thus, if you are earning $3 of interest annually on a $100 principal, then your annual interest rate is $3/100 = 0.03$ or 3%.

t is Time: The **time period** or term is the length of the financial transaction for which interest is charged or earned.

Formula 8.1

How It Works

Follow these steps when you calculate the amount of simple interest:

Step 1: Formula 8.1 has four variables, and you need to identify three for any calculation involving simple interest. If necessary, draw a timeline to illustrate how the money is being moved over time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If they are not already, you need to convert one of the two variables to the same units as the other.

Step 3: Apply Formula 8.1 and solve for the unknown variable. Use algebra to manipulate the formula if necessary.

Assume you have $500 earning 3% simple interest for a period of nine months. How much interest do you earn?

Step 1: Note that your principal is \$500, or $P = \$500$. The interest rate is assumed to be annual, so $r = 3\%$ per year. The time period is nine months.

Step 2: Convert the time period from months to years: $t = \frac{9}{12}$.

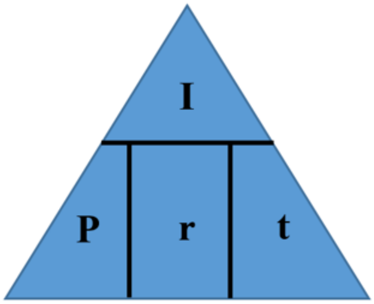
Step 3: According to Formula 8.1, $I = \$500 \times 3\% \times = \$11.25$. Therefore, the amount of interest you earn on the \$500 investment over the course of nine months is \$11.25.

Important Notes

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for Formula 8.1, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

- The time needs to be expressed annually as $\frac{9}{12}$ of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as $\frac{3\%}{12} = 0.25\%$ per month to match the number of months.

It does not matter which you do so long as you express both interest rate and time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving Formula 8.1 for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.



Paths To Success

Four variables are involved in the simple interest formula, which means that any three can be known, requiring you to solve for the fourth missing variable. To reduce formula clutter, the triangle technique illustrated here helps you remember how to rearrange the formula as needed.

Exercise $8.1.1$: Give It Some Thought

a. If you have a debt for which you haven't made any payments, yet it is being charged simple interest on the principal, would you owe more or less money in the future?

Answer

a. More, because the interest is earned and therefore is added to your savings account.
b. More, because you owe the principal and you owe the interest, which increases your total amount owing

Example $8.1.1$: How Much Interest Is Owed?

Julio borrowed \$1,100 from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

Solution

Calculate the amount of interest that Julio owes Maria (I).

What You Already Know

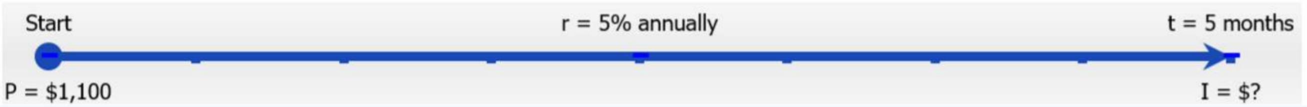
Step 1:

The terms of their agreement are as follows: $P = \$1,100, r = 5\%, t = 5$ months

How You Will Get There

Step 2:

The rate is annual, and the time is in months. Convert the time to an annual number.



Step 3:

Apply Formula 8.1.

Perform

Step 2:

Five months out of 12 months in a year is $\frac{5}{12}$ of a year, or $t = \frac{5}{12}$.

Step 3:

$$I = \$1,100 \times 5\% \times \frac{5}{12} = \$1,100 \times 0.05 \times 0.41\overline{6} = \$22.92$$

For Julio to pay back Maria, he must reimburse her for the $1,100 principal borrowed plus an additional $22.92 of simple interest as per their agreement.

Example 8.1.2: Do You Know Your Interest Rate?

A $3,500 investment earned $70 of interest over the course of six months. What annual rate of simple interest did the investment earn?

Solution

Calculate the annual interest rate (r).

What You Already Know

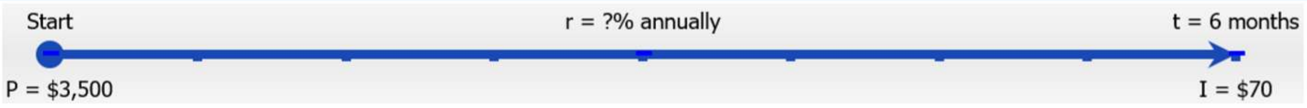
Step 1:

The principal, interest amount, and time are known: $P = \$3,500, I = \$70, t = 6$ months

How You Will Get There

Step 2:

The computed interest rate needs to be annual, so you must express the time period annually as well.



Step 3:

Apply Formula 8.1, rearranging for r.

Perform

Step 2:

Six months out of 12 months in a year is $\frac{6}{12}$ of a year, or $t = \frac{6}{12}$.

Step 3:

6

$$\$70 = \$3,500 \times r \times \frac{}{12}$$

$$r = \frac{\$70}{\$3,500 \times \frac{6}{12}} = \frac{\$70}{\$1,750} = 0.04 \text{ or } 4\%$$

For $3,500 to earn $70 simple interest over the course of six months, the annual simple interest rate must be 4%.

✓ Example 8.1.3: What Did You Start with?

What amount of money invested at 6% annual simple interest for 11 months earns $2,035 of interest?

Solution

Calculate the amount of money originally invested, which is known as the present value or principal, symbolized by P.

What You Already Know

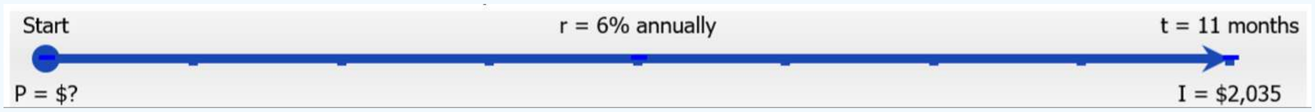
Step 1:

The interest rate, time, and interest earned are known: $r = 6\%, t = 11$ months, $I = \$2,035$

How You Will Get There

Step 2:

Convert the time from months to an annual basis to match the interest rate.



Step 3:

Apply Formula 8.1, rearranging for P.

Perform

Step 2:

Eleven months out of 12 months in a year is $\frac{11}{12}$ of a year, or $t = \frac{11}{12}$.

Step 3:

$$\$2,035 = P \times 6\% \times \frac{11}{12}$$

$$P = \frac{\$2,035}{6\% \times \frac{11}{12}} = \frac{\$2,035}{0.06 \times 0.91\overline{6}} = \$37,000$$

To generate $2,035 of simple interest at 6% over a time frame of 11 months, $37,000 must be invested.

✓ Example 8.1.4: how long?

For how many months must $95,000 be invested to earn $1,187.50 of simple interest at an interest rate of 5%?

Solution

Calculate the length of time in months (t) that it takes the money to acquire the interest.

What You Already Know

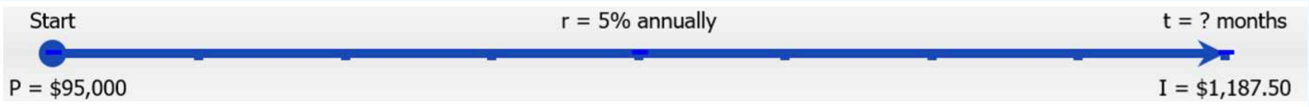
Step 1:

The amount of money invested, interest earned, and interest rate are known: $P = \$95,000.00, I = \$1,187.50, r = 5\%$

How You Will Get There

Step 2:

Express the time in months. Convert the interest rate to a "per month" format.



Step 3: Apply Formula 8.1, rearranging for t.

Perform

Step 2:

5% per year converted into a monthly rate is $r = \dfrac{0.05}{12}$

Step 3:

$$\$1,187.50 = \$95,000 \times \frac{0.05}{12} \times t$$

$$t = \frac{\$1,187.50}{\$95,000 \times \dfrac{0.05}{12}} = \frac{\$1,187.50}{\$95,000 \times 0.0041\overline{6}} = 3 \text{ months}$$

For $95,000 to earn $1,187.50 at 5% simple interest, it must be invested for a three-month period.

Time and Dates

In the examples of simple interest so far, the time period equaled an exact number of months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where $t = \dfrac{2}{12}$ or $t = 0.1$ of a year. However, if you use the exact number of days, the 31 days in July and 31 days in August total 62 days. In a 365-day year that is $t = \frac{62}{365}$ or $t = 0.169863$ of a year. Notice a difference of 0.003196 occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the exact number of days involved in the transaction.

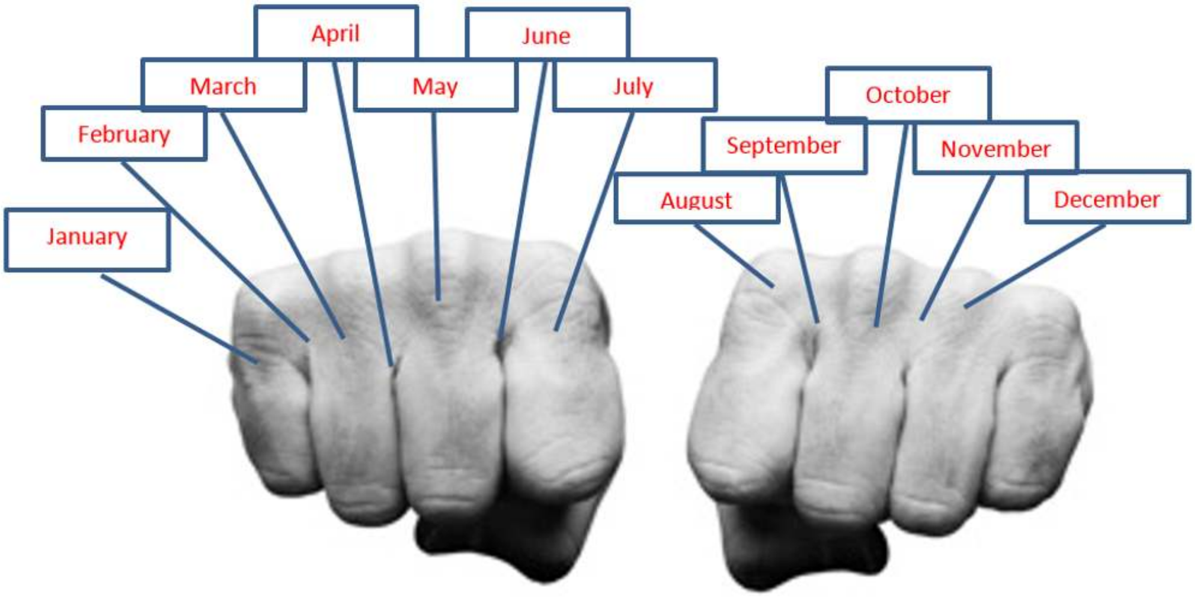
Figuring Out the Days

In practice, when counting the number of days in a transaction you include the first day but not the last day. This is because interest is calculated on the daily closing balance, not on interim balances throughout a day. For example, if you borrowed $500 on June 3 and paid it back with interest on June 5, the closing balance on June 3 and June 4 is $500. However, on June 5 a zero balance is restored. Therefore, you count June 3 and 4 as days of interest but not June 5, so you will owe two days of interest, not three.

There are three common ways to calculate the number of days involved in a transaction.

Method 1: Use Your Knuckles

You can use your knuckles as an aid to remembering which months of the year have more or fewer days.



Put your knuckles side by side and start on your outermost knuckle. Every knuckle represents a month with 31 days. Every valley between the knuckles represents a month that does not have 31 days (30 in all except February, which has 28 or 29). Now start counting months. Your first knuckle (pinkie finger) is January, which has 31 days. Next is your first valley, which is February and it does not have 31 days. Your next knuckle (ring finger) is March, which has 31 days … and so on. When you get to your last knuckle (your index finger), move to the other hand's first knuckle (an index finger again). Notice that July and August are the two months back to back with 31 days.

If you are trying to calculate the number of days between March 20 and May 4, use your knuckles to recall that March has 31 days and April has 30 days. The calculations are illustrated in the table here.

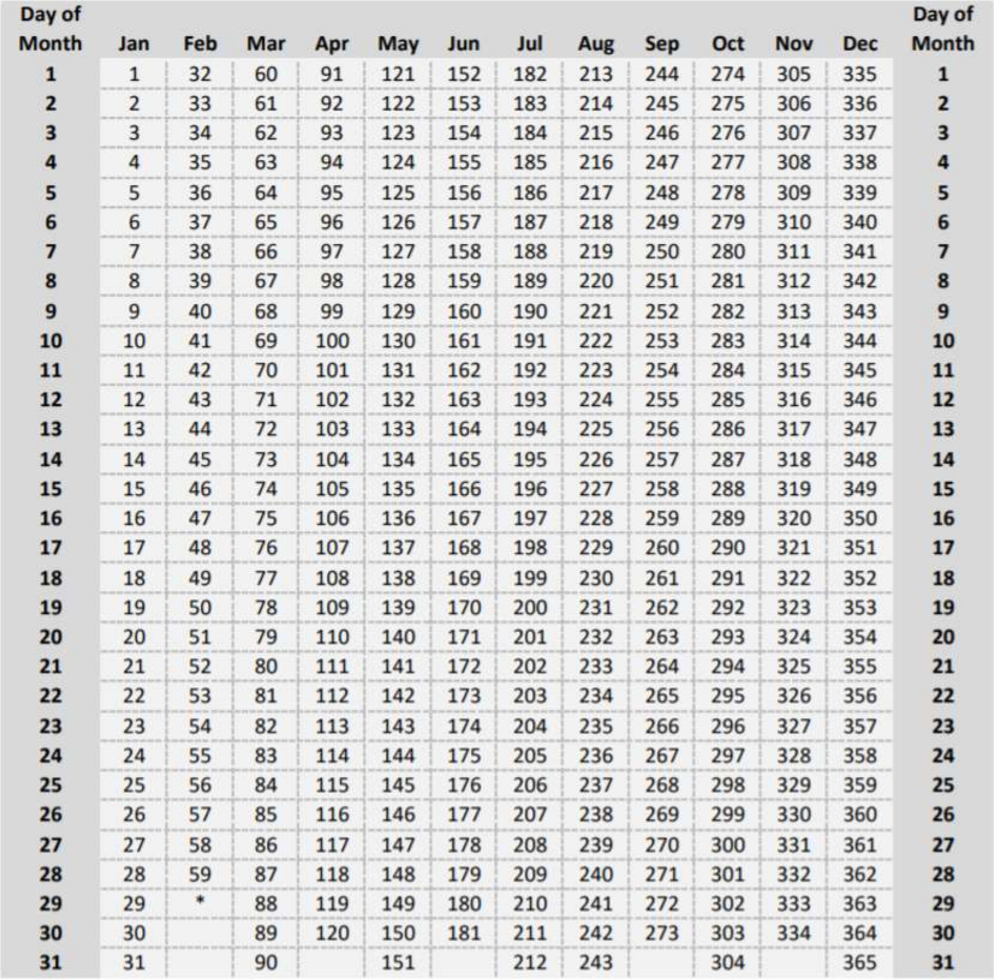
Start Date	End Date	Days between Dates
March 20	March 31	31 − 20 = 11 days
March 31 (which precedes April 1, so it is day "0" for April)	April 30	30 − 0 = 30 days
April 30 (which precedes May 1, so it is day "0" for May)	May 4	4 − 0 = 4 days
TOTAL		11 + 30 + 4 = 45 days

Why does the end date of one line become the start date of the next line? Recall that you count the first day, but not the last day. Therefore, on the first line March 31 has not been counted yet and must be counted on the second line. Ultimately a transaction extending from March 20 to May 4 involves 45 days as the time period.

Method 2: Use A Table of Serial Numbers

In the table on the next page, the days of the year are assigned a serial number. The number of days between any two dates in the same calendar year is simply the difference between the serial numbers for the dates.

Using the example in Method 1, when the dates are within the same calendar year, find the serial number for the later date, May 4 (Day 124) and the serial number for the earlier date, March 20 (Day 79). Then, find the difference between the two serial numbers ($124-79=45$days).

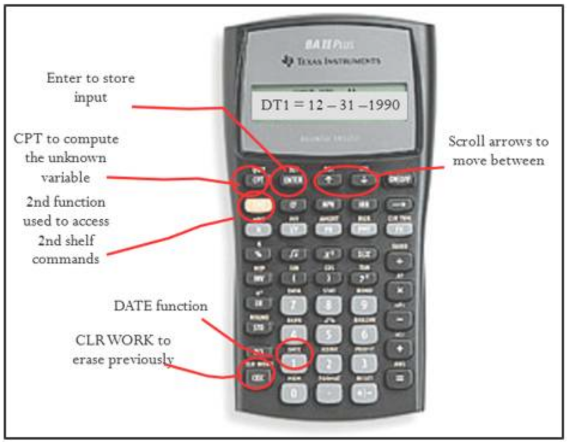


Day of Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29	*	88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

*For Leap years, February 29 becomes Day 60 and the serial number for each subsequent day is increased by 1.

Method 3: Use The BAII Plus Date Function

Your BAII Plus calculator is programmed with the ability to calculate the number of days involved in a transaction according to the principle of counting the first day but not the last day. The function "Date" is located on the second shelf above the number one. To use this function, open the window by pressing 2nd 1.


Enter to store input
CPT to compute the unknown variable
2nd function used to access 2nd shelf commands
DATE function
CLR WORK to erase previously
DT1 = 12 – 31 –1990
Scroll arrows to move between

- Scroll amongst the four display lines by using your ↑ and ↓ arrows. There are four variables:
    1. DT1 is the starting date of the transaction.
    2. DT2 is the ending date of the transaction.

   3. DBD is the days between dates, which is your t in Formula 8.1.
   4. ACT or 360 is the method of calculating the days between dates. In ACT (for “actual”), the calculator calculates the DBD based on the correct number of days in the month. In 360, the DBD is based on treating every month as having 30 days ($30 \times 12 = 360$ days). For all calculations, set the calculator to the ACT mode. Toggle this setting by pressing 2nd SET.
- To enter a date into the calculator, the calculator uses the format of MM.DDYY where:
  - MM is the numerical value of the month; you do not need to enter the leading zero on months with only one digit.
  - DD is the day of the month; you must always enter this value with two digits using a leading zero for days 1 through 9 in a month.
  - YY is the last two digits of the year; you must always enter this with two digits. If your question does not involve a year, select an appropriate year of your choice.
- As long as you know any two of the DT1, DT2, or DBD variables, you can calculate the third. Enter any two of the three variables, pressing ENTER after each and ensuring the window shows the output variable you are seeking, and then press CPT.

Using the example in Method 1, to calculate the time from March 20 to May 4 you first need to choose a year arbitrarily, perhaps 2011. Therefore, perform the following sequence:

$$\text{2nd DATE}$$

$$\text{2nd CLR Work (to erase previous calculations)}$$

$$\text{3.2011ENTER } \downarrow \text{(this is March 20, 2011)}$$

$$\text{5.0411ENTER } \downarrow \text{(this is May 4, 2011)}$$

$$\text{CPT}$$

$$\text{Answer: 45}$$

Note that the answer is the same as the one you arrived at manually using your knuckles.

Important Notes

When solving for t, decimals may appear in your solution. For example, if calculating t in days, the answer may show up as 45.9978 or 46.0023 days; however, interest is calculated only on complete days. This occurs because the interest amount (I) used in the calculation has been rounded off to two decimals. Since the interest amount is imprecise, the calculation of t is imprecise. When this occurs, round t off to the nearest integer.

Things To Watch Out For

Miscalculating the number of days in February has to be one of the most common errors. To help you remember how many days are in February, recall your knowledge of leap years.

- February has 28 days except for leap years, in which it has 29 days.
- Leap years are those years that are evenly divisible by four or 400. However, those years that are evenly divisible by 100 (excepting years that are evenly divisible by 400) are not leap years. Here are some examples to illustrate this rule:
  1. The year 1900 was not a leap year, since it is evenly divisible by 100 and not by 400.
  2. The year 1996 was a leap year, since it is evenly divisible by four and not by 100.
  3. The year 2000 was one of those exception years, and it was a leap year since it is evenly divisible by 400.
  4. The year 2004 was once again a leap year since it is evenly divisible by four and not by 100. Thus, in this century 2004, 2008, 2012, 2016, 2020, 2024, and so on by fours are all leap years with 29 days.
  5. The year 2100 will not be a leap year, since it is evenly divisible by 100.

Note that when it comes to the total number of days in a year, simple interest calculations ignore the 366th day in a leap year. Therefore, assume a year has 365 days in all calculations.

Paths To Success

When entering two dates either on the calculator or in Excel, the order in which you key in the dates is not important. If you happen to put the last day of the transaction into the first field (DT1 or Start Date) and the start day of the transaction into the last field (DT2 or End Date), then the number of days will compute as a negative number since the dates are reversed. Ignore the negative sign in these instances. Using the example, on your calculator if you had keyed May 4 into DT1 and March 20 into DT2, the DBD computes as −45, in which you ignore the negative sign to determine the answer is 45 days.

Example 8.1.5: Time Using Dates

Azul borrowed $4,200 from Hannah on November 3, 2011. Their agreement required Azul to pay back the loan on April 14, 2012, with 8% simple interest. In addition to the principal, how much interest does Azul owe Hannah on April 14?

Solution

Calculate the amount of simple interest (I) that Azul owes Hannah on his loan.

What You Already Know

Step 1:

The principal, simple interest rate, and dates are known: $P = \$4,200, r = 8\%$; Start Date = November 3, 2011; End Date = April 14, 2012

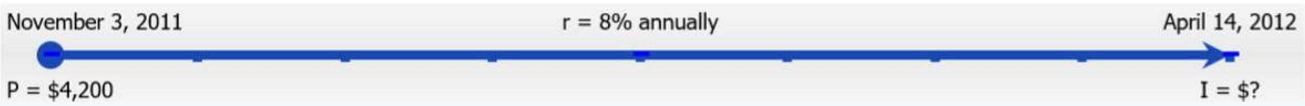
How You Will Get There

Step 1 (continued):

Calculate the number of days in the transaction. Note that this transaction involves February 2012, which is a leap year.

Step 2:

The rate is annual and the time is in days. Convert the time to an annual number.



Step 3:

Apply Formula 8.1.

Perform

Step 1 (continued):

Start Date	End Date	Days between Dates
November 3, 2011	November 30, 2011	30 − 3 = 27
November 30, 2011	December 31, 2011	31 − 0 = 31
December 31, 2011	January 31, 2012	31 − 0 = 31
January 31, 2012	February 29, 2012	29 − 0 = 29
February 29, 2012	March 31, 2012	31 − 0 = 31
March 31, 2012	April 14, 2012	14 − 0 = 14
TOTAL		27 + 31 + 31 + 29 + 31 + 14 = 163 days

Step 2:

163 days out of 365 days in a year is $t = \frac{163}{365}$

Step 3:

$$I = \$4,200 \times 8\% \times \frac{163}{365} = \$4,200 \times 0.08 \times 0.446575 = \$150.05$$

To get the number of days, use the DATE function:

DT1	DT2	DBD	Mode
11.0311	4.1412	Answer: 163	ACT

Azul borrowed the money for a period of 163 days and owes Hannah $150.05 of simple interest on his loan, in addition to repaying the principal of $4,200.

✓ Example 8.1.6: Figuring Out Dates Using Time

On September 13, 2011, Aladdin decided to pay back the Genie on his loan of $15,000 at 9% simple interest. If he paid the Genie the principal plus $1,283.42 of interest, on what day did he borrow the money from the Genie?

Solution

To determine the starting date, the time involved in this transaction, or t, must be calculated.

What You Already Know

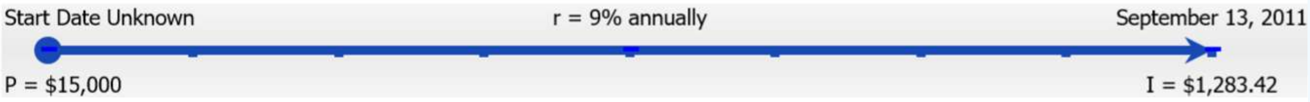
Step 1:

The principal, interest amount, simple interest rate, and end date are known: $P=\$15,000.00, I=\$1,283.42, r=9\%$; End Date = September 13, 2011

How You Will Get There

Step 2:

The time is in days, but the rate is annual. Convert the rate to a daily rate.



Step 3:

Apply Formula 8.1, rearranging for t.

Step 3 (continued):

Using the End Date, use the t to determine the Start Date.

Perform

Step 2:

Convert the interest rate to daily: $r=\frac{9\%}{365}$.

Step 3:

$$t=\frac{\$1,283.42}{\$15,000\times\frac{0.09}{365}}=346.998741 \text{ days } \rightarrow 347 \text{ days}$$

Step 3 (continued):

The number of days is close to a year. Go back 365 days and start at September 13, 2010. Notice that 347 days is 365 − 347 = 18 days short of a full year. Adding 18 days onto September 13, 2010, equals October 1, 2010. This is the start date.

Calculator Instructions

To calculate the start date, use the DATE function:

DT1	DT2	DBD	Mode
Answer: 10-01-2010	9.1311	347	ACT

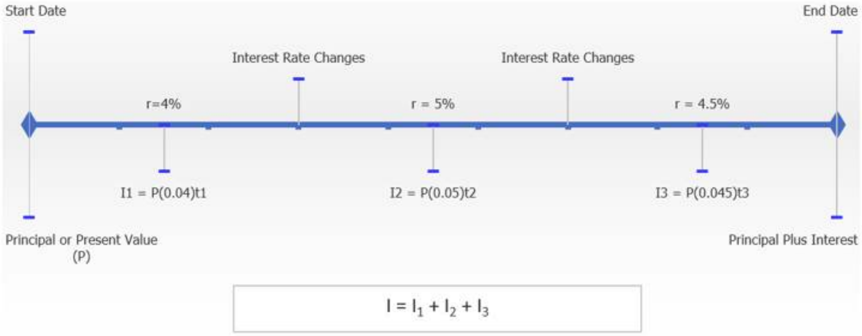
If Aladdin owed the Genie $1,283.42 of simple interest at 9% on a principal of $15,000, he must have borrowed the money 347 days earlier, which is October 1, 2010.

Variable Interest Rates

Not all interest rates remain constant. There are two types of interest rates:

1. **Fixed**. A fixed interest rate is an interest rate that is unchanged for the duration of the transaction.
2. **Variable**. A variable interest rate is an interest rate that is open to fluctuations over the duration of a transaction. This variable interest rate is usually tied to the prime rate, which is an interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans. Most people and companies are usually charged the prime rate plus a percentage (typically 0.5% to 5%) to arrive at the variable interest rate. The Bank of Canada periodically adjusts this rate because of economic and financial considerations in Canada.

In all of the previous examples in this section, the interest rate (r) was fixed for the duration of the transaction. When r fluctuates, you must break the question into time periods or time fragments for each value of r. In each of these time fragments, you calculate the amount of simple interest earned or charged and then sum the interest amounts to form the total interest earned or charged.



This figure illustrates this process with a variable rate that changes twice in the course of the transaction. To calculate the total interest amount, you need to execute Formula 8.1 for each of the time fragments. Then total the interest amounts from each of the three time fragments to calculate the interest amount (I) for the entire transaction.

Example 8.1.7: A Floating Rate

Julio borrowed $10,000 on May 17 at a variable interest rate of prime + 3% when the prime rate was 2¼%. On June 22 the prime rate increased by ¼%, and on September 2 it increased again by ½%. How much interest will Julio pay when he repays his loan on October 4?

Solution

Calculate the amount of interest that Julio owes for the entire time frame of his loan, or I, from May 17 to October 4.

What You Already Know

Step 1:

There is an initial interest rate and then it changes twice over the course of the transaction. That means there are three time periods and three different interest rates. The principal is also known: $P = \$10,000$.

Date	Interest Rate
May 17 to June 22	2¼% + 3% = 5¼%
June 22 to September 2	5¼% + ¼% = 5½%
September 2 to October	4 5½% + ½% = 6%

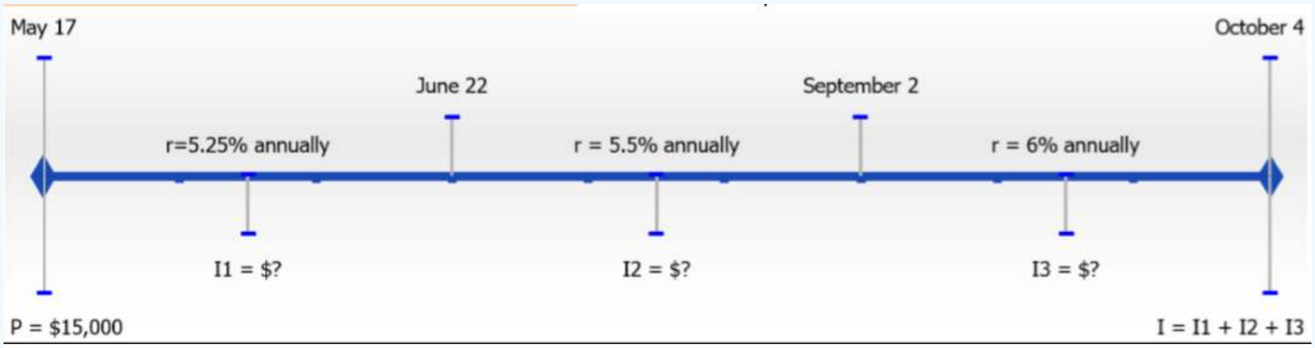
How You Will Get There

Step 1 (continued):

Calculate the number of days for each of the three time periods.

Step 2:

The rates are annual while each of the times are in days. Convert each of the times to an annual number.



Step 3:

For each time period, apply Formula 8.1.

Step 3 (continued):

Sum the three interest amounts to arrive at the total interest paid.

Dates	Calculations	Total # of Days	Convert t to annual	Calculate interest
	Step 1 (continued)		**Step 2**	**Step 3**
May 17 to June 22	May 17 to May 31 = 14 days May 31 to June 22 = 22 days	14 + 22 = 36 days	$\frac{36}{365}$	$I_1 = \$15,000(0.0525)\left(\frac{36}{365}\right)$ $= \$77.671232$
June 22 to September 2	June 22 to June 30 = 8 days June 30 to July 31 = 31 days July 31 to August 31 = 31 days August 31 to Sept 2 = 2 days	8 + 31 + 31 + 2 = 72 days	$\frac{72}{365}$	$I_2 = \$15,000(0.055)\left(\frac{72}{365}\right)$ $= \$162.739726$
September 2 to October 4	Sept 2 to Sept 30 = 28 days Sept 30 to October 4 = 4 days	28 + 4 = 32	$\frac{32}{365}$	$I_3 = \$15,000(0.06)\left(\frac{32}{365}\right)$ $= \$78.904109$
Step 3 (continued): TOTAL				$319.32

Calculator Instructions

Arbitrarily pick the year 2011 and use the DATE function:

Time Fragment	DT1	DT2	DBD	Mode
1	5.1711	6.2211	Answer: 36	ACT
2	6.2211	9.0211	Answer: 72	ACT
3	9.0211	10.0411	Answer: 32	ACT

Julio owes $319.32 of simple interest on his October 4 repayment date.

Contributors and Attributions

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8.2: Moving Money Involving Simple Interest

Can you calculate the amount of money required to meet a future goal? Assume you will graduate college with your business administration diploma in a few months and have already registered at your local university to continue with your studies toward a bachelor of commerce degree. You estimate your total tuition, fees, and textbooks at $8,000. After investigating some short-term investments, you find the best simple rate of interest obtainable is 4.5%. How much money must you invest today for it to grow with interest to the needed tuition money?

In the previous section, you calculated the amount of interest earned or charged on an investment or loan. While this number is good to know, most of the time investors are interested solely in how much in total, including both principal and interest, is either owed or saved. Also, to calculate the interest amount in Formula 8.1, you must know the principal. When people plan for the future, they know the future amount of money that they need but do not know how much money they must invest today to arrive at that goal. This is the case in the opening example above, so you need a further technique for handling simple interest.

This section explores how to calculate the principal and interest together in a single calculation. It adds the flexibility of figuring out how much money there is at the beginning of the time period so long as you know the value at the end, or vice versa. In this way, you can solve almost any simple interest situation.

Maturity Value (or Future Value)

The maturity value of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the "maturing" of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the future value.

The Formula

For any financial transaction involving simple interest, the following is true:

$$\text{Amount of money at the end} = \text{Amount of money at the beginning} + \text{Interest}$$

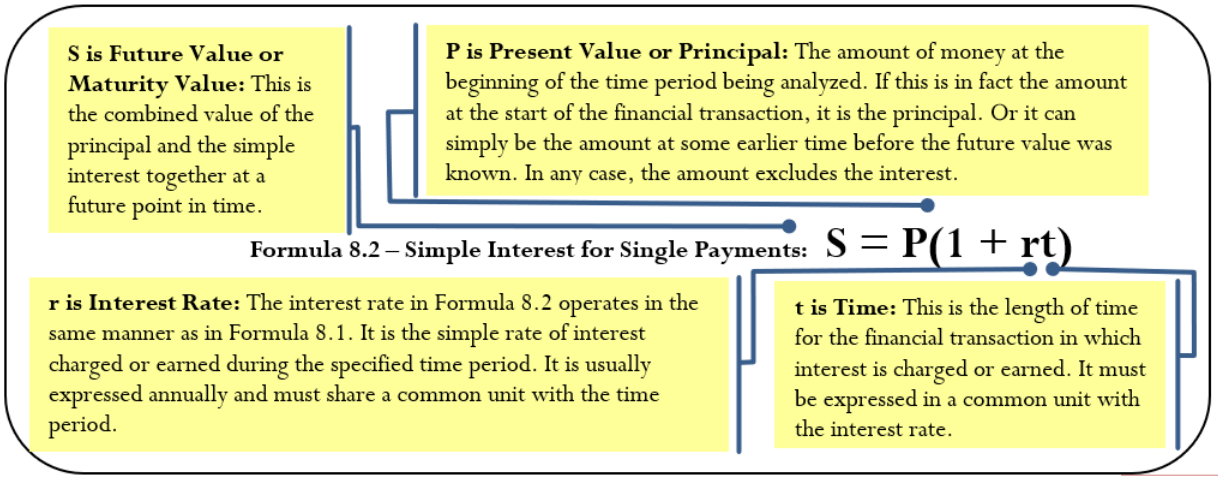
Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol S:

$$S = P + I$$

Substituting in Formula 8.1, $I = Prt$, yields the equation

$$S = P + Prt$$

Simplify this equation by factoring P on the right-hand side to arrive at Formula 8.2.



S is Future Value or Maturity Value: This is the combined value of the principal and the simple interest together at a future point in time.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed. If this is in fact the amount at the start of the financial transaction, it is the principal. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

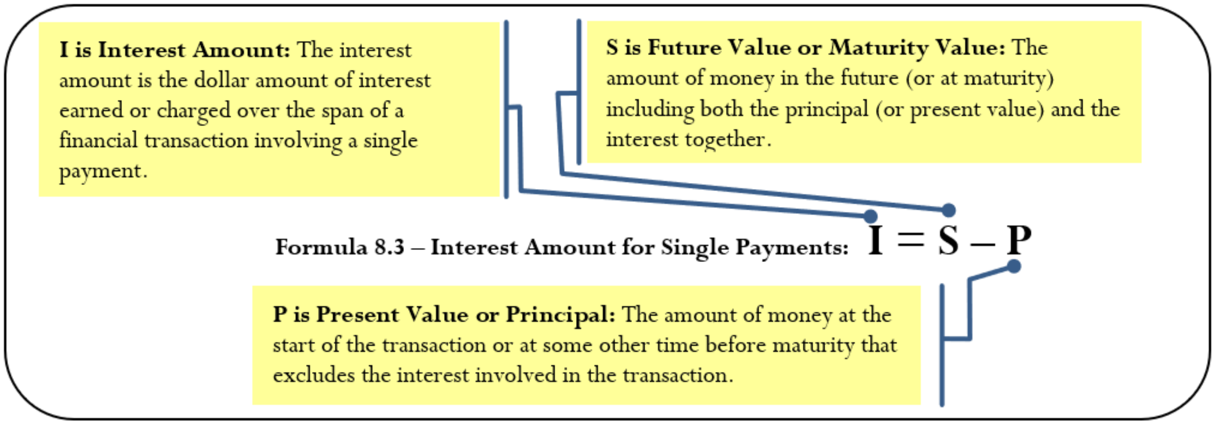
Formula 8.2 – Simple Interest for Single Payments: $S = P(1 + rt)$

r is Interest Rate: The interest rate in Formula 8.2 operates in the same manner as in Formula 8.1. It is the simple rate of interest charged or earned during the specified time period. It is usually expressed annually and must share a common unit with the time period.

t is Time: This is the length of time for the financial transaction in which interest is charged or earned. It must be expressed in a common unit with the interest rate.

Formula 8.2

Depending on the financial scenario, what information you know, and what variable you need to calculate, you may need a second formula offering an alternative method of calculating the simple interest dollar amount.



I is Interest Amount: The interest amount is the dollar amount of interest earned or charged over the span of a financial transaction involving a single payment.

S is Future Value or Maturity Value: The amount of money in the future (or at maturity) including both the principal (or present value) and the interest together.

Formula 8.3 – Interest Amount for Single Payments: $\mathbf{I = S - P}$

P is Present Value or Principal: The amount of money at the start of the transaction or at some other time before maturity that excludes the interest involved in the transaction.

Formula 8.3

How It Works

Follow these steps when working with single payments involving simple interest:

Step 1: Formula 8.2 has four variables, any three of which you must identify to work with a single payment involving simple interest.

- If either the P or S is missing but the I is known, apply Formula 8.3 to determine the unknown value.
- If only the interest amount I is known and neither P nor S is identified, recall that the interest amount is a portion and that the interest rate is a product of rate and time (e.g., 6% annually for nine months is an earned interest rate of 4.5%). Apply Formula 2.2 on rate, portion, and base to solve for base, which is the present value or P.
- If necessary, draw a timeline to illustrate how the money is being moved through time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If not, you need to convert one of the two variables to achieve the commonality.

Step 3: Apply Formula 8.2 and solve for the unknown variable, manipulating the formula as required.

Step 4: If you need to calculate the amount of interest, apply Formula 8.3.

Assume that today you have $10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Step 1: The principal is $P = \$10,000$, the simple interest rate is 7% per year, or $r = 0.07$, and the time is $t = 11$ months.

Step 2: The time is in months, but to match the rate it needs to be expressed annually as $t = \frac{11}{12}$.

Step 3: Applying Formula 8.2 to calculate the future value including interest,

$S = \$10,000 \times (1 + 0.07\times) = \frac{11}{12}\$10,641.67$. This is the total amount after 11 months.

Step 4: Applying Formula 8.3 to calculate the interest earned, $I = \$10,641.67 - \$10,000.00 = \$641.67$. You could also apply Formula 8.1 to calculate the interest amount; however, Formula 8.3 is a lot easier to use. The $10,000 earns $641.67 in simple interest over the next 11 months, resulting in $10,641.67 altogether.

Things To Watch Out For

As with Formula 8.1, the most common mistake in the application of Formula 8.2 is failing to ensure that the rate and time are expressed in the same units. Before you proceed, always check these two variables!

Paths To Success

When solving Formula 8.2 for either rate or time, it is generally easier to use Formula 8.3 and Formula 8.1 instead. Follow these steps to solve for rate or time:

1. If you have been provided with both the present value and future value, apply Formula 8.3 to calculate the amount of interest (I).
2. Apply and rearrange Formula 8.1 to solve for either rate or time as required.

Exercise 8.2.1: Give It Some Thought

In each of the following situations, determine which statement is correct.

a. If you have a future value of $5,000 and interest is involved, then your present value is
    i. More than $5,000
    ii. The same
    iii. Less than $5,000
b. If you had a principal of $10,000 earning 10% simple interest for any period less than a year, you earn
    i. More than $1,000 interest
    ii. Exactly $1,000 interest
    iii. Less than $1,000 interest
c. If you took out a loan for $2,500 at a 5% simple interest rate for six months and you made no payments during that time frame, you owe
    i. More than $2,500
    ii. Exactly $2,500
    iii. Less than $2,500

Answer

1. Less than $5,000. When being charged or earning interest, the present value is always less than the future value.
2. Less than $1,000 interest. The $10,000 earns exactly $1,000 interest per year, so any time frame less than a year produces a smaller interest amount.
3. More than $2,500. When interest is being charged, the future value is always larger than the present value

Example 8.2.1: Saving for a Down Payment on a Home

You just inherited $35,000 from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns 4¼% simple interest? How much interest will you have earned?

Solution

Calculate the amount of money four months from now including both the principal and interest earned. This is the maturity value (S). Also calculate the interest earned (I).

What You Already Know

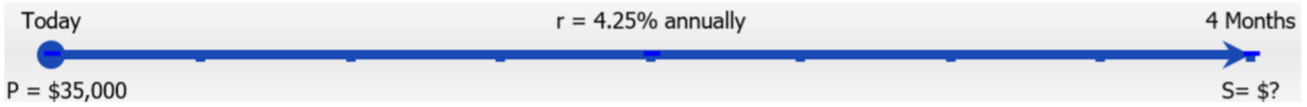
Step 1:

The principal, interest rate, and time frame are known: $P = \$35,000, r = 4¼\%$ per year $t = 4$ months

How You Will Get There

Step 2:

While the rate is annual, the time is in months. Convert the time to an annual number.



Step 3:

Apply Formula 8.2.

Step 4:

Apply Formula 8.3 to calculate the interest amount.

Perform

Step 2:

Four months out of 12 months in a year is $t = \frac{4}{12}$

Step 3:

$$S = \$35,000 \times \left(1 + 4\frac{1}{4}\% \times \frac{4}{12}\right) = \$35,000 \times (1 + 0.0425 \times 0.\overline{3}) = \$35,495.83$$

Step 4:

$$I = \$35,495.83 - \$35,000.00 = \$495.83$$

Four months from now you will have \$35,495.83 as a down payment toward your house, which includes \$35,000 in principal and \$495.83 of interest.

✓ Example $8.2.2$: Saving for Tuition

Recall the section opener, where you needed \$8,000 for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

Solution

Calculate the principal amount of money today (P) that you must invest such that it will earn interest and end up at the \$8,000 required for the tuition.

What You Already Know

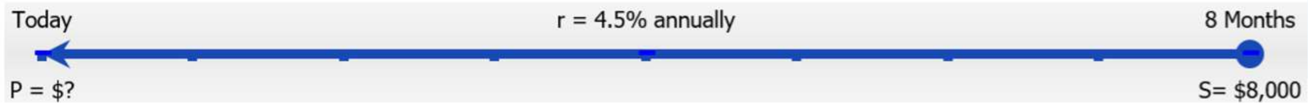
Step 1:

The future value, interest rate, and time period are known: $S = \$8,000, r = 4.5\%$ per year $t = 8$ months

How You Will Get There

Step 2:

While the rate is annual, the time is in months. Convert the time to an annual number.



Step 3:

Apply Formula 8.2, rearranging for P.

Perform

Step 2:

Eight months out of 12 months in a year is $t = \frac{8}{12}$

Step 3:

$$\$8,000 = P\left(1 + 4.5\% \times \frac{8}{12}\right)$$
$$P = \frac{\$8,000}{(1 + 0.045 \times 0.\overline{6})} = \$7,766.99$$

If you place \$7,766.99 into the investment, it will grow to \$8,000 in the eight months.

✓ Example 8.2.3: What Exactly Are You Being Offered?

You are sitting in an office at your local financial institution on August 4. The bank officer says to you, "We will make you a great deal. If we advance that line of credit and you borrow $20,000 today, when you want to repay that balance on September 1 you will only have to pay us $20,168.77, which is not much more!" Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

Solution

Determine the rate of interest that you would be charged on your line of credit, or r.

What You Already Know

Step 1:

The principal, maturity amount, and time frame are known: $P=\$20,000.00, S=\$20,168.77, t=$ August 4 to September 1

How You Will Get There

Step 1 (continued):

Calculate the number of days in the transaction.

Step 2:

Since interest rates are usually expressed annually, convert the time from days to an annual number.

Step 3:

Use the combination of Formula 8.3 and Formula 8.1. First calculate the interest amount using Formula 8.3.

Step 3 (continued):

Then apply Formula 8.1, rearranging for r.

Perform

Step 1 (continued):

Start Date	End Date	Days between Dates
August 4	August 31	31 − 4 = 27
August 31	September 1	1 − 0 = 1
TOTAL		27 + 1 = 28 days

Step 2:

28 days out of 365 days in a year is $t=\frac{28}{365}$

Step 3:

$$I=\$20,168.77-\$20,000.00=\$168.77$$

Step 3 (continued):

$$r=\frac{\$168.77}{\$20,000.00\times\frac{28}{365}}=0.110002 \text{ or } 11.0002\%$$

Calculator Instructions

Assume the year 2011 and use the DATE function:

DT1	DT2	DBD	Mode
08.0411	09.0111	Answer: 28	ACT

The interest rate on the offered line of credit is 11.0002% (note that it is probably exactly 11%; the extra 0.0002% is most likely due to the rounded amount of interest used in the calculation).

Equivalent Payments

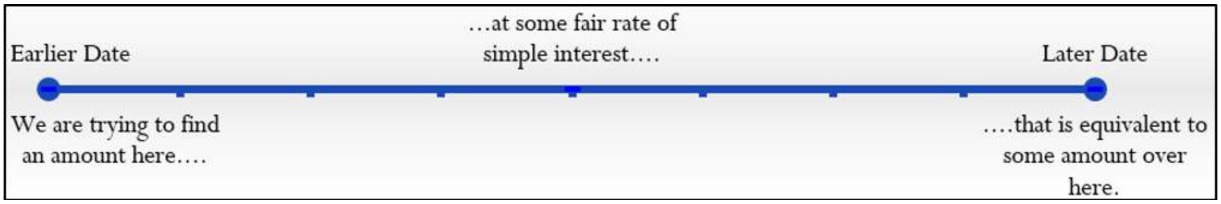
Life happens. Sometimes the best laid financial plans go unfulfilled. Perhaps you have lost your job. Maybe a reckless driver totaled your vehicle, which you now have to replace at an expense you must struggle to fit into your budget. No matter the reason, you find yourself unable to make your debt payment as promised.

On the positive side, maybe you just received a large inheritance unexpectedly. What if you bought a scratch ticket and just won $25,000? Now that you have the money, you might want to pay off that debt early. Can it be done?

Whether paying late or paying early, any amount paid must be equivalent to the original financial obligation. As you have learned, when you move money into the future it accumulates simple interest. When you move money into the past, simple interest must be removed from the money. This principle applies both to early and late payments:

- **Late Payments**. If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe $100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of $10 should be added, making your debt payment $110. This is no different from your friend receiving the $100 today and investing it himself at 10% interest so that it accumulates to $110 in one year.
- **Early Payments**. If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend $110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of $10 should be deducted, making your debt payment today $100. If your friend then invests this money at 10% simple interest, one year from now he will have the $110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means $100 today is the same thing as having $110 one year from now. This illustrates the concept that two payments are equivalent payments if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.


Earlier Date
…at some fair rate of simple interest….
Later Date
We are trying to find an amount here….
….that is equivalent to some amount over here.

How It Works

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

Paths To Success

Being financially smart means paying attention to when you make your debt payments. If you receive no financial benefit for making an early payment, then why make it? The prudent choice is to keep the money yourself, invest it at the best interest rate possible, and pay the debt off when it comes due. Whatever interest is earned is yours to keep and you still fulfill your debt obligations in a timely manner!

✓ Example $8.2.4$: Making a Late Payment

Erin owes Charlotte $1,500 today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. How much does Erin need to pay nine months from now? What is the amount of her late penalty?

Solution

A late payment is a future value amount (S). The late penalty is equal to the interest (I).

What You Already Know

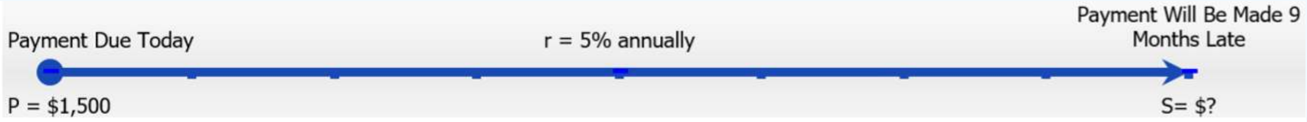
Step 1:

The payment today, the agreed-upon interest rate, and the late payment timing are known: $P = \$1,500, r = 5\%$ annually, $t = 9$ months

How You Will Get There

Step 2:

While the rate is annual, the time is in months. Convert the time to an annual number.



Step 3:

Apply Formula 8.2.

Step 4:

Apply Formula 8.3 to calculate the late interest penalty amount.

Perform

Step 2:

Nine months out of 12 months in a year is $t = \frac{9}{12}$.

Step 3:

$$S = \$1,500 \times \left(1 + 5\% \times \frac{9}{12}\right) = \$1,500 \times (1 + 0.05 \times 0.75) = \$1,556.25$$

Step 4:

$$I = \$1,556.25 - \$1,500.00 = \$56.25$$

Erin's late payment is for $1,556.25, which includes a $56.25 interest penalty for making the payment nine months late.

✓ Example 8.2.5: Making an Early Payment

Rupert owes Aminata two debt payments: $600 four months from now and $475 eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. What amount should Rupert pay today? What is the total amount of his early payment benefit?

Solution

An early payment is a present value amount (P). Both payments will be moved to today and summed. The early payment benefit will be the total amount of interest removed (I).

What You Already Know

Step 1:

The two payments, interest rate, and payment due dates are known: $r = 7\%$ annually

First Payment: $S = \$600, t = 4$ months from now

Second Payment: $S = \$475, t = 11$ months from now

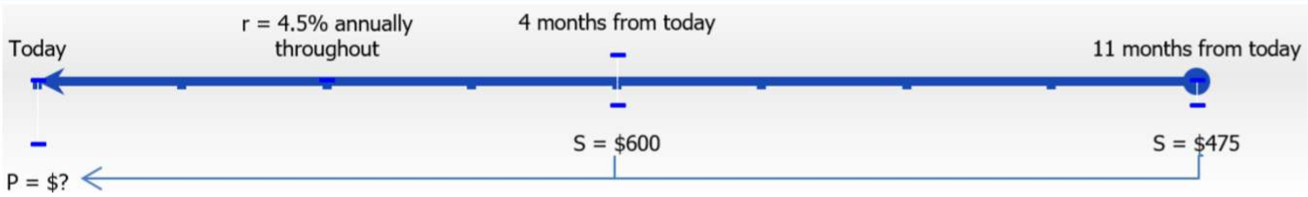
Replacement payment is being made today.

How You Will Get There

Do steps 2 and 3 for each payment:

Step 2:

While the rate is annual, the time is in months. Convert the time to an annual number.



Step 3:

Apply Formula 8.2, rearranging it to solve for P. Once you calculate both P, they can be summed to arrive at the total payment.

Step 4:

For each payment, apply Formula 8.3 to calculate the associated early interest benefit. Total the two values of I to get the early interest benefit overall.

Perform

Payment #1:

Step 2:

Four months out of 12 months in a year is $t = \frac{4}{12}$

Step 3:

$$\begin{aligned} \$600 &= P\left(1+7\% \times \frac{4}{12}\right) \\ P &= \frac{\$600}{(1+0.07 \times 0.\overline{3})} \\ &= \$586.32 \end{aligned}$$

Payment #2:

Step 2:

Eleven months out of 12 months in a year is $t = \frac{11}{12}$

Step 3:

$$\begin{aligned} \$475 &= P\left(1+7\% \times \frac{11}{12}\right) \\ P &= \frac{\$475}{(1+0.07 \times 0.91\overline{6})} = \$446.36 \end{aligned}$$

Totals:

P today $= \$586.32 + \$446.36 = \$1,032.68$

Step 4:

Payment #1: $I = S–P = \$600–\$586.32 = \$13.68$

Payment #2: $I = S–P = \$475–\$446.36 = \$28.64$

$$I = \$13.68 + \$28.64 = \$42.32$$

To clear both debts today, Rupert pays $1,032.68, which reflects a $42.32 interest benefit reduction for the early payment.

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8.3: Savings Accounts And Short-Term GICs

When the stock market offers potentially huge returns, why would anyone put money into an investment that earns just a few percent annually? The answer of course is that sometimes stocks decrease in value. For example, after reaching a historic high in June 2008, the Toronto Stock Exchange plunged 40% over the course of the next six months. So if you are not willing to take the risk in the stock market, there are many options for investing your money that are safe and secure, particularly savings accounts and short-term guaranteed investment certificates (GICs). Some of the benefits and drawbacks associated with safe and secure investments are listed below.

Benefits

- Investments are virtually 100% safe, meaning that the chances of losing your investment are almost zero.
- They always have a positive rate of interest.
- It is easy to access the money when the investor needs it.
- Any interest earned is better than zero interest.
- It is suitable for small amounts.

Drawbacks

- Interest rates are extremely low and offer little growth for your money.
- Many come with stiff fees and interest rate penalties if certain conditions are not met.
- Minimum balances are a common requirement, otherwise the interest rate is zero.
- Balances are only insured up to $100,000.

In this section, you are exposed to both savings accounts and short-term GICs. You will learn a little about the characteristics of each along with how to perform simple interest calculations on these financial tools.

Savings Accounts

A savings account is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

1. **How Interest Is Calculated**. There are two common methods for calculating simple interest:
    i. Accounts earn simple interest that is calculated based on the daily closing balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with $500 in the account and deposit $3,000 at 9:00 a.m., then withdraw the $3,000 at 4:00 p.m., your closing balance is $500. That is the principal on which interest is calculated, not the $3,500 in the account throughout the day.
    ii. Accounts earn simple interest based on a minimum monthly balance in the account. For example, if in a single month you had a balance in the account of $900 except for one day, when the balance was $500, then only the $500 is used in calculating the entire month's worth of interest.
2. When Interest Is Deposited. Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.
3. Insurance against Loss. Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to $100,000 in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO), which also guarantees up to $100,000. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.

4. Interest Rate Amounts. Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn. At the time of writing, interest rates on savings accounts ranged from a low of 0.05% to a high of 1.95%. Though this is not much, it is better than nothing and certainly better than losing money!

While a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat rate savings accounts and tiered savings accounts. Each of these is discussed separately.

How It Works

Flat-Rate Savings Accounts.

A flat-rate savings account has a single interest rate that applies to the entire balance. The interest rate may fluctuate in synch with short-term interest rates in the financial markets.

Follow these steps to calculate the monthly interest for a flat-rate savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a flat-rate table as illustrated here. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Date	Closing Balance in Account	# of Days	Simple Interest Earned
			$I = Prt$
Total Monthly Interest Earned			

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: Calculate the number of days that the closing balance is maintained for each row.

Step 5: Apply Formula 8.1, $I = Prt$, to each row in the table. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: Sum the Simple Interest Earned column and round off to two decimals.

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

✓ Example 8.3.1: Savings at the Royal Bank

The RBC High Interest Savings Account pays 0.75% simple interest on the daily closing balance in the account and the interest is paid on the first day of the following month. On March 1, the opening balance in the account was $2,400. On March 12, a deposit of $1,600 was made. On March 21, a withdrawal of $2,000 was made. Calculate the total simple interest earned for the month of March.

Solution

Calculate the total interest amount for the month (I).

What You Already Know

Step 1:

The following transactions, dates, and interest rate are known: $r = 0.75\%$ per year, March 1 opening balance = $2,400, March 12 deposit = $1,600, March 21 withdrawal = $2,000

How You Will Get There

Step 2:

Set up a flat-rate table.

Step 3:

Determine the date ranges for each balance throughout the month and calculate the closing balances.

Step 4:

For each row of the table, calculate the number of days involved.

Step 5:

Apply Formula 8.1 to calculate simple interest on each row.

Step 6:

Sum the Simple Interest Earned column.

Dates	Closing Balance in Account	# of Days	Simple Interest Earned (I = Prt)
Step 2 and Step 3		**Step 4**	**Step 5**
March 1 to March 12	\$2,400	12 − 1 = 11	$I = \$2,400(0.0075)\left(\frac{11}{365}\right)$ $I = \$0.542465$
March 12 to March 21	\$2,400 + \$1,600 = \$4,000	21 − 12 = 9	$I = \$4,000(0.0075)\left(\frac{9}{365}\right)$ $I = \$0.739726$
March 21 to April 1	\$4,000 − \$2,000 = \$2,000	31 + 1 − 21 = 11	$I = \$2,000(0.0075)\left(\frac{11}{365}\right)$ $I = \$0.452054$
Step 6: Total Monthly Interest Earned			$I = \$0.542465 + \$0.739726 + \$0.452$ $I = \$1.73$

For the month of March, the savings account earned a total simple interest of \$1.73, which was deposited to the account on April 1.

How It Works

Tiered Savings Accounts.

A tiered savings account pays higher rates of interest on higher balances in the account. This is very much like a graduated commission on gross earnings, as discussed in Section 4.1. For example, you might earn 0.25% interest on the first \$1,000 in your account and 0.35% for balances over \$1,000. Most of these tiered savings accounts use a portioning system. This means that if the account has \$2,500, the first \$1,000 earns the 0.25% interest rate and it is only the portion above the first \$1,000 (hence, \$1,500) that earns the higher interest rate.

Follow these steps to calculate the monthly interest for a tiered savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a tiered interest rate table as illustrated below. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one. Adjust the number of columns to suit the number of tiered rates. Fill in the headers for each tiered rate with the balance requirements and interest rate for which the balance is eligible.

Dates	Closing Balance in Account	# of Days	Tier Rate #1 Balance Requirements and Interest Rate	Tier Rate #2 Balance Requirements and Interest Rate	Tier Rate #3 Balance Requirements and Interest Rate
			Eligible P = I = Prt	Eligible P = I = Prt	Eligible P = I = Prt
Total Monthly Interest Earned					

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: For each row, calculate the number of days that the closing balance is maintained.

Step 5: Assign the closing balance to the different tiers, paying attention to whether portioning is being used. In each cell with a balance, apply Formula 8.1, where $I = Prt$. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: To calculate the Total Monthly Interest Earned, sum all interest earned amounts from all tier columns and round off to two decimals.

✓ Example $8.3.2$: A Rate Builder Tiered Account

The Rate Builder savings account at your local credit union pays simple interest on the daily closing balance as indicated in the table below:

Balance	Interest Rate
\$0.00 to \$500.00	0% on entire balance
\$500.01 to \$2,500.00	0.5% on entire balance
\$2,500.01 to \$5,000.00	0.95% on this portion of balance only
\$5,000.01 and up	1.35% on this portion of balance only

In the month of August, the opening balance on an account was \$2,150.00. Deposits were made to the account on August 5 and August 15 in the amounts of \$3,850.00 and \$3,500.00. Withdrawals were made from the account on August 12 and August 29 in the amounts of \$5,750.00 and \$3,000.00. Calculate the simple interest earned for the month of August.

Solution

Calculate the total interest amount for the month of August (I).

What You Already Know

Step 1:

The interest rate structure is in the table above.

The transactions and dates are also known:

August 1 opening balance = \$2,150.00, August 5 deposit = \$3,850.00, August 12 withdrawal = \$5,750.00, August 15 deposit = \$3,500.00, August 29 withdrawal = \$3,000.00

How You Will Get There

Step 2:

Set up a tiered interest rate table with four columns for the tiered rates.

Step 3:

Determine the date ranges for each balance throughout the month and calculate the closing balances.

Step 4:

Calculate the number of days involved on each row of the table.

Step 5:

Assign the closing balance to each tier accordingly. Apply Formula 8.1 to any cell containing a balance.

Step 6:

Total up all of the interest from all cells of the table.

Perform

Dates	Closing Balance in Account	# of Days	0% \$0 to \$500 Entire Balance	0.5% \$500.01 to \$2,500 Entire Balance	0.95% \$2,500.01 to \$5,000 This portion only	1.35% \$5,000.01 and up This portion only
Step 2 and Step 3		**Step 4**	**Step 5**			
August 1 to August 5	\$2,150.00	5 − 1 = 4		$P = \$2,150.00$ $I = \$2,150(0.005)\left(\frac{4}{365}\right)$ $I = \$0.117808$		
August 5 to August 12	\$2,150.00 + \$3,850.00 = \$6,000.00	12 − 5 = 7		$P = \$2,500.00$ $I = \$2,500(0.005)\left(\frac{7}{365}\right)$ $I = \$0.239726$	$P = \$2,500.00$ $I = \$2,500(0.0095)\left(\frac{7}{365}\right)$ $I = \$0.455479$	$P = \$1,000.00$ $I = \$1,000(0.0135)\Big($ $I = \$0.258904$
August 12 to August 15	\$6,000.00 − \$5,750.00 = \$250.00	15 −12 = 3	P = \$250.00 I = \$0.00			
August 15 to August 29	\$250.00 + \$3,500.00 = \$3,750.00	29 − 15 = 14		$P = \$2,500.00$ $I = \$2,500(0.005)\left(\frac{14}{365}\right)$ $I = \$0.479452$		
August 29 to September 1	\$3,750.00 − \$3,000.00 = \$750.00	31 + 1 − 29 = 3		$P = \$750.00$ $I = \$750(0.005)\left(\frac{3}{365}\right)$ $I = \$0.030821$		
Step 6: Total Monthly Interest Earned			$I = \$0.117808 + \$0.239726 + \$0.455479 + \$0.258904 + \$0.00 + \$0.479452 + \$$ $I = \$2.04$			

For the month of August, the tiered savings account earned a total simple interest of \$2.04, which was deposited to the account on September 1.

Short-Term Guaranteed Investment Certificates (GICs)

A guaranteed investment certificate (GIC) is an investment that offers a guaranteed rate of interest over a fixed period of time. In many countries around the world, a GIC is often called a time or term deposit. GICs are found mostly at commercial banks, trust companies, and credit unions. Just like savings accounts, they have a very low risk profile and tend to have much lower rates than are available through other investments such as the stock market or bonds. They are also unconditionally guaranteed, much like savings accounts. In this section, you will deal only with short-term GICs, defined as those that have a time frame of less than one year.

The table below summarizes three factors that determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

	Higher Interest Rates	Lower Interest Rates
Principal Amount	Large	Small
Time	Longer	Shorter
Redemption Privileges	Nonredeemable	Redeemable

1. **Amount of Principal**. Typically, a larger principal is able to realize a higher interest rate than a smaller principal.
2. **Time**. The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
3. **Redemption Privileges**. The two types of GICs are known as redeemable and nonredeemable. A redeemable GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A nonredeemable GIC "locks in" your money for the agreed-upon term. Accessing that money before the end of the term usually

incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

To summarize, if you want to receive the most interest it is best to invest a large sum for a long time in a nonredeemable short-term GIC.

How It Works

Short-term GICs involve a lump sum of money (the principal) invested for a fixed term (the time) at a guaranteed interest rate (the rate). Most commonly the only items of concern are the amount of interest earned and the maturity value. Therefore, you need the same four steps as for single payments involving simple interest shown in Section 8.2.

✓ Example $8.3.3$: GIC Choices

Your parents have $10,000 to invest. They can either deposit the money into a 364-day nonredeemable GIC at Assiniboine Credit Union with a posted rate of 0.75%, or they could put their money into back-to-back 182-day nonredeemable GICs with a posted rate of 0.7%. At the end of the first 182 days, they will reinvest both the principal and interest into the second GIC. The interest rate remains unchanged on the second GIC. Which option should they choose?

Solution

For both options, calculate the future value (S), of the investment after 364 days. The one with the higher future value is your parents' better option.

What You Already Know

Step 1:

For the first GIC investment option: $P=\$10,000, r=0.75\%$ per year $t=364$ days

For the second GIC investment option: Initial $P=\$10,000, r=0.7\%$ per year $t=182$ days each

How You Will Get There

Step 2:

The rate is annual, the time is in days. Convert the time to an annual number.

Step 3 (1st GIC option):

Calculate the maturity value (S_1) of the first GIC option after its 364-day term by applying Formula 8.2.

Step 3 (2nd GIC option, 1st GIC):

Calculate the maturity value (S_2) after the first 182-day term by applying Formula 8.2.

Step 3 (2nd GIC option, 2nd GIC):

Reinvest the first maturity value as principal for another term of 182 days applying Formula 8.2 and calculate the final future value (S_3).

Perform

Step 2:

Transforming both time variables, $t=\frac{364}{365}$ and $t=\frac{182}{365}$

Step 3 (1st GIC option): $S_1=\$10,000\left(1+(0.0075)\left(\frac{364}{365}\right)\right)=\$10,074.79$

Step 3 (2nd GIC option, 1st GIC): $S_2=\$10,000\left(1+(0.007)\left(\frac{182}{365}\right)\right)=\$10,034.90$

Step 3 (2nd GIC option, 2nd GIC): $S_3=\$10,034.90\left(1+(0.007)\left(\frac{182}{365}\right)\right)=\$10,069.93$

The 364-day GIC results in a maturity value of $10,074.79, while the two back-to-back 182-day GICs result in a maturity value of $10,069.93. Clearly, the 364-day GIC is the better option as it will earn $4.86 more in simple interest.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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8.4: Application - Promissory Notes

Have you ever lent a friend or family member some money on a promise they would pay you back on a specific date? Then you are already experienced with promissory notes!

Just before class a fellow student and close friend announces that the two of you should go for coffee afterwards. He seems a little tense, but he says there is no trouble, so you contain your curiosity until you are seated in a quiet corner of your local Starbucks. Then he asks if he can borrow $5,000 from you to pursue a new entrepreneurial venture, explaining that he already tried to get the money from a bank but too much red tape was involved and the interest rate was outrageous. You have the money saved up in your account. After your friend lays out all of the details, you feel satisfied that he has a good idea. Before lending your friend the money, though, it might be worthwhile to put this down in writing just in case matters turn sour, right? And by loaning your friend the money, you would be forgoing at least the interest on your savings and feel that it is fair to charge your friend the same interest rate so that you are not hurt financially. A promissory note seems a good path to take.

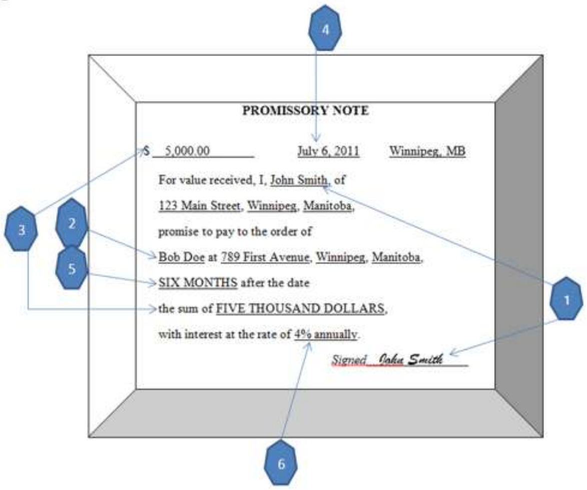
This section explains the characteristics of short-term promissory notes. You will learn how to calculate the maturity value of a promissory note along with its selling price if you want to cash it in before it matures.

What Are Promissory Notes?

Promissory notes are borrowing tools that allow both companies and people to get financing from a source other than a bank. The money is borrowed from other people or companies who are willing to provide the financing. The Bills of Exchange Act defines a promissory note as "an unconditional promise in writing made by one person to another person, signed by the maker, engaging to pay, on demand or at a fixed or determinable future time, a sum certain in money to, or to the order of, a specified person or to bearer.1 Promissory notes are commonly referred to just as notes.

A promissory note is not the same as an IOU (I owe you). An IOU simply acknowledges a debt but contains no promise of repayment, no timelines, and no consequences. A promissory note includes these elements.

In the opening scenario, it is probably best to create a promissory note that clearly details your agreement and terms. This allows both of you to have peace of mind in the transaction, and in the event something goes wrong both of you have a legally binding written record of your transaction.



PROMISSORY NOTE

$ 5,000.00 July 6, 2011 Winnipeg, MB

For value received, I, John Smith, of
123 Main Street, Winnipeg, Manitoba,
promise to pay to the order of
Bob Doe at 789 First Avenue, Winnipeg, Manitoba,
SIX MONTHS after the date
the sum of FIVE THOUSAND DOLLARS,
with interest at the rate of 4% annually.

Signed John Smith

The illustration here provides an example of a promissory note. Observe the following characteristics:

1. **Borrower**. This is the individual who is borrowing the money, sometimes known as the maker, promisor, or payor. It is best to provide details such as the name and address of the borrower to clearly establish an identity.
2. **Lender**. This is the individual who is loaning the money (and should be repaid), sometimes known as the payee or issuer. As with the borrower, detailed name and address information clearly establish the identity of this person.
3. **Face Value**. This is the borrowed principal. It appears numerically and in writing to prevent it from being altered. In simple interest calculations, this is the P.
4. **Issue Date (Start Date)**. This is the date on which the promissory note is issued and forms the starting date of the transaction.

5. **Term**. This is the length of time for which the face value is borrowed from the lender. In simple interest calculations, this is the basis for the time period (t).
6. **Interest Rate**. This is the percentage interest being charged along with the respective time frame for the interest (monthly, annually, etc.). In simple interest calculations, this is the rate (r).

Note in the illustration that the lender has decided to not have any additional consequence for failing to pay back the $5,000 after the six months. As written, any late payment continues to be charged 4% annual interest only.

Promissory notes are issued outside the normal banking system, so it is up to the lender to determine the interest rate. This results in two types of promissory notes:

1. **Noninterest-Bearing Notes**. In these cases, the lender has decided not to charge any interest, hence making the interest rate 0%. Alternatively, the lender and borrower may decide to include an amount of interest in the face value specified on the note. For example, the lender may borrow $5,000, but the note is written for $5,100 with no interest. In the case of a noninterest-bearing note, of course, no calculation of interest on the promissory note is needed to determine the amount that is due, as the face value equals the maturity value.
2. **Interest-Bearing Notes**. In these cases, the lender has decided to charge interest on the face value. This requires you to calculate the simple interest on the face value. Therefore, the maturity value is the face value plus a simple interest amount. The method for this calculation is discussed next.

Maturity Value for Interest-Bearing Notes

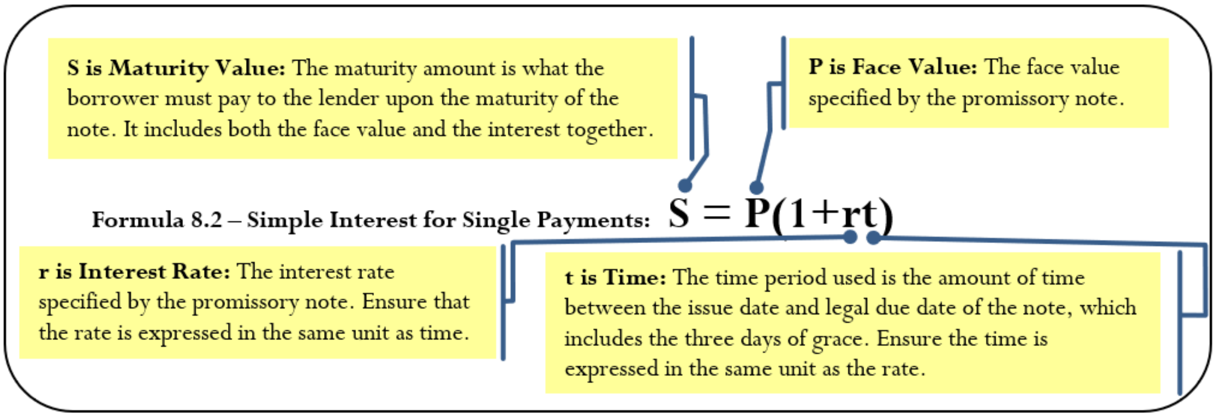
Based on all of the elements included in the promissory note, the maturity date or end date of the promissory note must be determined before the borrower and lender can calculate simple interest. This is the date upon which the promissory note needs to be repaid. However, by Canadian law this is not the due date. The legal due date of a note is instead three days after the term specified, allowing the borrower time to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend, when it might not be possible to make the payment. Interest is still charged during those three days, though. The three days' grace period can be waived by the borrower by stating "no grace" on the promissory note.

The following rules are used to determine the due date:

1. **Specific Date or Number of Days**. If the note states a specific maturity date or details the exact number of days, then the due date is three days later than the maturity date.
2. **Time Period in Months**. If the note states a number of months, then it is due on the same date in the month in which it matures plus three days. In the illustration, the note was issued on July 6, 2011. Six months later is January 6, 2012, plus three days gives a legal due date of January 9, 2012. A special situation arises, though, if the note is issued near or at the end of a month. For example, if the note is issued instead on August 31, 2011, six months later is February 31, 2012. Since this date does not exist, the rule is to use the last day of the month in which it matures. Therefore, the note has a maturity date of February 29, 2012, and that plus three days gives a legal due date of March 3, 2012.

The Formula

Once you know the legal due date, calculate the time involved in the transaction using the methods discussed in Section 8.1. You calculate the maturity value using Formula 8.2 from Section 8.2, which is reproduced below and adapted for promissory notes.



S is Maturity Value: The maturity amount is what the borrower must pay to the lender upon the maturity of the note. It includes both the face value and the interest together.

P is Face Value: The face value specified by the promissory note.

Formula 8.2 – Simple Interest for Single Payments: $S = P(1+rt)$

r is Interest Rate: The interest rate specified by the promissory note. Ensure that the rate is expressed in the same unit as time.

t is Time: The time period used is the amount of time between the issue date and legal due date of the note, which includes the three days of grace. Ensure the time is expressed in the same unit as the rate.

How It Works

You use the same four steps to calculate the maturity value of a single payment as when calculating the maturity value of a promissory note. The only nuance requiring close attention lies in determining the time period.

Working with the previous promissory note illustration, calculate the amount that John Smith owes to Bob Doe along with the amount of interest.

Step 1: Note that this promissory note is interest-bearing because of the interest rate information provided on the note. The face value or $P=\$5,000$. The interest rate or $r=4\%$ annually. To calculate the time, it is noted above that the legal due date of this promissory note is six months plus three days later, which is January 9, 2012. Using an appropriate date method from Section 8.1, the days between the dates equal 187 days.

Step 2: Expressing the time annually to match the annual rate, $t=187/365$.

Step 3: Substituting into Formula 8.2, $S=\$5,000(1+0.04(187/365))=\$5,102.47$. Therefore, John needs to repay $5,102.47 to Bob on January 9, 2012.

Step 4: The amount of interest on the note is $I=\$5,102.47-\$5,000=\$102.47$.

✓ Example $8.4.1$: Maturity Value of an Interest-Bearing Promissory Note

Corey borrowed $7,500 from Miller Enterprises on March 31, 2011. An interest-bearing promissory note was created, indicating that the money was to be repaid eight months later at an interest rate of 6% annually. Calculate the maturity value of the promissory note on the legal due date.

Solution

Calculate the maturity value of the promissory note (S) upon the legal due date using the stated interest rate.

What You Already Know

Step 1:

The face value, interest rate, term, and issue date are known:

$P=\$7,500, r=6\%$ per year $t=8$ months, Issue date = March 31, 2011

How You Will Get There

Step 1 (continued):

Calculate the legal due date by determining the end of the term and adding three days. Transform this date into the number of days.

Step 2:

While the rate is annual, the time is in days. Convert the time to an annual number.

Step 3:

Apply Formula 8.2.

Perform

Step 1 (continued):

Eight months after the issue date of March 31, 2011, is November 31, 2011. Since November has only 30 days, use the last day of November as the maturity date. Adding three days' grace, the legal due date is December 3, 2011. Therefore, the timeframe is March 31, 2011, to December 3, 2011. By counting the days in between (1 + 30 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 2) or using a calculator, you find that there are 247 days between these dates.

Step 2:

247 days out of 365 days in a year is $t=\frac{247}{365}$

Step 3:

$$S=\$7,500\left(1+0.06\left(\frac{247}{365}\right)\right)=\$7,804.52$$

Calculator Instructions

For the days between dates in step 1, use the DATE function:

DT1	DT2	DBD	Mode
03.3111	12.0311	Answer: 247	ACT

Corey owes Miller Enterprises $7,804.52 upon the legal due date for the promissory note.

Selling a Promissory Note Before It Matures

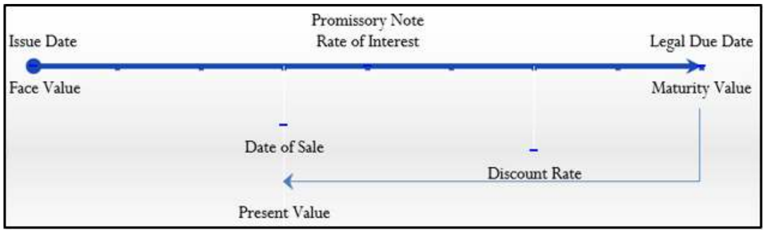
Two months ago you established a promissory note to lend your friend the $5,000 to help him start his entrepreneurial venture. Now you have unexpectedly lost your job and find yourself in need of the money you loaned out. However, the note does not require your friend to repay the debt for another four months. Is there any way you can access your money?

During the life of a promissory note, its holder (the lender) may be unable to wait until the end of the term to get paid back and so sell the note to another investor. It is generally up to both parties to negotiate a fair interest rate for the sale of the note. This interest rate might be based on prevailing interest rates in the market at the time of the sale or just a mutually agreed-upon rate. This negotiated interest rate is sometimes called a discount rate, since the process to sell the note requires the removal of interest from the maturity value of the promissory note.

How It Works

When selling a promissory note before its maturity date, always follow a three-step procedure:

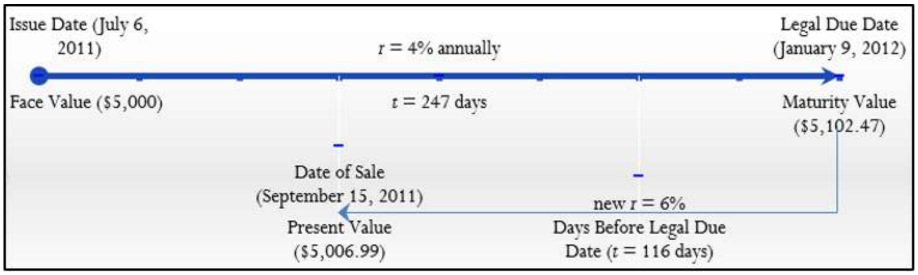
Step 1: Identify the promissory note information and selling information. For the promissory note, this includes the face value of the promissory note, the agreed-upon interest rate, and the legal due date. For selling the note, the discount rate and the time of the sale must be known. If necessary, draw a timeline to illustrate the transaction as illustrated in the figure below.


Issue Date
Promissory Note
Rate of Interest
Legal Due Date
Face Value
Maturity Value
Date of Sale
Discount Rate
Present Value

Step 2: Calculate the maturity value of the promissory note itself using the face value, promissory note rate of interest, and the time period involved with the legal due date.

Step 3: Calculate the present value of the note at the date of sale using the maturity value of the note, the new discount rate, and the time between the date of the sale and the legal due date.

Assume in the example that you negotiated an interest rate of 6% with the buyer of the promissory note, and that the date of sale was September 15, 2011. The timeline illustrates the two steps:


Issue Date (July 6, 2011)
r = 4% annually
Legal Due Date (January 9, 2012)
Face Value ($5,000)
t = 247 days
Maturity Value ($5,102.47)
Date of Sale (September 15, 2011)
new r = 6%
Present Value ($5,006.99)
Days Before Legal Due Date (t = 116 days)

Step 1: For the promissory note, the face value is $5,000, interest rate is 4% annually, and the time is from July 6, 2011, to January 9, 2012, previously calculated as 187 days. For selling the note, the discount rate is 6%. The selling date is September 15, 2011, which works out to 116 days before the legal due date.

Step 2: Applying Formula 8.2, calculate the maturity value of the promissory note on the legal due date. From previous calculations, the promissory note is worth $5,102.47 on January 9, 2012, when the borrower repays the note. This is the value that an investor purchasing the note receives in the future.

Step 3: Now sell the note using the discount rate and time before the legal due date. Applying Formula 8.2, rearrange and solve for $P = \$5,102.47/(1+(0.06\times)) = \$5,006.99$.

The $5,006.99 is how much the buyer of the promissory note pays based on the negotiated interest rate. This is sometimes called the proceeds, which is the amount of money received from a sale. The buyer now holds the promissory note and receives the full payment of $5,102.47 on January 9, 2012, from the borrower. This earns the buyer a 6% rate of interest on the investment.

Important Notes

If the original promissory note is noninterest-bearing, remember you can skip the second step above, since the face value of the note equals the maturity value upon the legal due date.

Things To Watch Out For

When working with the sale of a promissory note to another investor, the most common mistake is to mix up the interest rates. Remember that you use the original, stated interest rate of the promissory note in calculating the maturity value on the legal due date. When calculating the selling price, you must use the new negotiated rate to discount the maturity value.

✓ Example $8.4.2$: Deciding You Need the Money Sooner

On April 30, 2011, Bernie purchased some furniture from Fran’s Fine Furniture on a 10-month no payment plan. A promissory note was created establishing an interest rate of 11% and it included the clause “no grace.” The furniture purchase was for $2,234.40. On August 12, 2011, Fran’s Fine Furniture sold the promissory note to an investor at an agreed-upon interest rate of 13%. What were the proceeds of the sale?

Solution

Calculate the proceeds of the sale. This represents a present value calculation (P) based on the maturity value of the original promissory using the discount rate.

What You Already Know

Step 1:

The following information about the promissory note and sale are known:

Original Promissory Note: Issue Date = April 30, 2011, $P = \$2,234.40$, Term = 10 months, no grace, $r = 11\%$ per year

Sale of Promissory Note: Sale Date = August 12, 2011, Discount $r = 13\%$ per year

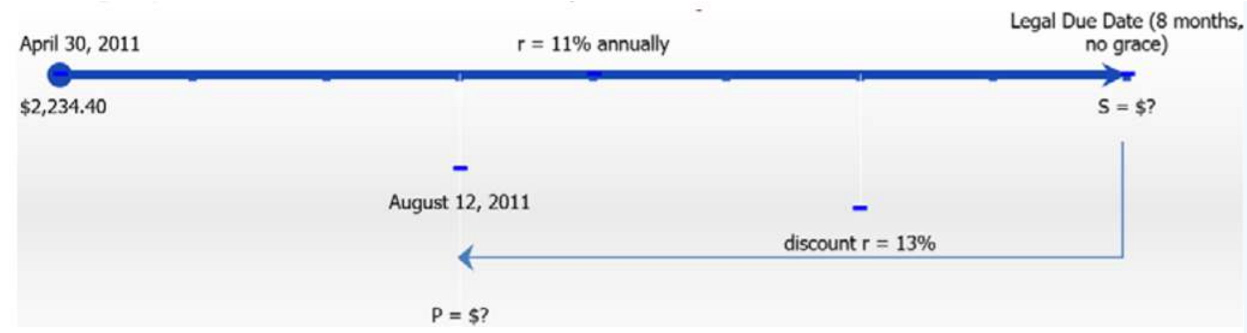
How You Will Get There

Step 1 (continued):

Calculate the legal due date, accounting for the no grace clause, and transform it into the number of days expressed annually to match the interest rate. Also calculate the number of days between the legal due date and the sale date, expressing it annually to match the interest rate.

Step 2:

To arrive at the maturity value, apply Formula 8.2.



Step 3:

To arrive at the present value using the new interest rate, apply Formula 8.2, rearranging for P.

Perform

Step 1 (continued):

Ten months from April 30, 2011, is February 30, 2012, which is adjusted to the last day of the month, or February 29, 2012. No grace period is added. By counting the days in between (1 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 31 + 31 + 29) or using a calculator, you find that there are 305 days between the dates. Thus $t = \frac{305}{365}$ expressed annually.

Step 1 (continued):

Using a selling date of August 12, 2011, and the legal due date of February 29, 2012, count the days in between (19 + 30 + 31 + 30 + 31 + 31 + 29) or use a calculator to determine there are 201 days between the dates. Thus, $t = \frac{201}{365}$ expressed annually.

Step 2:

$$S = \$2,234.40\left(1 + 0.11\left(\frac{305}{365}\right)\right) = \$2,439.78$$

Step 3:

$$P = \frac{\$2,439.78}{1 + 0.13\left(\frac{201}{355}\right)} = \$2,276.79$$

Calculator Instructions

Use the DATE function to calculate number of days between:

	DT1	DT2	DBD	Mode
Original Promissory Note	04.3011	02.2912	Answer: 305	ACT
Sale of Promissory Note	08.1211	√	Answer: 201	ACT

When Fran’s Fine Furniture sold the note to the investor, the proceeds of the sale were $2,276.

✓ Example 8.4.3: The Sale of a Noninterest Bearing Promissory Note

A noninterest-bearing promissory note was issued on January 31 with a face value of $10,000 and a term of eight months. On May 28 the note was sold to an investor at an agreed-upon interest rate of 5% per year. What was the sale price of the promissory note? Assume February is in a non–leap year.

Solution

Calculate the proceeds of the sale. This represents a present value calculation (P) based on the maturity value of the original promissory using the discount rate.

What You Already Know

Step 1:

The following about the promissory note and sale are known:

Original Promissory Note: Issue Date =January 31 Term = 8 months, $P = \$10,000$, $r = 0\%$

Sale of Promissory Note: Sale Date = May 28 Discount $r = 5\%$ per year

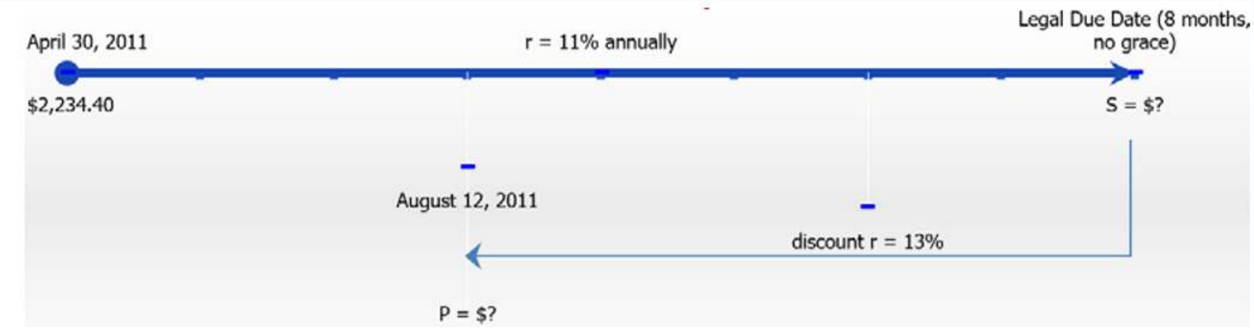
How You Will Get There

Step 1 (continued):

Calculate the legal due date by adding three days. Also calculate the number of days between the legal due date and the sale date, expressing it annually to match the interest rate.

Step 2:

The promissory note is noninterest-bearing, so the maturity value equals the face value.



Step 3: To arrive at the present value using the new interest rate, apply Formula 8.2, rearranging for P.

Perform

Step 1 (continued):

Add eight months to January 31, arriving at September 31. Adjust this to the last day of the month, September 30. Adding three days' grace gives you a legal due date of October 3.

Step 1 (continued):

Using a selling date of May 28 and the legal due date of October 3, count the days in between (4 + 30 + 31 + 31 + 30 + 2) or use a calculator to determine that there are 128 days between the dates. Thus, $t = \frac{128}{365}$ expressed annually.

Step 2:

$$P = S = \$10,000$$

Step 3:

$$P = \frac{\$10,000}{1 + 0.05\left(\frac{128}{365}\right)} = \$9,827.68$$

Calculator Instructions

Assume a non-leap year of 2011 and use the DATE function to calculate number of days

DT1	DT2	DBD	Mode
04.2811	10.0311	Answer: 128	ACT

> The promissory note was sold for $9,827.68 on May 28.

References

1. Bills of Exchange Act, R.S.C., 1985, c. B-4, Part IV Promissory Notes, §176. Available at http://laws-lois.justice.gc.ca/eng/a...B-4/index.html.

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- Template:ContribBusiMathOlivier

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8.5: Application - Loans

Almost everybody needs a little financial help from time to time. Perhaps you want to purchase a big-ticket item like a big-screen 3D HDTV without having the cash in full today to pay for it up front. However, the television is drastically on sale this week and if you wait to purchase it you will lose out and potentially have to pay full price next month. Or maybe as you flip through all of your bills you notice that though you will eventually have enough income to cover them, some of your payments fall due a few days before you will actually receive your income. What will you do for those few days until the income is deposited into your account?

Businesses also find themselves in similar situations. Maybe a supplier is offering a special deal on a product line, but the business does not have the cash to stock up. Also typical of business operations is that they make purchases in advance of sales, so they need to spend the money before they can receive the revenue to pay for their expenses. How can a business get access to short-term financing?

In this section, short-term financing in the form of demand loans is explored. This will include looking at loans where payments are variable as well as loans possessing fixed payments. The section ends by looking briefly at a loan you may already have—a student loan.

Demand Loans and Characteristics

A demand loan is a short-term loan that generally has no specific maturity date, can be paid at any time without any interest penalty, and where the lender can demand repayment in full at any time. It allows borrowing when needed and repayment when money permits, subject to the following six characteristics:

1. **Credit Limit**. This establishes the maximum amount that can be borrowed.
2. **Variable Interest Rate**. Almost all demand loans use variable simple interest rates based on the prime rate. Only the best, most secure customers can receive prime, while others usually get "prime plus" some additional amount.
3. **Fixed Interest Payment Date**. Interest is always payable on the same date each and every month. For simplicity, the payment is usually tied to a chequing or savings account, allowing the interest payment to occur automatically.
4. **Interest Calculation Procedure**. Interest is always calculated using a simple interest procedure based on the daily closing balance in the account. This means the first day but not the last day is counted.
5. **Security**. Loans can be secured or unsecured. Secured loans are those loans that are guaranteed by an asset such as a building or a vehicle. In the event that the loan defaults, the asset can be seized by the lender to pay the debt. Unsecured loans are those loans backed up by the general goodwill and nature of the borrower. Usually a good credit history or working relationship is needed for these types of loans. A secured loan typically enables access to a higher credit limit than an unsecured loan.
6. **Repayment Structure**. The repayment of the loan is either variable or fixed.
    a. A variable repayment structure allows the borrower to repay any amount at any time, although a minimum requirement may have to be met such as "at least 2% of the current balance each month." A current balance is the balance in an account plus any accrued interest. Accrued interest is any interest amount that has been calculated but not yet placed (charged or earned) into an account.
    b. A fixed repayment structure requires a fixed payment amount toward the current balance on the same date each and every month.

Types of Simple Interest Financing

While many types of financial tools use simple interest, these are the four most common:

1. **Personal Line of Credit (LOC)**. A demand loan for individuals is generally unsecured and is granted to those individuals who have high credit ratings and an established relationship with a financial institution. Since it is unsecured, the credit limit is usually a small amount, such as $10,000. Repayment is variable and usually has a minimum monthly requirement based on the current balance.
2. **Home Equity Line of Credit (HELOC)**. This is a special type of line of credit for individuals that is secured by residential homeownership. Typically, an amount not exceeding 80% of the equity in a home is used to establish the credit limit, thus enabling an individual access to a large amount of money. The interest rates tend to follow mortgage interest rates and are lower than personal lines of credit. Repayment is variable, usually involving only the accrued interest every month.
3. **Operating Loans**. An operating loan is the business version of a line of credit. It may or may not be secured, depending on the nature of the business and the strength of the relationship the business has with the financial institution. Repayment can be

either variable or fixed.

4. **Student Loans**. A loan available to students to pursue educational opportunities. Although these are long-term in nature, the calculation of interest on a student loan uses simple interest techniques. These loans are not true demand loans since a student loan cannot be called in at any time. Repayment is fixed monthly.

Repayment Schedules

When you work with short-term loans, regardless of the repayment structure, you should always set up a repayment schedule. A repayment schedule is a table that details the financial transactions in an account including the balance, interest amounts, and payments. The table below presents the table structure used for setting up a repayment schedule.

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I)	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Start Date								(1)
(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)

Each number in the table corresponds to an entry below that explains how to use each column or row.

1. The first row appears if the schedule has an opening balance. In these instances, list the start date in the first column and the opening balance in the last column.
2. List the date of any transaction. This could include a payment, advance, interest rate change, or an accrued interest payment.
3. Carry this number forward from (1).
4. Record the interest rate that applies to the date interval in the first column.
5. Calculate the number of days between the date on the previous row and the current row. Remember to count the first day but not the last day. Express the number annually to match the interest rate.
6. Compute the interest charges for the date interval using Formula 8.1, $I = Prt$. Use (3), (4), and (5) as your values for the formula.
7. Enter the cumulative total of any unpaid or accrued interest as of the current row's date. This amount is the sum of (6) from the current row plus any number recorded in this column from the row above.
8. Enter the amount of the transaction occurring for this row of the table. Payments should be recorded as positives (credits) as they will decrease the balance, and advances should be recorded as negatives (debits) as they will increase the balance. If this is an interest payment date, one of the following two events will happen:
   a. Only the interest payment occurs on this date. Copy the accrued interest from (7) into this column.
   b. An additional payment or advance is made on this date. Add the accrued interest to the advance or payment, placing the sum into this column.

   Regardless of which event happens, cross out the accrued interest amount in (7) as it is considered paid, so the accrued interest balance is reduced to zero once again.
9. One of two events will happen in this column:
   a. On an interest payment date, subtract the accrued interest from the amount in (8).
   b. On a date other than an interest payment date, any payment or advance will have its total amount applied to the principal. Therefore, carry the amount in (8) across to this column.
10. Subtract the amount in (9) from the amount in this column on the previous row to create the new balance in the account. Copy this amount to the next row in (3).

How It Works

Although the calculations of a repayment schedule are relatively straightforward, the complexity of the repayment schedule sometimes causes grief. When a repayment schedule is required, follow these steps:

Step 1: Set up the repayment schedule as per the example table.

Step 2: Record the start date and the opening balance for the loan.

Step 3: In chronological order, make new row entries in the schedule by filling in the details provided in the question. You require a new row in the table whenever one of the following three events occurs:

a. A payment or advance is made.
b. An interest payment date occurs.
c. The interest rate changes.

Step 4: Starting with the first row, work left to right across the table, filling in all information. Pay particular attention to the nuances of the "Payment or Advance" and "Principal Amount" columns as discussed previously. Once a row is complete, move to the next row until you fill in the entire table.

Step 5: Calculate any totals requested such as total interest or total principal paid.

Important Notes

For simplicity in writing the numbers into repayment schedules and performing calculations, it is this textbook's practice to round all interest calculations to two decimals throughout the table. In real-world applications, you must keep track of all decimals in the account at all times.

Things To Watch Out For

Because of the size of the repayment schedule and the large amount of information involved in the calculations, the number-one error is what most people call a “silly” error. It means that a wrong date is recorded, a wrong amount is written down, a payment is recorded as an advance or vice versa, or simply the wrong button is pressed on a calculator. Take the time to ensure you read and record the correct numbers and that you pause when performing calculations. For example, advances mean the balance should get bigger, while payments mean the balance should get smaller. Just by thinking for a second about the basic principles of debt you should be able to catch those silly errors.

✓ Example 8.5.1: Operating Loan, Fixed Repayment

On July 15, when the prime rate was set at 4%, Canadian Footwear took out an operating loan from CIBC for $8,000 at prime plus 1.25%. The terms of the loan require a fixed payment of $1,500 on the 15th of every month until the loan is repaid. The prime rate climbed by 0.5% on September 29. Create a repayment schedule for the loan and calculate the total interest paid.

Solution

Create the repayment schedule for the loan and sum the interest charges for the entire operating loan.

What You Already Know

For a fixed repayment loan, capture three important categories of information:

Opening Balance	Payments	Dates of Interest Rate Changes†
July 15: $8,000	August 15: $1,500 September 15: $1,500 October 15: $1,500 November 15: $1,500 December 15: $1,500 January 15: $????? *	July 15: 5.25% September 29: 5.75%

*Note: You can presume there is a payment on January 15 since the first five payments total up to $7,500, which falls short of repaying the loan. The last payment, on January 15, is still unknown as it will be based on the remaining balance in the account plus any accrued interest.

† Calculated as prime plus 1.25%.

How You Will Get There

Step 1:

Set up a repayment table.

Step 2:

Fill in the start date and opening balance.

Step 3:

Chronologically fill in all information, with one transaction per row of the table.

Step 4:

Work left to right and top to bottom throughout the table.

Step 5:

Once you have completed this step, sum the interest charges paid on the 15th of every month.

Perform

Steps 1-4:

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged ($I = Prt$)	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
July 15								\$8,000.00
Aug 15	\$8,000.00	5.25%	31/365 (1)	\$35.67 (2)	~~\$35.67~~	\$1,500.00	\$1,464.33 (3)	\$6,535.67 (4)
Sept 15	\$6,535.67	5.25%	31/365	\$29.14	~~\$29.14~~	\$1,500.00	\$1,470.86	\$5,064.81
Sept 29	\$5,064.81	5.25%	14/365	\$10.20	\$10.20	\$0.00	\$0.00	\$5,064.81
Oct 15	\$5,064.81	5.75%	16/365	\$12.77	~~\$22.97~~ (5)	\$1,500.00	\$1,477.03	\$3,587.88
Nov 15	\$3,587.88	5.75%	31/365	\$17.52	~~\$17.52~~	\$1,500.00	\$1,482.48	\$2,105.30
Dec 15	\$2,105.30	5.75%	30/365	\$9.95	~~\$9.95~~	\$1,500.00	\$1,490.05	\$615.25
Jan 15	\$615.25	5.75%	31/365	\$3.00	~~\$3.00~~	\$618.25 (6)	\$615.25 (7)	\$0.00

Select calculations and comments from the above table:

(1) This is the number of days from the date on the previous row to the current row, or July 15 to August 15. Count the first day, but not the last.

(2) $I = Prt = \$8,000(0.0525)(31/365) = \$35.67$

(3) \$1,500.00 − \$35.67 = \$1,464.33. The accrued interest of \$35.67 is now paid and can be crossed out.

(4) \$8,000.00 − \$1,464.33 = \$6,535.67

(5) \$10.20 + \$12.77 = \$22.97

(6) The last payment must clear the balance owing and the accrued interest: \$615.25 + \$3.00 = \$618.25

(7) \$618.25 − \$3.00 = \$615.25

Step 5:

Total interest charges = \$35.67 + \$29.14 + \$22.97 + \$17.52 + \$9.95 + \$3.00 = \$118.25

To clear the operating loan it takes six payments: five of \$1,500 each and a final payment of \$618.25. The loan incurs total interest paid of \$118.25.

✓ Example 8.5.2: HELOC, Variable Repayment, No Minimum Requirement

Lynne has access to a HELOC that requires only the payment of accrued interest on the first of every month. On March 1, the opening balance on her HELOC was $15,000. She took advances of $6,000 and $10,000 on March 21 and May 4, respectively. She made additional payments of $11,000 and $15,000 on April 15 and June 17. The interest rate on her HELOC sits at prime plus 2%. On March 1, the prime rate was 3%. On April 26, it rose by 0.5%. Determine the total interest paid on her HELOC from March 1 to July 1.

Solution

This is a HELOC using simple interest. Calculate the total interest that she had to pay in March, April, May, and June and then sum the total interest paid.

What You Already Know

Capture five important categories of information:

Opening Balance	Advances	Payments
March 1: $15,000	March 21: $6,000 May 4: $10,000	April 15: $11,000 June 17: $15,000

Interest Payment	Dates Dates of Interest Rate Changes*
April 1 May 1 June 1 July 1	March 1: 5% April 26: 5.5%

*Calculated as prime plus 2%.

How You Will Get There

Step 1:

Set up a repayment table.

Step 2:

Fill in the start date and opening balance.

Step 3:

Chronologically fill in all information, with one transaction per row of the table.

Step 4:

Work left to right and top to bottom throughout the table.

Step 5:

Once you have completed this step, sum the accrued interest payments on April 1, May 1, June 1, and July 1.

Perform

Steps 1- 4

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged ($I = Prt$)	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Mar 1		5%						$15,000.00

Mar 21	$15,000 (1)	5%	20/365 (2)	$41.10 (3)	$41.10 (4)	-$6000	-$6000 (5)	$21,000 (6)
Apr 1	$21,000	5%	11/365	$31.64	~~$72.74~~ (7)	$72.74 (8)	$0.00 (9)	$21,000
Apr 15	$21,000	5%	14/365	$40.27	$40.27	$11,000	$11,000	$10,000
Apr 26	$10,000	5%	11/365	$15.07	$55.34	$0.00	$0.00	$10,000
May 1	$10,000	5.5%	5/365	$7.53	~~$62.87~~	$62.87	$0.00	$10,000
May 4	$10,000	5.5%	3/365	$4.52	$4.52	-$10,000	-$10,000	$20,000
June 1	$20,000	5.5%	28/365	$84.38	~~$88.90~~	$88.90	$0.00	$20,000
June 17	$20,000	5.5%	16/365	$48.22	$48.22	$15,000	$15,000	$5,000
July 1	$5,000	5.5%	14/365	$10.55	~~$58.77~~	$58.77	$0.00	$5,000

Select calculations and comments from the above table:

(1) This is the balance from the last column in the row above carried forward = $15,000.

(2) The date interval from the previous row to this row is March 1 to March 21: $t = 21 - 1 = 20$ days.

(3) $I = Prt = \$15,000(0.05)(20/365) = \$41.10$

(4) The accrued interest from the row above plus (3). $0 + $41.10 = $41.10

(5) Not an interest payment date, so the full payment is applied to principal = −$6,000

(6) Previous balance in the column above minus the principal portion from (5). $15,000 − (−$6,000) = $21,000

(7) The accrued interest from the row above plus interest from this row. $41.10 + $31.64 = $72.74

(8) This is an interest payment date. Carry (7) across and cross it out, reducing the accrued interest balance to zero.

(9) This is an interest payment date. Take (8) and subtract (7). $72.74 − $72.74 = $0.00

Step 5:

April 1 + May 1 + June 1 + July 1 = $72.74 + $62.87 + $88.90 + $58.77 = $283.28.

The total interest paid on the HELOC from March 1 to July 1 is $283.28.

riable Repayment, with Minimum Requirement

Everlyne has a personal LOC with her bank with a maximum credit limit of $10,000. The interest rate is prime plus 3.5%, and the current prime rate is 4.5%. Regardless of any account transaction activity, her bank requires on the first of every month for her to pay "the greater of 5% of the current balance or $100" from her chequing account. She is allowed to exceed her maximum credit limit, but if she does the entire balance is subject to 21% interest until such time as the balance is restored below the credit limit. On October 1, the opening balance on her LOC was $2,000. She took advances of $5,000, $6,000, and $1,000 on October 21, November 13, and December 1, respectively. She made payments of $1,000, $3,000, and $8,500 on November 1, November 20, and December 15, respectively. The prime rate decreased by 1/4% on November 5. Calculate her total required payments (not her voluntary payments) and the portion of those payments that went toward interest for the months of October, November, and December.

Solution

Calculate a personal LOC using simple interest to determine the total required payments on November 1, December 1, and January 1 and the amount of interest included in those payments.

What You Already Know

Capture five important categories of information:

Opening Balance	Advances	Payments
Oct 1: $2,000	October 21: $5,000 Nov. 13: $6,000 December 1: $1,000	November 1: $1,000 November 20: $3,000 December 15: $8,500

Interest Payment	Dates Dates of Interest Rate Changes*
November 1 December 1 January 1	October 1: 8% November 5: 7.75% Interest rate jumps to 21% if credit limit is exceeded

*Calculated as prime plus 4.5%

Perform

Steps 1- 4

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged ($I = Prt$)	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Oct 1								$2,000
Oct 21	$2,000	8%	20/365	$8.77	$8.77	-$5,000	-$5,000	$7,000
Nov 1	$7,000	8%	11/365	$16.88	~~$25.65~~	$1,000 + $351.28 = $1,351.28 (1)	$1,325.63 (2)	$5,674.37
Nov 5	$5,674.37	8%	4/365	$4.97	$4.97	$0.00	$0.00	$5,674.37
Nov 13	$5,674.37	7.75%	8/365	$9.64	$14.61	-$6,000	−$6,000	$11,674.37
Nov 20	$11,674.37	21% (3)	7/365	$47.02	$61.63	$3,000	$3,000	$8,674.37
Dec 1	$8,674.37	7.75%	11/365	$20.26	~~$81.89~~	−$1,000 + $437.81 = −$562.19 (4)	−$644.08 (5)	$9,318.45
Dec 15	$9,318.45	7.75%	14/365	$27.70	$27.70	$8,500	$8,500	$818.45
Jan 1	$818.45	7.75%	17/365	$2.95	~~$30.65~~	$100 (6)	$69.35 (7)	$749.10

Select calculations and comments from the table:

(1) The required payment is the greater of 5% of the current balance or $100 regardless of her monthly account activity. The current balance is $7,000 + $25.65 = $7,025.65. Five percent of that number is $351.28. The $1,000 extra payment plus the required payment of $351.28 totals $1,351.28.

(2) The $1,351.28 − $25.65 = $1,325.63 toward the principal. The accrued interest is now paid and is crossed out.

(3) Note that by exceeding the credit limit of \$10,000, she changes the interest rate to the penalty rate.

(4) The current balance is \$8,674.37 + \$81.89 = \$8,756.26, of which 5% is \$437.81. Therefore, with an advance of \$1,000 and a payment of \$437.81 on the same date, the net daily result is an advance of \$562.19.

(5) −\$562.19−\$81.89=−\$644.08, which is added to the principal. The accrued interest is paid and is crossed out.

(6) The current balance is \$818.45 + \$30.65 = \$849.10, of which 5% is \$42.46, so pay \$100 - the greater amount.

(7) \$100 − \$30.65 = \$69.35. The accrued interest is now paid and is crossed out.

Step 5:

Total Required Payments = Nov 1 + Dec 1+ Jan 1 = \$351.28 + \$437.81 + \$100.00 = \$889.09

Total Interest Charges = Nov 1 + Dec 1 + Jan 1 = \$25.65 + \$81.89 + \$30.65 = \$138.19

Throughout the months of October, November, and December, the total required payments on the LOC came to \$889.09. Of that amount, \$138.19 went toward the interest charges.

Student Loans

A student loan is a special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education. The Canada Student Loans Program (CSLP) is administered by the federal government and run by the National Student Loan Service Centre (NSLSC) under contract to Human Resources and Skills Development Canada (HRSDC).

If and when you take out a student loan, the following characteristics are in place for full-time students (defined as having a 60% workload or more):

a.

2. Fixed interest rate of prime + 5% (where prime is determined on the day that your grace period ends).

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8.6: Application - Treasury Bills and Commercial Papers

Where does the government go to borrow money? The evening news you are watching announces that Canada’s national debt has ballooned from $458 billion in 2008 to over $654 billion in August 2012. This means the government has borrowed this money from somewhere. You begin to wonder where ... it can’t exactly walk into the country’s largest bank (RBC) and ask for a $654 billion loan! Although RBC is Canada's largest banking institution and in 2012 it ranked 70th on the *Forbes* Global 2000 list of the world's biggest public companies, it still does not have enough money to lend to the government. If a government cannot get a loan from a financial institution, then how do federal and provincial governments borrow money in such large amounts?

Treasury Bills: The Basics

The answer lies in **treasury bills**, better known as T-bills, which are short-term financial instruments that both federal and provincial governments issue with maturities no longer than one year. Approximately 27% of the national debt is borrowed through T-bills.

Who purchases these T-bills? It is a two-tiered process. First, major investment companies and banks purchase the T-bills in very large denominations (such as $1 million or $10 million). Then these organizations break the T-bills down into bite-size chunks (such as $1,000 denominations) and sell them to their customers and other investors like you and me, at a profit of course. In essence, it is you and I who finance the debt of Canada!

Here are some of the basics about T-bills:

1. The Government of Canada regularly places T-bills up for auction every second Tuesday. Provincial governments issue them at irregular intervals.
2. The most common terms for federal and provincial T-bills are 30 days, 60 days, 90 days, 182 days, and 364 days.
3. T-bills do not earn interest. Instead, they are sold at a discount and redeemed at full value. This follows the principle of "buy low, sell high." The percentage by which the value of the T-bill grows from sale to redemption is called the **yield** or rate of return. From a mathematical perspective, the yield is calculated in the exact same way as an interest rate is calculated, and therefore the yield is mathematically substituted as the discount rate in all simple interest formulas. Up-to-date yields on T-bills can be found at www.bankofcanada.ca/en/rates/monmrt.html.
4. The **face value of a T-bill** (also called *par value*) is the maturity value, payable at the end of the term. It includes both the principal and yield together.
5. T-bills do not have to be retained by the initial investor throughout their entire term. At any point during a T-bill’s term, an investor is able to sell it to another investor through secondary financial markets. Prevailing yields on T-bills at the time of sale are used to calculate the price.

Commercial Papers - The Basics

A **commercial paper** (or paper for short) is the same as a T-bill except that it is issued by a large corporation instead of a government. It is an alternative to short-term bank borrowing for large corporations. Most of these large companies have solid credit ratings, meaning that investors bear very little risk that the face value will not be repaid upon maturity.

For a corporation, papers tend to be a cheaper source of financing than borrowing from a bank. The yields paid to investors are less than the interest rate the corporation is charged by the bank. The corporation can also make its financing even cheaper by offering the commercial paper directly to investors to avoid intermediary brokerage fees and charges. Finally, commercial papers are not subject to strict financial requirements and registration, such as stocks and bonds require; therefore, they pose easy access to funds in the short-term for a corporation.

Commercial papers carry the same properties as T-bills. The only fundamental differences lie in the term and the yield:

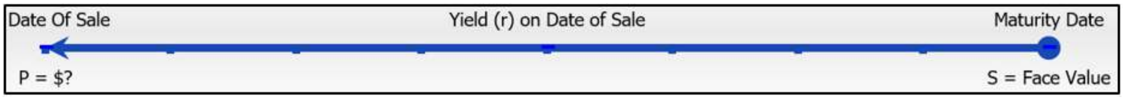
1. The terms are usually less than 270 days but can range from 30 days to 364 days. The most typical terms are 30 days, 60 days, and 90 days.
2. The yield on commercial papers tends to be slightly higher than on T-bills since corporations do carry a higher risk of default than governments.

How It Works

Mathematically, T-bills and commercial papers operate in the exact same way. The future value for both of these investment instruments is always known since it is the face value. Commonly, the two calculated variables are either the present value (price) or the yield (interest rate). The yield is explored later in this section. Follow these steps to calculate the price:

Step 1: The face value, yield, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify the following:

1. The face value (S).
2. The yield (r) on the date of the sale, which is always expressed annually. Remember that mathematically the yield is the same as the discount rate.
3. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually to match the annual yield.



Step 2: Apply Formula 8.2, rearranging and solving for the present value, which is the price of the T-bill or commercial paper. This price is always less than the face value.

Paths To Success

When you calculate the price of a T-bill or commercial paper, the only important pieces of information are the face value, the yield or discount rate *on the date of sale*, and how many days remain until maturity. Any prior history of the T-bill or commercial paper does not matter in any way. What the market rate was yesterday or last week or upon the date of issue does not impact today's price. Likewise, the number of days since the date of issue does not matter. What an investor paid for it three weeks ago is irrelevant. When calculating the price, pay attention only to the current market information and the future. Ignore the history!

? Exercise 8.6.1: Give It Some Thought

a. All other variables remaining stable, what happens to the price (present value) of a T-bill or commercial paper if the discount rate increases?
b. An investor purchases a T-bill when the yield is 3%. A few months later, the investor sells the T-bill when the yield has dropped to 2%. Will the investor realize a yield of 3%, more than 3%, or less than 3% on the investment?

Answer

a. The price is lower because more yield must be realized on the investment.
b. More than 3%. Since the T-bill yield is lower when sold, the price of the T-bill increases, resulting in the investor receiving more money on the investment. This raises the investor's return on investment

✓ Example 8.6.1: Price of a Treasury Bill

A Government of Canada 182-day issue T-bill has a face value of $100,000. Market yields on these T-bills are 1.5%. Calculate the price of the T-bill on its issue date.

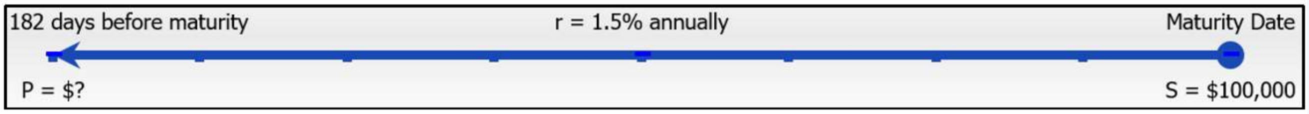
Solution

Calculate the price that an investor will pay for the T-bill, or P.

What You Already Know

Step 1:

The T-bill maturity value, term, and yield are known: $S=\$100,000, r=1.5\%, t=182/365$



How You Will Get There

Step 2:

Apply Formula 8.2, rearranging for P.

Perform

Step 2:

$$\begin{aligned}\$100,000 &= P\left(1+0.015\times\frac{182}{365}\right)\\ P &= \frac{\$100,000}{1+(0.015)\left(\frac{182}{365}\right)}\\ &= \$99,257.61\end{aligned}$$

An investor will pay $99,257.61 for the T-bill. If the investor holds onto the T-bill until maturity, the investor realizes a yield of 1.5% and receives $100,000.

✓ Example $8.6.2$: Selling a Commercial Paper during Its Term

Pfizer Inc. issued a 90-day, $250,000 commercial paper on April 18 when the market rate of return was 3.1%. The paper was sold 49 days later when the market rate of return was 3.63%. Calculate the price of the commercial paper on its date of sale.

Solution

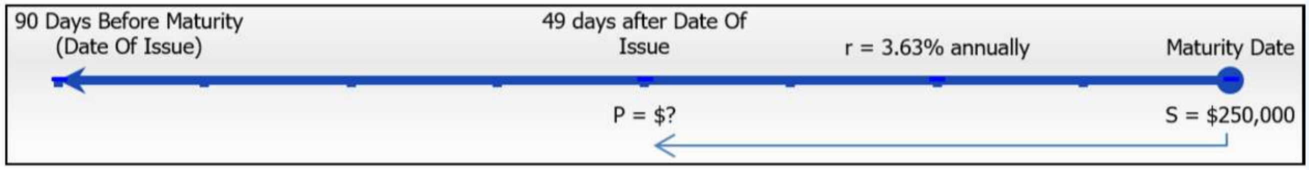
Calculate the price that an investor pays for the commercial paper on its date of sale, or P.

What You Already Know

Step 1:

Note that the historical rate of return of 3.1% is irrelevant to the price of the commercial paper today. The number of days elapsed since the date of issue is also unimportant. The number of days before maturity is the key piece of information.

The commercial paper maturity value, term, and yield are: $S=\$250,000, r=3.63\%, t=90-49=41$ days or 41/365



How You Will Get There

Step 2:

Apply Formula 8.2, rearranging for P.

Perform

Step 2:

$$\begin{aligned}\$250,000 &= P\left(1+0.0363\times\frac{41}{365}\right)\\ P &= \frac{\$250,000}{1+(0.0363)\left(\frac{41}{365}\right)}\\ &= \$248,984.76\end{aligned}$$

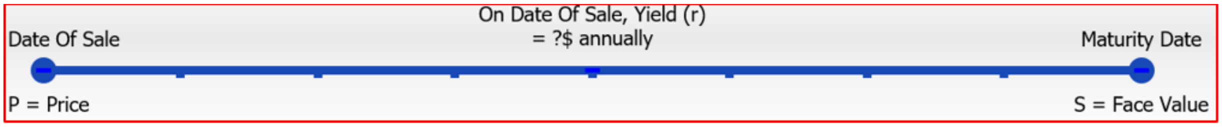
An investor pays $248,984.76 for the commercial paper on the date of sale. If the investor holds onto the commercial paper for 41 more days (until maturity), the investor realizes a yield of 3.63% and receives $250,000.

How It Works

Calculating a Rate of Return: Sometimes the unknown value when working with T-bills and commercial papers is the yield, or rate of return. In these cases, follow these steps to solve the problem:

Step 1: The face value, price, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify:

1. The face value (S)
2. The price on the date of the sale (P)


Date Of Sale
On Date Of Sale, Yield (r)
= ?$ annually
Maturity Date
P = Price
S = Face Value

3. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually so that the calculated yield will be annual.

Step 2: Apply Formula 8.3, $I = S - P$, to calculate the interest earned during the investment.

Step 3: Apply Formula 8.1, $I = Prt$, rearranging for \(r\) to solve for the interest rate (or yield or rate of return).

Paths To Success

When you solve for yield, you will almost always find that the purchase price has been rounded to two decimals. This means the calculation of the yield is based on an imprecise number. The results are likely to show small decimals such as 3.4005%. In this example, the most likely yield is 3.40% with the 0.0005% appearing because of rounding. Yields on T-bills and commercial papers typically do not have more than two decimal places, which means that decimals in the third position or more are most likely caused by a rounding error and should be rounded off to two decimals accordingly.

✓ Example 8.6.3: Figuring Out Rates of Return for Multiple Investors

Marlie paid $489,027.04 on the date of issue for a $500,000 face value T-bill with a 364-day term. Marlie received $496,302.21 when he sold it to Josephine 217 days after the date of issue. Josephine held the T-bill until maturity. Determine the following:

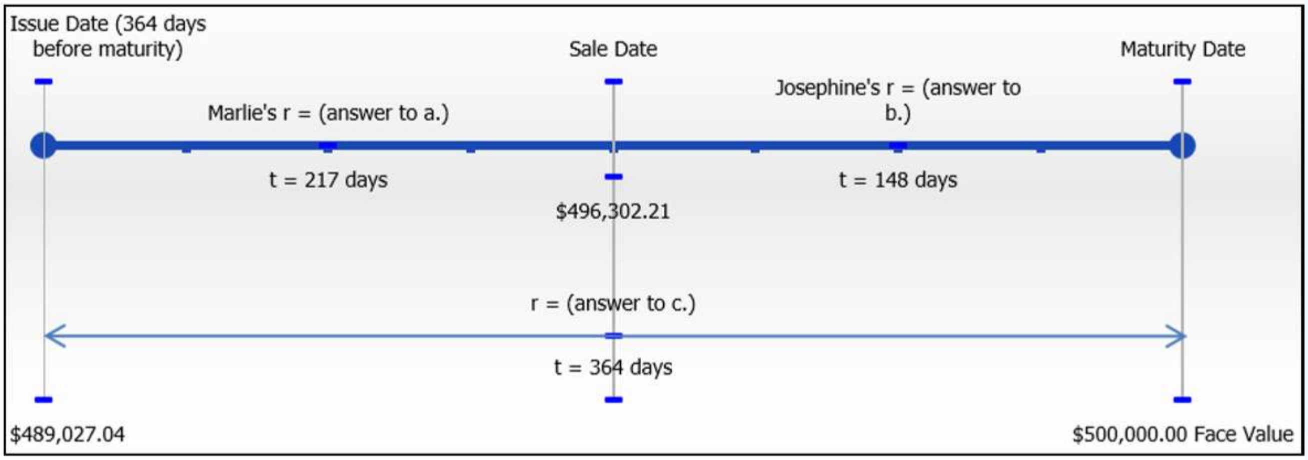
a. Marlie's actual rate of return
b. Josephine's actual rate of return
c. If Marlie held onto the T-bill for the entire 364 days instead of selling it to Josephine, what would his rate of return have been?
d. Comment on the answers to (a) and (c).

Solution

Calculate three yields or rates of return (r) involving Marlie and the sale to Josephine, Josephine herself, and Marlie without the sale to Josephine. Afterwards, comment on the rate of return for Marlie with and without the sale.

What You Already Know

Step 1:


Issue Date (364 days
before maturity)
Sale Date
Maturity Date
Marlie's r = (answer to a.)
Josephine's r = (answer to
b.)
t = 217 days
t = 148 days
$496,302.21
r = (answer to c.)
t = 364 days
$489,027.04
$500,000.00 Face Value

The present values, maturity value, and terms are known:

a. Marlie with sale: $P = \$489,027.04, S = \$496,302.21, t = 217/365$

b. Josephine: $P = \$496,302.21, S = \$500,000.00, 364 - 217 = 147$ days remaining $t = 147/365$
c. Marlie without sale: $P = \$489,027.04, S = \$500,000.00, t = 364/365$

How You Will Get There

Step 2:

For each situation, calculate the I by applying Formula 8.3.

Step 3:

For each situation, apply Formula 8.1, rearranging for r.

Step 4:

Compare the answers for (a) and (c) and comment.

Perform

Step 2: Calculate I

a. Marlie with sale to Josephine: $I = \$496,302.21 - \$489,027.04 = \$7,275.17$
b. Josephine by herself: $I = \$500,000.00 - \$496,302.21 = \$3,697.79$
c. Marlie without sale to Josephine: $I = \$500,000.00 - \$489,027.04 = \$10,972.96$

Step 3: Calculate r

a. Marlie with sale to Josephine: $r = \dfrac{\$7,275.17}{(\$489,027.04)\left(\dfrac{217}{365}\right)} = 2.50\%$
b. Josephine by herself: $r = \dfrac{\$3,697.79}{(\$496,302.21)\left(\dfrac{147}{365}\right)} = 1.85\%$
c. Marlie without sale to Josephine: $r = \dfrac{\$10,972.96}{(\$489,027.04)\left(\dfrac{364}{365}\right)} = 2.25\%$

When Marlie sold the T-bill after holding it for 217 days, he realized a 2.50% rate of return. Josephine then held the T-bill for another 148 days to maturity, realizing a 1.85% rate of return. If Marlie hadn't sold the note to Josephine and instead held it for the entire 364 days, he would have realized a 2.25% rate of return.

Step 4:

The yield on the date of issue was 2.25%. Marlie realized a higher rate of return because the interest rates in the market decreased during the 217 days he held it (to 1.85%, which is what Josephine is able to obtain by holding it until maturity). This raises the selling price of the T-bill. If his investment of \$489,027.04 grows by 2.25% for 217 days, he has \$6,541.57 in interest. The additional \$733.60 of interest (totaling \$7,275.17) is due to the lower yield in the market, increasing his rate of return to 2.50% instead of 2.25%.

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8.S: Summary

Key Concepts Summary

8.1: Principal, Rate, Time (How Does Interest Work?)

- Calculating the amount of simple interest either earned or charged in a simple interest environment
- Calculating the time period when specific dates or numbers of days are involved
- Calculating the simple interest amount when the interest rate is variable throughout the transaction

8.2: Moving Money Involving Simple Interest (Move and Nobody Gets Hurt)

- Putting the principal and interest together into a single calculation known as maturity value
- Altering a financial agreement and establishing equivalent payments

8.3: Application: Savings Accounts and Short-Term GICs (Safe and Secure)

- How to calculate simple interest for flat-rate and tiered savings accounts
- How to calculate simple interest on a short-term GIC

8.4: Application: Promissory Notes (A Promise Is a Promise)

- The characteristics of a promissory note
- Calculating the maturity value of a promissory note
- Selling a promissory note before its maturity date

8.5: Application: Loans (The Bank Comes Knocking)

- Demand loans, their characteristics, and the common forms they take
- Establishing a repayment schedule for demand loans
- The characteristics of a student loan and how it is repaid

8.6 Application: Treasury Bills and Commercial Papers (When Governments and Businesses Borrow)

- The characteristics of treasury bills
- The characteristics of commercial papers
- Calculating the price of T-Bills and commercial papers
- Calculating the yield of T-Bills and commercial papers

The Language of Business Mathematics

accrued interest

Any interest amount that has been calculated but not yet placed (charged or earned) into an account.

commercial paper

A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

compound interest

A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

current balance

The balance in an account plus any accrued interest.

demand loan

A short-term loan that generally has no specific maturity date, may be paid at any time without any interest penalty, and where the lender may demand repayment at any time.

discount rate

An interest rate used to remove interest from a future value.

equivalent payments

Two payments that have the same value on the same day factoring in a fair interest rate.

face value of a T-bill

The maturity value of a T-bill, which is payable at the end of the term. It includes both the principal and interest together.

fixed interest rate

An interest rate that is unchanged for the duration of the transaction.

future value

The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the maturity value.

guaranteed investment certificate (GIC)

An investment that offers a guaranteed rate of interest over a fixed period of time.

interest amount

The dollar amount of interest that is paid or earned.

interest rate

The rate of interest that is charged or earned during a specified time period.

legal due date of a note

Three days after the term specified in an interest-bearing promissory note is the date when a promissory note becomes legally due. This grace period allows the borrower to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend.

maturity date

The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.

maturity value

The amount of money at the end of a transaction, which includes both the interest and the principal together.

present value

The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.

prime rate

An interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans.

principal

The original amount of money that is borrowed or invested in a financial transaction.

proceeds

The amount of money received from a sale.

promissory note

An unconditional promise in writing made by one person to another person to pay a sum of money on demand or at a fixed or determinable future time.

repayment schedule

A table that details the financial transactions in an account, including the balance, interest amounts, and payments.

savings account

A deposit account that bears interest and has no stated maturity date.

secured loan

Those loans that are guaranteed by an asset such as a building or a vehicle that can be seized to pay the debt in case of default.

simple interest

A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.

student loan

A special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education.

time period

The length of the financial transaction for which interest is charged or earned. It may also be called the term.

treasury bills

Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.

unsecured loan

Those loans backed up by the general goodwill and nature of the borrower.

variable interest rate

An interest rate that is open to fluctuations over the duration of a transaction.

yield

The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.

The Formulas You Need to Know

Symbols Used

S = maturity value or future value in dollars

I = interest amount in dollars

P = principal or present value in dollars

r = interest rate (in decimal format)

t = time or term

Formulas Introduced

Formula 8.1: Simple Interest: $I = Prt$

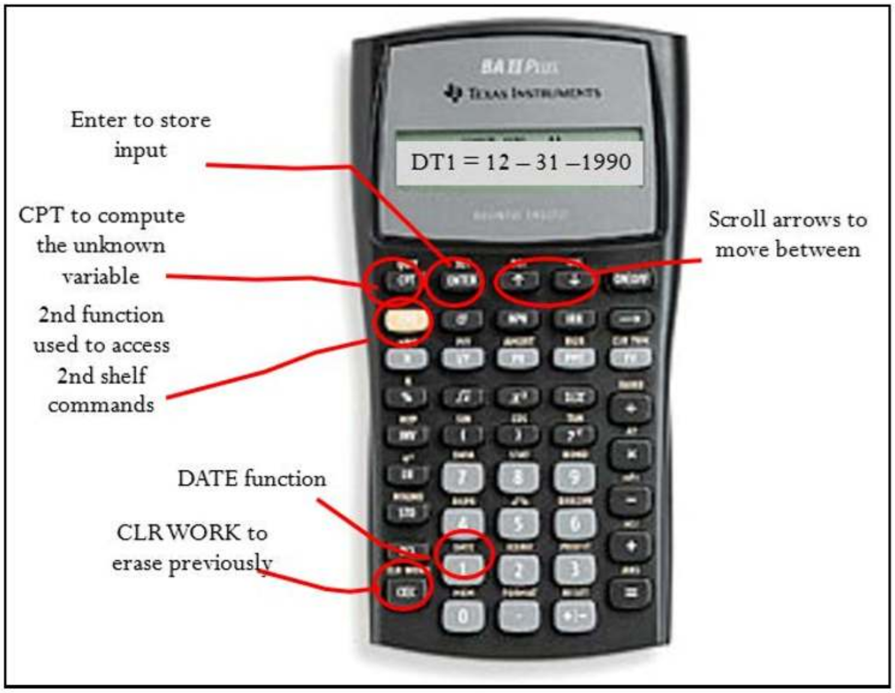
Formula 8.2: Simple Interest for Single Payments: $S = P(1 + rt)$

Formula 8.3: Interest Amount for Single Payments: $I = S - P$

Technology

Calculator

The following calculator functions were introduced in this chapter:


BA II Plus
Texas Instruments
DT1 = 12 – 31 –1990
Enter to store input
CPT to compute the unknown variable
2nd function used to access 2nd shelf commands
DATE function
CLR WORK to erase previously
Scroll arrows to move between

Date Function

- 2nd Profit to access.
- Enter two of the three variables (DT1, DT2, DBD) by pressing Enter after each input and using and ¯ to scroll through the display. The variables are:
    - DT1 = The starting date of the transaction
    - DT2 = The ending date of the transaction
    - DBD = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
    - ACT / 360 = A setting for determining how the calculator determines the DBD. In Canada, you should maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press 2nd SET.
- Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press CPT on the unknown (when it is on the screen display) to compute the answer.

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8.CS: Case Study - Managing A Company's Investments

The Situation

As with most mid-size to large companies, Lightning Wholesale has a finance department to manage its money. The structure of Lightning Wholesale's financial plan allows each department to have its own bank account from which all expenses, purchases, and charges are deducted. This same account has all revenues and interest deposited into it.

The manager of the sporting goods department wants a summary of all interest amounts earned or charged to her department for the year 2013. This will allow her to better understand and assess the financial policies of the company and make any necessary changes for 2014.

The Data

Month	Total Sales Revenue	Cost of Goods Sold	Operating Expenses
January	$1,798	$2,102	$156
February	$2,407	$2,242	$100
March	$2,568	$2,606	$222
April	$2,985	$2,719	$258
May	$3,114	$2,831	$269
June	$3,242	$2,887	$280
July	$3,306	$3,363	$286
August	$3,852	$4,203	$333
September	$4,815	$6,306	$525
October	$7,222	$11,210	$625
November	$12,839	$8,408	$1,111
December	$9,630	$1,569	$833

Select Dates	T-Bill Yields	Prime rate
December 31, 2012	2.85%	4.25%
January 31, 2012	2.90%	4.25%
February 28, 2013	2.87%	4.25%
March 31, 2013	3.04%	4.75%
April 30, 2013	3.09%	4.75%
May 31, 2013	3.11%	4.75%
June 30, 2013	3.07%	4.75%
July 31, 2013	2.95%	4.50%
August 31, 2013	2.92%	4.50%
September 30, 2013	3.40%	5.00%
October 31, 2013	3.45%	5.00%
November 30, 2013	3.47%	5.00%

(All numbers in thousands of dollars)

Important Information

- The balance in the sporting goods bank account on December 31, 2012, was $2,245,636.45.
- The bank account permits a negative balance, which the bank treats as an operating loan. The interest rate on any operating loan is prime +0.5%. Accrued interest is placed into the account on the last day of each month.
- When a positive balance exists, the bank pays interest at 0.85% on the first $50,000 in the account and 1.2% only on the portion above $50,000. Interest is deposited on the last day of each month.
- On the last day of each month, the finance department purchases T-bills in the market that will mature by the last day of the next month. The face value of T-bills are bought in denominations of exactly $100,000 in a quantity as permitted by the current balance in the bank account. If the bank account has a negative balance (i.e., it is using its operating loan), no T-bills are purchased that month. For example, if the bank account has a balance of $350,000 on March 31, three $100,000 T-bills will be purchased with 30 days left to maturity.
- Assume all revenues are deposited to the account at the end of the corresponding month.
- Assume all cost of goods sold and operating expenses are deducted at the end of the corresponding month.
- For simplicity, assume the balance in the bank account remains unchanged throughout each month.

Your Tasks

The manager wants a report that summarizes the following information from December 31, 2012, to December 31, 2013:

a. The total interest earned through T-bill investments.
b. The total interest earned from the bank account.
c. The total interest charged by any operating loans.
d. The final balance in the bank account as of December 31, 2013, when no purchases of T-bills for January 2014 have been made.

In order to meet the manager’s requirements, work through 2013 month by month starting from December 31, 2012, by following the steps below. Once arriving at December 31, 2013, use the answers to provide the four pieces of information requested by the manager.

1. Using the opening balance, determine the face value of T-bills that can be purchased. If the balance is negative, no T-bills are purchased, so skip to step 4.
2. Calculate the purchase price of the T-bills using the current market yield and the number of days until the end of the next month. The difference between the purchase price and the face value is the total interest earned for the month by T-bills.
3. Deduct the purchase price of the T-bills from the balance in the account.
4. Examine the account balance.
5. If the balance is positive, calculate the interest earned for the month based on the tiered interest rate structure. This is the total interest earned by the investment for the month.
6. If the balance is negative, charge interest to the account for the month using the interest rate charged by the bank. This is the total interest charged by the operating loan for the month.
7. To figure out the balance at the end of the next month, take the balance from step 4, add the face value of the T-bills that are maturing at the end of the month, add any interest earned from the bank account (step 4a), deduct any interest charged on the operating loan (step 4b), add the revenues for the month, and deduct the expenses and cost of goods sold for the month.
8. Go back to step 1 and repeat for the next month.
9. When all months are complete, calculate the required totals and present the requested summary information to the manager.

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8.E: Simple Interest- Working With Single Payments and Applications (Exercises)

8.1: Principal, Rate, Time

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

Interest Amount	Principal or Present Value	Interest Rate	Time
1. ?	$130,000.00	8%	9 months
2. $4,000.00	?	2% per month	8 months
3. $1,437.50	$57,500.00	? per year	4 months
4. $103.13	$1,250.00	9%	? months
5. ?	$42,250.00	1½% per month	½ year
6. $1,350.00	?	6¾%	3 months

Applications

7. Brynn borrowed $25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?
8. Alex took out a variable rate $20,000 loan on July 3 at prime + 4% when prime was set at 3%. The prime rate increased ½% on August 15. How much simple interest does Alex owe if the loan is paid back on September 29?
9. How much simple interest is earned on $50,000 over 320 days if the interest rate is:
   a. 3%
   b. 6%
   c. 9%
   d. What relationship is evident between simple interest amounts and rates in your three answers above?
10. If you placed $2,000 into an investment account earning 3% simple interest, how many months does it take for you to have $2,025 in your account?
11. Cuthbert put $15,000 into a nine-month term deposit earning simple interest. After six months he decided to cash the investment in early, taking a penalty of 1% on his interest rate. If he received $393.75 of interest, what was the original interest rate before the penalty on his term deposit?
12. If you want to earn $1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?
13. On March 3, when the posted prime rate was 4%, an investment of $10,000 was placed into an account earning prime + 2½%. Effective June 26, the prime rate rose by ¾%, and on October 4 it rose another ½%. On November 2, it declined by ¼%. If the money was withdrawn on December 15, how much simple interest did it earn?
14. If $6,000 principal plus $132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?
15. Jessica decided to invest her $11,000 in two back-to-back three-month term deposits. On the first three-month term, she earned $110 of interest. If she placed both the principal and the interest into the second three-month term deposit and earned $145.82 of interest, how much higher or lower was the interest rate on the second term deposit?

Challenge, Critical Thinking, & Other Applications

16. The simple interest formula applies to any instance of constant unit growth. Assume that 200 units of product are sold today and you forecast a constant 20-unit sales increase every four weeks. What will be the sales in one year's time (assume a year has exactly 52 weeks)?
17. Cade received an invoice for $5,200 dated August 13 with terms of 2/10, net 30 with a stated penalty of 2% simple interest per month. If Cade pays this invoice on December 13, what penalty amount is added to the invoice?
18. Marrina is searching for the best way to invest her $10,000. One financial institution offers 4.25% on three-month term deposits and 4.5% on six-month term deposits. Marrina is considering either doing two back-to-back three-month term deposits or just

taking the six-month deposit. It is almost certain that interest rates will rise by 0.5% before her first three-month term is up. She will place the simple interest and principal from the first three-month term deposit into the second three-month deposit. Which option should Marrina pursue? How much better is your recommended option?

19. Goliath borrowed $20,000 on January 3 at prime + 4.5%. The prime rate changed throughout his term (of a non–leap year) as follows:

Date Effective	Prime Rate
January 3	2.25%
February 17	2.5%
April 30	3%
June 15	2.75%
September 15	3.25%

   a. If he repaid his loan on December 24, what amount of simple interest should Goliath be charged?
   b. What equivalent fixed rate of simple interest did he get charged on his loan?

20. Evaluate each of the following $10,000 investment alternatives and recommend the best alternative for investing any principal regardless of the actual amount. Assume in all cases that the principal and simple interest earned in prior terms are placed into subsequent investments.

Alternative 1	Alternative 2	Alternative 3
6% for 1 year	5% for 6 months, then 7% for 6 months	5.25% for 3 months, then 5.75% for 3 months, then 6.25% for 3 months, then 6.75% for 3 months

What percentage more interest is earned in the best alternative versus the worst alternative?

8.2: Moving Money Involving Simple Interest

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

Interest Amount	Principal / Present Value	Interest Rate	Time	Maturity / Future Value
1. ?	$16,775.00	0.5% per month	6 months	?
2. ?	?	9%	2 months	$61,915.00
3. $1,171.44	?	5¾%	? months	$23,394.44
4. $2,073.67	$41,679.00	?	227 days	?

Questions 5–8 involve either early or late payments. Calculate the following for each:

a. The dollar amount of the early or late payment
b. The amount of interest that forms the late penalty or early benefit

Original Payments	Proposed Payments	Interest Rate
5. $1,700 today	Seven months from today	8%
6. $3,800 two years from now	Today	6.35%

Original Payments	Proposed Payments	Interest Rate
7. $950 two months from now; $1,000 seven months from now	A single payment one year from now	4.95%
8. $2,150 three months from now; $1,875 thirteen months from now	A single payment today	11%

Applications

9. On January 23 of a non–leap year, a loan is taken out for $15,230 at 8.8% simple interest. What is the maturity value of the loan on October 23?
10. An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was $10,267.21 for a loan that spanned April 14 to July 31 at 9.1%, how much was the principal and how much was the interest?
11. Suppose Robin borrowed $3,600 on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid $3,694.63?
12. How many weeks will it take $5,250 to grow to $5,586 at a simple interest rate of 10.4%? Assume 52 weeks in a year.
13. Jayne needs to make three payments to Jade requiring $2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?
14. Markus failed to make three payments of $2,500 scheduled one year ago, nine months ago, and six months ago. As his creditor has successfully sued Markus in small claims court, the judge orders him to pay his debts. If the court uses a simple interest rate of 1.5% per month, what amount should the judge order Markus to pay today?

Challenge, Critical Thinking, & Other Applications

15. Over a period of nine months, $40.85 of simple interest is earned at 1.75% per quarter. Calculate both the principal and maturity value for the investment.
16. The Home Depot is clearing out lawn mowers for $399.75 in an end-of-season sale that ends on September 30. Alternatively, you could wait until April 1 (assume February is in a non–leap year) and pay $419.95 on the same lawn mower. Assume your money can earn simple interest of 4.72% on a short-term investment.
    a. Which option should you choose?
    b. How much money will you have saved on September 30 by making that choice?
17. Tula lent $15,000 to a business associate on August 12 at 6% simple interest. The loan was to be repaid on November 27; however, the business associate was unable to make the payment. As a consequence, they agreed that the associate could repay the debt on December 29 and that Tula would add $200 to the balance that was due on November 27. What rate of simple interest is Tula charging on the late payment?
18. Merina is scheduled to make two loan payments to Bradford in the amount of $1,000 each, two months and nine months from now. Merina doesn't think she can make those payments and offers Bradford an alternative plan where she will pay $775 seven months from now and another payment seven months later. Bradford determines that 8.5% is a fair interest rate. What is the amount of the second payment?
19. Alia took out three back-to-back short-term simple interest investments. On November 10, Alia had saved up a total of $12,986.75. Her interest rate for each investment is listed below:

Dates	Interest Rate
March 2 to June 19	4.13%
June 19 to October 7	4.63%
October 7 to November 10	4.43%

How much did Alia originally invest on March 2? Assume that the principal and interest from a prior investment are both placed into the next investment.

20. An accountant just received a payment from a client on January 26 in the amount of $14,500. The cheque stub states the payment is to be applied toward the client's three outstanding loans. The three loans were for $4,500 on May 12, $6,750 on July 11, and $8,000 on August 23. All loans are being charged simple interest of 6.85%. What is the remaining balance on the client's account?

8.3: Savings Accounts And Short-Term GICs

Mechanics

1. A savings account at your local credit union holds a balance of $5,894 for the entire month of September. The posted simple interest rate is 1.35%. Calculate the amount of interest earned for the month (30 days).
2. If you place $25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?
3. In November, the opening balance on a savings account was $12,345. A deposit of $3,000 was made to the account on November 19, and a withdrawal of $3,345 was made from the account on November 8. If simple interest is paid at 0.95% based on the daily closing balance, how much interest for the month of November is deposited to the account on December 1?
4. A 320-day short-term GIC earns 0.78% simple interest. If $4,500 is invested into the GIC, what is the maturity value and how much interest is earned?

Applications

5. In February of a leap year, the opening balance on your savings account was $3,553. You made two deposits of $2,000 each on February 5 and February 21. You made a withdrawal of $3,500 on February 10, and another withdrawal of $750 on February 17. If simple interest is calculated on the daily closing balance at a rate of 1.45%, how much interest do you earn for the month of February?
6. An investor places $30,500 into a short-term 120-day GIC at the Bank of Montreal earning 0.5% simple interest. The maturity value is then rolled into another short-term 181-day GIC earning 0.57% simple interest. Calculate the final maturity value.
7. The Mennonite Savings and Credit Union posts the following rate structure for its Daily Interest Savings Account. The entire balance is subject to the highest posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$10,000.00	0.1%
$10,000.01–$25,000.00	0.21%
$25,000.01–$50,000.00	0.34%
$50,000.01–$100,000.00	0.85%
$100,000.01 and up	1.02%

The opening balance in February of a non–leap year was $47,335. The transactions on this account were a deposit of $60,000 on February 8, a withdrawal of $86,000 on February 15, and a deposit of $34,000 on February 24. What interest for the month of February will be deposited to the account on March 1?

8. An investor with $75,000 is weighing options between a 200-day GIC or two back-to-back 100-day GICs. The 200-day GIC has a posted simple interest rate of 1.4%. The 100-day GICs have a posted simple interest rate of 1.35%. The maturity value of the first 100-day GIC would be reinvested in the second 100-day GIC (assume the same interest rate upon renewal). Which alternative is best and by how much?
9. Canadian Western Bank offers a Summit Savings Account with posted interest rates as indicated in the table below. Only each tier is subject to the posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$5,000.00	0%
$5,000.01–$1,000,000.00	1.05%
$1,000,000.01 and up	0.80%

December's opening balance was $550,000. Two deposits in the amount of $600,000 each were made on December 3 and December 21. Two withdrawals in the amount of $400,000 and $300,000 were made on December 13 and December 24, respectively. What interest for the month of December will be deposited to the account on January 1?

10. Sun Life Financial Trust offers a 360-day short-term GIC at 0.65%. It also offers a 120-day short-term GIC at 0.58%. You are considering either the 360-day GIC or three consecutive 120-day GICs. For the 120-day GICs, the entire maturity value would be "rolled over" into the next GIC. Assume that the posted rate increases by 0.1% upon each renewal. If you have $115,000 to invest, which option should you pursue and how much more interest will it earn?

Challenge, Critical Thinking, & Other Applications

11. The Regular Savings Account at Steinbach Credit Union posts the following tiered rate structure. Interest is calculated on the entire balance for any given tier.

Balance	Interest Rate
$0–$100,000.00	1.75%
$100,000.01–$250,000.00	1.85%
$250,000.01 and up	1.95%

For any given month, simple interest is calculated on the lowest monthly closing balance, but the interest is not deposited until January 1 of the following year. On January 1, 2014, the opening balance on an account was $75,000. The following transactions took place throughout the year:

Date	Activity	Amount
February 14	Deposit	$133,000
March 30	Deposit	$42,000
June 15	Withdrawal	$110,000
August 1	Deposit	$212,000
October 29	Withdrawal	$300,000

Calculate the annual amount of interest earned on this account that will be deposited on January 1, 2015.

12. Interest rates can fluctuate throughout the year. Working with the information from question 11, recalculate the annual amount of interest if all of the posted interest rates were adjusted as follows throughout the year:

Effective Date	Change	Percent
March 1	Increase	0.1%
July 1	Decrease	0.15%
September 1	Increase	0.25%
November 1	Decrease	0.05%

13. Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have $50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.
14. A Citibank Savings Account posts the following simple interest rate structure. All interest is paid on entire balances, and interest is deposited to the account every quarter.

Balance	Interest Rate
$0–$2,500.00	0.05%
$2,500.01–$5,000.00	0.1%
$5,000.01 and up	0.75%

On January 1, 2013, the opening balance was $3,300. The following transactions took place during the first quarter (January 1 to March 31):

Deposits	Withdrawals
January 4 $350	January 15 $500
January 29 $1,800	January 19 $1,000
February 7 $2,300	February 20 $1,300
March 2 $2,400	March 7 $4,000
March 22 $1,100	March 25 $2,000

Calculate the total amount of interest that will be deposited to the account on April 1, 2013.

15. You are considering the following short-term GIC investment options for an amount of $90,000.

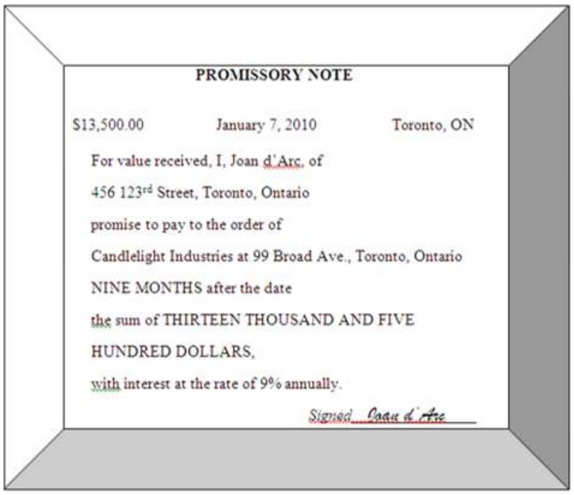
	Option 1	Option 2	Option 3	Option 4
Institution	Coast Capital Savings	MCAN Mortgage Corp.	DUCA Financial Services	Sun Life Financial Trust
Term	360 day	2 × 180 day	3 × 120 day	4 × 90 day
Posted Rate	0.8%	0.75%	0.72%	0.715%

In all cases, assume that the posted rates remain unchanged and that the entire maturity value will be reinvested in the next short-term GIC. Calculate the total maturity value for each option at the end of 360 days.

8.4: Application - Promissory Notes

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics



PROMISSORY NOTE

$13,500.00 January 7, 2010 Toronto, ON

For value received, I, Joan d'Arc, of
456 123rd Street, Toronto, Ontario
promise to pay to the order of
Candlelight Industries at 99 Broad Ave., Toronto, Ontario
NINE MONTHS after the date
the sum of THIRTEEN THOUSAND AND FIVE
HUNDRED DOLLARS,
with interest at the rate of 9% annually.

Signed Joan d'Arc

1. Examine this promissory note.
    a. Identify the six components of the promissory note.
    b. Determine the legal due date.
    c. Calculate the maturity value of the note.

For questions 2–12, solve for the unknown variables (identified with a ?) based on the information provided.

Issue Date	Term	Legal Due Date
2. January 16, 2011	320 days	?
3. November 29, 2010	?	November 1, 2011
4. ?	120 days	April 11, 2012

Issue Date	Legal Due Date	Face Value	Term	Interest Rate	Maturity Value
5. October 4, 2011	?	$33,550	90 days	7.1%	?
6. April 18, 2011	?	$72,950	150 days	?	$75,014.09
7. January 29, 2011	?	?	270 days	4.8%	$8,805.16
8. ?	December 23, 2011	$29,900	?	6.25%	$31,553.72

Issue Date	Face Value	Term	Interest Rate	Maturity Value	Discount Rate	Sale Date	Proceeds
9. March 2, 2011	$4,500	180 days	18.1%	?	21.5%	June 20, 2011	?
10. Dec. 15, 2010	?	350 days	12.2%	$16,764.25	13.75%	?	$16,196.81
11. July 3, 2011	$43,000	165 days	4.36%	?	?	Oct. 18, 2011	$43,490.78
12. Nov. 12, 2010	$18,750	340 days	?	$19,904.10	8.35%	May 27, 2011	?

Applications

13. A $20,000 promissory note is issued on June 12 for a five-month term carrying an annual interest rate of 5.5%. Calculate the maturity value of the note.
14. A noninterest-bearing note is issued on February 12 of a non–leap year in the amount of $14,250. It has a term of 10 months. It is sold on September 22 with a negotiated interest rate of 4%. Determine the proceeds of the sale.
15. A $4,700 promissory note is issued on November 12 for an 11-month term (February is a leap year) carrying an annual interest rate of 12%. It is sold on July 3 with a negotiated interest rate of 16%. Determine the proceeds of the sale.
16. A $12,600 promissory note was issued on March 13 with a term of 120 days. It acquired $541.37 of interest. What was the annual interest rate on the promissory note?
17. A $34,250 promissory note with 5.9% annual interest issued on April 30 had a value of $35,296.36 on its legal due date.
    a. What is the term of the promissory note?
    b. If the note was sold to an investor on August 28 for $34,853.60, what was the negotiated rate of interest on the sale?

Challenge, Critical Thinking, & Other Applications

18. An investor purchased a promissory note on June 15 for $27,329.96 using a negotiated interest rate of 10.5%. The promissory note had been issued on April 2 for a term of eight months at an interest rate of 9.25%. Calculate the original face value of the promissory note.
19. A 150-day promissory note has 18.2% annual interest. Forty-five days after the issue, the owner of the note has two offers to purchase the note. The first offer is to purchase the note immediately at an annual interest rate of 19.3%. The second offer is to

purchase the note 15 days later at an annual interest rate of 21%. If money can be invested at 5% simple interest, which offer should the owner accept? What percentage more proceeds are realized that way?

20. Teagan is a good friend of the manager of Borling Industries and is looking to borrow $6,000 from the company on January 12, 2012, for a term of anywhere from 6 to 11 months. Both parties are agreeable to the following possibilities with respect to term and the corresponding interest rate.

Term	Interest Rate
6 months	9%
7 months	7.75%
8 months	7%
9 months	6.15%
10 months	5.5%
11 months	5%

a. If Teagan wants to pay the least amount of total interest on his promissory note, which term should he pursue? What amount of interest will he pay?
b. If Borling Industries wants to maximize the total interest on the promissory note, which term should it pursue? What amount of interest will it receive?

8.5: Application - Loans

Mechanics

Fill in the partial repayment schedules with the missing values.

1. Woodgrain Industries took out an operating loan with RBC for $20,000 at a fixed interest rate of 8% on September 14. The operating loan requires a monthly fixed payment of $800 on the 14th of every month. Create the first three months of its repayment schedule.

Date	Balance before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Sep 14								$20,000
Oct 14		8%				$800		
Nov 14		8%				$800		
Dec 14		8%				$800		

2. Gayle has a HELOC with MCAP Financial Corporation at an interest rate of prime + 3%. Her current balance owing on November 1 is $13,750 and she is required to make interest-only payments on the first of every month. The prime rate is set at 3.75%. She makes one payment of $2,500 on January 19. Create three months of her repayment schedule.

Date	Balance before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Nov 1								$13,750
Dec 1		6.75%						
Jan 1		6.75%						
Jan 19		6.75%				$2,500		

Date	Balance before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Nov 1								$13,750
Dec 1		6.75%						
Jan 1		6.75%						
Jan 19		6.75%				$2,500		
Feb 1		6.75%						

3. Grant has a personal LOC with TD Canada Trust at prime + 4.5%, where the current prime rate is 3.75%. On March 31, his balance owing was $5,000. On April 4 he advanced $1,000 and on April 24 he paid $2,250. The prime rate declined by 0.5% on April 14. He is required to make the complete interest payment at the end of every month. Create the repayment schedule for April.

Date	Balance before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
Mar 31								$5,000.00
Apr 4		8.25%				-$1,000		
Apr 14		8.25%						
Apr 24		7.75%				$2,250		
Apr 30		7.75%						

4. Amarjeet graduated from the University of Calgary on May 2 and has student loans totalling $35,000. The prime rate upon graduation was 4.5%. He has decided to pay in full the interest charged during the grace period (i.e., he is not converting it to principal) before starting monthly payments of $600 at the fixed interest rate. Calculate the total interest paid during the grace period and the first three months of his repayment schedule.

Date	Balance before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
June 1								$35,000
Nov 30 (inclusive)		7%					$0.00	
Dec 31		9.5%				$600.00		
Jan 31		9.5%				$600.00		
Feb 29		9.5%				$600.00		

Applications

5. A $7,500 demand loan was taken out on March 4 at a fixed interest rate of 7.72% with fixed monthly payments of $1,200. The first monthly repayment is due April 4 and the 4th of every month thereafter. Prepare a full repayment schedule for the loan.
6. Paintball Paradise took out a $17,500 operating loan on February 15 (in a non–leap year) at an interest rate of prime + 3.5% when the prime rate was 3.75%. The loan requires fixed monthly payments of $3,000 each made on the 15th of the month starting with March 15. If Paintball Paradise plans to make an additional payment of $4,500 (without penalty) on May 4 and the prime rate rises by 0.75% on June 28, construct a full repayment schedule for the loan.
7. Vertical Adventures has an open line of credit with a zero balance at its credit union using a fixed interest rate of 7.35%. On the last day of every month, the accrued interest must be paid. On July 8 and August 14, the company made advances of $15,000 and $12,000, respectively. On July 30, it made a payment of $10,000. Vertical Adventures will restore its zero balance on August 31. Construct a full repayment schedule from July 8 to August 31.
8. Scotiabank approved a $250,000 line of credit for Buhler Industries at prime + 1%. It requires only the repayment of accrued interest on the 27th of each month, which is automatically deducted from the chequing account of Buhler Industries. Buhler took out an advance on December 2 for $200,000 and made a payment of $125,000 on January 12. The prime rate was 6.5% initially and increased to 7.5% on January 4. Calculate the total interest charged to Buhler Industries on December 27 and January 27.

9. On May 9 Rainbow Daycare established an open line of credit with its bank and immediately withdrew $25,000. The interest rate was set at prime + 4.75%, and the prime rate was 6.75%. The line of credit requires a payment on the 9th of every month in the amount of "$1,000 or 5% of the current balance, whichever is greater." Rainbow Daycare took another advance of $15,000 on June 2. It made a payment of $28,000 on July 16. The prime rate decreased by 0.5% on June 30. Create a repayment schedule from May 9 to August 9 and calculate the total interest paid by Rainbow Daycare.
10. Lacy has a $40,000 student loan when she graduates on May 4, and the prime rate is set at 5.25%. She has decided at the end of the grace period to convert the interest to principal, and she sets her fixed monthly payment at $850. She opts for the variable rate on her student loan. Create the first four repayments of her repayment schedule. Calculate the total interest charged for both the grace period and the four payments combined. Assume February does not involve a leap year.
11. Genevieve has a $32,000 student loan when she graduates on September 20, at which time the prime rate is 6%. She has decided at the end of the grace period to convert the interest to principal, and she sets her fixed monthly payment at $910. She opts for the fixed rate on her student loan. The prime rate rises by 0.5% on November 3 and by another 0.5% on February 18. Create the first six repayments of her repayment schedule. Calculate the total interest charged for both the grace period and the six payments combined. Assume February has 29 days.

Challenge, Critical Thinking, & Other Applications

12. Home Run Sports has an operating loan with CIBC. The interest rate is set at prime + 3.25%, and on the 17th of every month a payment in the amount of 4% of the current balance is automatically taken from the borrower’s chequing account. The prime rate is currently 5.75%. On March 7, Home Run Sports took a first advance of $6,000. It took additional advances of $10,000, $8,000, and $12,000 on March 28, April 17, and May 24, respectively. It made payments of $4,000, $2,500, and $10,000 on April 10, May 17, and May 29. The prime rate increased by 0.25% on March 31 and by an additional 0.25% on May 31. Create a repayment schedule and calculate the total interest paid from March 7 to June 17.
13. Eyeland Optical has a line of credit with its credit union with an established credit limit of $24,000 and interest set at prime + 4.1%. Eyeland Optical is allowed to exceed its credit limit, but if it does so the entire line of credit becomes subject to 28% interest. Payment terms require the repayment of the accrued interest only on the 11th of every month. On July 8, Eyeland Optical made its first withdrawal of $10,000. It made further advances of $15,000, $15,000, and $8,000 on August 11, August 20, and August 28, respectively. It made payments of $3,500, $2,000, and $28,000 on July 31, August 15, and September 11, respectively. On October 11, Eyeland Optical restored a zero balance. The prime rate was initially 3.5%, rose by 0.25% on July 30, and declined by 0.5% on September 1. Create a repayment schedule and calculate the total interest charges from July 8 to October 11.
14. Pierre has a $40,000 student loan when he graduates on December 22. The prime rate is currently 4.25%. He decides that at the end of the grace period he will convert the interest to principal, take the variable interest rate, and set his fixed monthly payment at $750. The prime rate rises by 0.5% on each of February 15, April 15, June 30, August 15, and November 30. He makes additional payments of $500 on each of September 3, October 21, and December 6. Create a repayment schedule and calculate the total interest charged to Pierre from January 1 to December 31. Assume February has 29 days.
15. Use the following information for a student loan (assume February does not involve a leap year):

Prime Rate: 5% on June 1, rising by 0.25% on September 13, rising by 0.5% on January 25, rising by 1% on March 18.

Grace Period: June 1 to November 30 (inclusive)

Total Student Loan as of June 1: $13,000

Chosen Fixed Payment: $350

There are essentially four options on this loan:

Option	Grace Period Interest on November 30	Interest Rate
1	Pay it off	Variable
2	Convert to principal	Variable
3	Pay it off	Fixed
4	Convert to principal	Fixed

For each option, create a complete repayment schedule and calculate the total interest charges. How much less interest than the worst option does the best option incur?

8.6: Application - Treasury Bills & Commercial Papers

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–7, solve each of the following T-bills or commercial papers for the unknown variables (identified with a ?) based on the information provided.

Purchase Price on Date of Issue	Term (days)	Yield	Face Value
1. ?	90	1.75%	$100,000
2. $240,663.83	364	?	$250,000
3. ?	270	5.67%	$500,000
4. $295,796.45	182	?	$300,000
5. $73,307.58	?	4.63%	$75,000
6. $9,931.92	?	2.78%	$10,000
7. $190,175.88	364	5.18%	?

Applications

8. A 60-day, $90,000 face value commercial paper was issued when yields were 2.09%. What was its purchase price?
9. A Government of Canada V39065 issue 90-day T-bill achieved its highest rate of return on May 24, 2000, with a yield of 5.74%. It realized its lowest rate of return on February 26, 2010, with a yield of 0.16%. Calculate the purchase price of a $100,000 T-bill on each of these dates. In dollars, how much more yield did an investor realize in 2000 than in 2010?
10. A Government of Canada V121780 issue 364-day T-bill achieved its highest rate of return on May 17, 2000, with a yield of 6.31%. It realized its lowest rate of return on May 6, 2009, with a yield of 0.42%. Calculate the purchase price of a $55,000 T-bill on each of these dates. In dollars, how much more yield did an investor realize in 2000 than in 2009?
11. Pawan is the marketing manager for Cyanamid Canada. His company will execute a marketing program half a year from now that requires a $500,000 investment. If his finance department had $485,000 to invest today into a 182-day $500,000 face value commercial paper yielding 4.55%, would it have enough money to purchase the commercial paper? Show calculations to support your answer.
12. A 182-day Province of Manitoba T-bill with a face value of $250,000 was issued 102 days ago, when the yield was 4.3%. What is its purchase price today if the current rate of return is 4.53%?
13. William purchased a $110,000 270-day commercial paper on its date of issue, when the yield was 3.89%. He sold it 178 days later when yields had increased to 4.03%. How much money did William earn on his investment?
14. Dollar Thrifty Automotive Group issued a $1,000,000, 180-day commercial paper. A bank purchased the paper for $975,560.21 on the issue date. What was the yield for the commercial paper on its issue date?
15. A 90-day Province of Ontario T-bill with a $35,000 face value matures on December 11. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, when the yield is 4.94%, and sells the T-bill on November 25, when the yield is 4.83%, calculate how much money Farrah earned and the rate of return she realized.
16. On August 8, Harriet invested in a $150,000 Canadian Wheat Board commercial paper on its date of issue with a 220-day term at a yield of 5.98%. On October 15, she sold the commercial paper to another investor when the current market rate was 5.75%. Calculate the amount of money earned and the rate of return realized.

Challenge, Critical Thinking, & Other Applications

17. Philippe purchased a $100,000 Citicorp Financial 220-day commercial paper for $96,453.93. He sold it 110 days later to Damien for $98,414.58, who then held onto the commercial paper until its maturity date.
    a. What is Philippe's actual rate of return?

b. What is Damien's actual rate of return?
c. What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?
d. Comment on your answers to the above.

18. It is interesting to note another application of the percent change Formula 3.1. Recall, $\Delta\% = \frac{\text{New-Old}}{\text{Old}} \times 100$. In the case of simple interest and solving for rate of return, assign $S = \text{New}$, $P = \text{Old}$, and $r = \Delta\%$. Therefore, you rewrite the formula as $r = \frac{S-P}{P} \times 100$. It is critical to recognize that the r is based on the implied time period between the S and P. If the S is 100 days after the P, then the r is a "% per 100 days." Since r is expressed annually, you must convert it by dividing by the period and multiplying by 365. Recalculate questions 2 and 4, showing how to arrive at the same answer by using the percent change formula.

Spreadsheet Applications

19. Calculate the purchase price of a $10,000 T-bill with a yield of 2.95% using different terms of 30 days, 60 days, 90 days, 182 days, and 364 days. What do you observe about the purchase price when comparing these various terms?
20. Take a $100,000, 364-day T-bill that had a rate of return of 4.73% upon issue.
    a. Calculate the purchase price on the date of issue, and also calculate the purchase price if the rate of return had been higher or lower by 1% on the date of issue.
    b. Repeat part (a) on the 120th day and the 240th day after the date of issue.
    c. Compare your answers to (a) and (b) above and comment on your findings.

Review Exercises

Mechanics

1. If $4,000 is borrowed from April 3 to June 22 at a simple interest rate of 3.8%, how much interest is paid on the loan?
2. A savings account pays flat-rate interest of 1.45%. If a balance of $3,285.40 is maintained for the entire month of August, how much interest does the savings account earn?
3. What is the legal due date and maturity value on that date for a 125-day $51,000 promissory note issued on May 14 at
4. A full-time student graduated from college on December 16, 2013, with $26,500 in outstanding student loans. Calculate the interest accrued during his grace period if the prime rate is 4.7%.
5. A 182-day $1,000,000 Government of British Columbia T-bill was issued when the market rate of return was 4.21%. Calculate the purchase price of the T-bill on its issue date.
6. If you place $8,000 into a 300-day short-term GIC at Scotiabank earning 0.95% simple interest, how much will you receive when the investment matures?

Applications

7. Proper accounting procedures require accountants to separate principal and interest components on any loan. Allocate the principal and interest portions of a $24,159.18 payment clearing a 147-day loan at 8.88%.
8. Cadillac Fairview withdrew $115,000 from its operating loan account on September 4 to perform some needed maintenance on one of its properties. The operating loan requires interest at prime + 3% and fixed $25,000 monthly payments starting October 1. The company thinks it can make an additional payment of $35,000 on November 18. The prime rate on the date of withdrawal was 3.6%, and it increased by 0.5% on October 27. Construct a full repayment schedule for this loan.
9. As part of your financial plan for retirement, you purchased a 270-day $25,000 commercial paper on its date of issue, July 14, when market yields were 2.94%. 234 days later, you sold the note when market yields were 2.76%. What rate of return did you realize on your investment?
10. Alterna Savings and Bank offers a Daily Interest Savings Account with posted interest rates as indicated in the table below. The entire balance qualifies for the posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$5,000.00	0.10%
$5,000.01–$10,000.00	0.15%
$10,000.01–$25,000.00	0.20%

Balance	Interest Rate
$25,000.01 and up	0.25%

BOS Designer Candles Inc. has an opening balance in July of $17,500. Three deposits in the amounts of $6,000, $4,000, and $1,500 were made on July 4, July 18, and July 22, respectively. Two withdrawals in the amount of $20,000 and $8,000 were made on July 20 and July 27, respectively. What interest for the month of July will Alterna deposit to the account on August 1?

11. Shannon has a $68,000 student loan when she graduates on August 28, 2014, and the prime rate is set at 4.9%. She will convert the interest to principal at the end of her grace period, and she elects to take the variable rate on her student loan. She sets her fixed monthly payment at $1,400. The prime rate decreases by 0.5% on January 15 and rises by 0.75% on April 25. Compute the first six repayments of her repayment schedule. Calculate the total interest charged for both the grace period and the six payments combined.
12. Sturm put $48,700 into a 10-month term deposit, but needed to withdraw the funds after five months to deal with a family emergency. The credit union penalized him 2.35% off of his interest rate for the early withdrawal and deposited $49,602.98 into his account.
    a. What was Sturm's original interest rate?
    b. How much interest, in dollars, was he penalized for the early withdrawal?
13. Jerry's Concrete allows his customers to create six-month promissory notes on any stamped concrete driveway project with interest at 8.9%. Three months after completing a $17,300 job on March 2, 2014, Jerry had some liquidity problems and sold the promissory note to a financial institution at an interest rate of 10.35%. Calculate Jerry's proceeds on the sale.
14. Three hundred days from now you will be departing on a backpacking trip through Europe. You need $4,000 in spending money to take with you. Today, you currently have saved $3,960.
    a. If you place your money in consecutive 100-day short-term GICs earning 1.02%, will you meet your goal? Assume the full maturity values are reinvested into the next GIC.
    b. What would the interest rate need to be if you had placed your money into a single 300-day short-term GIC to reach your goal?

Challenge, Critical Thinking, & Other Applications

15. A credit union posts the following tiered interest rate structure for its savings accounts. Only each tier is subject to the posted rate and is computed using the daily opening balance in the account. Interest is deposited to the account on the last day of every month.

Balance	Interest Rate
$0–$4,000.00	0.15%
$4,000.01–$8,000.00	0.25%
$8,000.01 and up	0.45%

On August 1, the opening balance was $6,400. Three deposits of $2,000, $3,500, and $1,500 were made on August 3, August 10, and August 27, respectively. Two withdrawals of $7,000 and $1,900 were made on August 6 and August 21, respectively. Compute the total interest earned for the month of August.

16. Island Lakes Dental (ILD) found itself with an urgent need to replace its X-ray machine when its existing machine suddenly became inoperable. It borrowed $43,000 on May 12 from its operating loan with interest set at prime + 2.4%. The prime rate was 2.75% and increased by 0.5% on June 5. The loan requires payment of the balance in full when the balance is less than $5,000, or 10% of the current balance on the first of every month. ILD made payments of $15,000 on May 29 and $21,000 on June 21. Create a full repayment schedule for its operating loan and calculate the total interest charged on the purchase.
17. Pendragon Inc. has an operating loan with a balance of $52,000 on September 1 with interest set at prime + 4.75%. On the first of every month the operating loan requires repayment of the accrued interest only. The current prime rate is 4.75% and will decrease to 4.25% on October 7. On September 15, it has a 270-day $20,000 GIC with 3.8% interest maturing. On October 15,

it has a 320-day $18,000 GIC with 3.68% interest maturing. It will use these maturing investments to pay down its operating loan. Construct a repayment schedule from September 1 to November 1 only.

18. A 364-day, $50,000 face value T-bill is issued when market yields are 2.85%. The T-bill is sold to another investor every 91 days until maturity, with yields of 3.1%, 2.98%, and 3.15% on each of the dates of sale, respectively. Compute the purchase price for each investor, including the date of issue. For each investor, calculate the actual rate of return realized on their investment.
19. Brant and Sylvia have a prime + 2.35% HELOC with a $30,000 balance on March 1 with accrued interest payable on the first of every month. They also have a savings account with a $10,000 balance on March 1 earning 0.85% payable on the first of every month. On March 18 they deposited $4,000 into the savings account. On both March 5 and March 30 they transferred $3,000 from their savings to their HELOC. Prime was initially at 3.75% but increased to 4.25% on March 22. Calculate the balance on April 1 in both the savings account and the HELOC.
20. Norbert is looking to make a short-term investment for 120 days with the $25,000 he just inherited from his father’s estate. His options are as follows. Compute the maturity values of each of the options and rank his choices.
    a. Two back-to-back 60-day GICs earning 2.25% today, with a forecasted rate of 2.5% 60 days from now. Assume the full maturity value is rolled over into the next investment.
    b. One 120-day GIC earning 2.35%.
    c. A 90-day $25,000 face value T-bill earning 2.28% followed by a 30-day $25,000 face value T-bill earning a projected 2.38%. Assume any leftover funds are invested to earn 0.85% interest.

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CHAPTER OVERVIEW

9: Compound Interest- Working With Single Payments

9.0: Introduction
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9.0: Introduction

Do you dream of owning a home? What about a car or a home theatre system? Big purchases such as these require long-term financial planning. Compound interest means that you will pay substantially more money for your purchases or earn more money on investments than you would with simple interest.

Compound interest is used for most transactions lasting one year or more. In simple interest, interest is converted to principal at the end of the transaction. Therefore, all interest is based solely on the original principal amount of the transaction. **Compound interest**, by contrast, involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest. How significant is the interest when it compounds? Let’s examine your big purchases from above:

- A home theatre system with the big-screen TV, Blu-Ray player, stereo surround sound, and more can retail for $5,000. Did you know that if you put that amount on the retail store's 18% interest credit card and pay it off monthly for three years, you will pay over $1,500 of compound interest?
- The average Canadian new car price is $25,683.1 Unless you have cash, you will join the ranks of other Canadians in taking six years to pay it off through $450 monthly payments, resulting in almost $7,000 of compound interest charges!
- The average Canadian home price is $365,000,2 though of course it varies widely across our country. If it takes 25 years to pay off your mortgage at 6% interest, your $365,000 house becomes a $700,000 acquisition thanks to compound interest. That is almost twice the original price!

And it is not any different for businesses. Whether they are local companies or multinational conglomerates they must invest and borrow at compound interest rates in their attempt to achieve long-term financial strategies. Some examples of these business activities include the following:

- Borrowing $480,000 to open a new Tim Hortons’ single restaurant franchise operation.3
- Spending $1,000,000 on a fleet of rigs and semi-trailers for product distribution.
- Constructing new production plants or warehouses costing $10,000,000 or more.

The money for these types of transactions does not appear out of thin air. It must be borrowed or taken from savings, and either approach involves compound interest.

Throughout the rest of this textbook, you will study compound interest as it relates to three distinct but interconnected concepts.

1. Calculating interest on a single amount (called a lump-sum amount or single payment). Chapter 9 introduces the basic mathematics, which are applied in Chapter 10.
2. Calculating interest on a series of regular, equal payments, called annuities. Chapter 11 introduces the basic mathematics, which are applied in Chapter 12.
3. Specialized applications including amortization, mortgages, bonds, sinking funds, and internal rates of return are covered in Chapter 13 through Chapter 15.

Now turn the page to get started with the basics.

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3. Tim Hortons, “Frequently Asked Questions,” accessed May 13, 2010, http://www.timhortons.com/ca/en/join...se_ca_faq.html.

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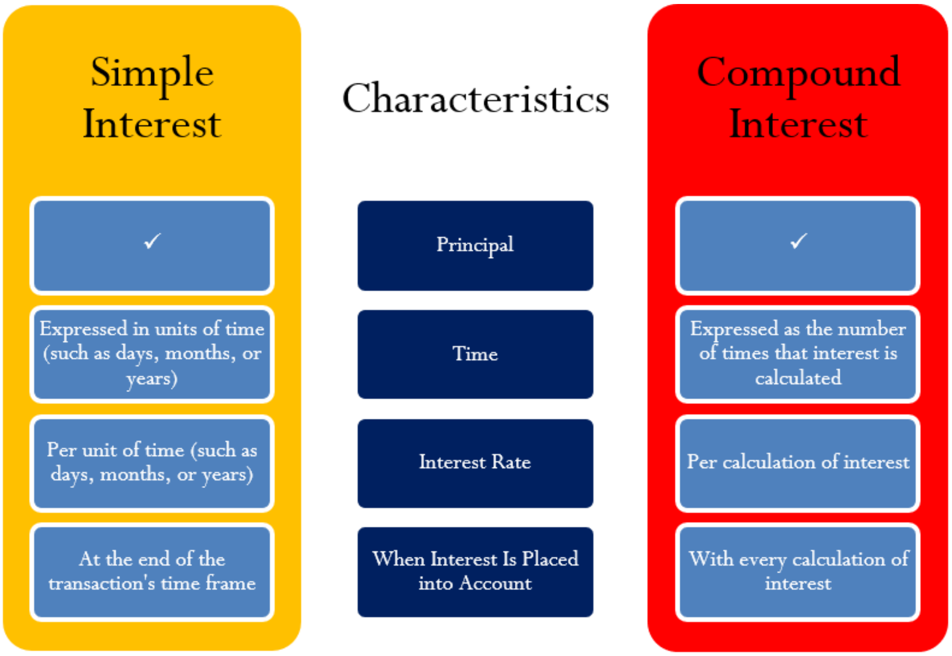
9.1: Compound Interest Fundamentals

Five years ago you started a long-term GIC. The bank statement shows that you originally placed $15,000 into the account earning 5.95% in annually compounded interest. You now have $20,026.09. In the first year of the GIC, you earned $892.50 in interest. Subsequent years earned annual interest amounts of $945.60, $1,001.87, $1,061.48, and finally $1,124.64. Of course, you like that the interest amount increased each year even though you did not make any more deposits to the GIC, but why did this happen?

The Concept of Compounding

Simple interest, compound interest—what is the big difference? As you can see in the figure below, simple and compound interest calculations share the same fundamentals of time, interest rate, and placing interest into the account. But the accumulation is not the same, and over time the growth of compound interest will far outpace that of simple interest.



Simple Interest	Characteristics	Compound Interest
✓	Principal	✓
Expressed in units of time (such as days, months, or years)	Time	Expressed as the number of times that interest is calculated
Per unit of time (such as days, months, or years)	Interest Rate	Per calculation of interest
At the end of the transaction's time frame	When Interest Is Placed into Account	With every calculation of interest

In both forms of interest, the principal is the starting amount that accumulates interest for a length of time at a specified interest rate. But you calculate simple interest in direct proportion to the starting amount, the interest rate, and the time period, whereas the calculation for compound interest is, well, not so simple!

The critical difference is the placement of interest into the account. Under simple interest, you convert the interest to principal at the end of the transaction's time frame. For example, in a six-month simple interest GIC the balance in its account at any point before the maturity date is the original principal and nothing more. Only upon maturity does the interest appear. In contrast, a five-year compound interest GIC such as the one discussed in the section opener receives an interest deposit annually. After one year the principal increases from $15,000 to $15,892.50. This higher principal in the second year explains why the interest earned in the second year then increases as well.

Invest $1,000 for 25 years at both 10% simple interest and 10% interest compounded annually. As demonstrated in the figure and table below, $1,000 invested at 10% per year of simple interest has a $3,500 balance after 25 years. This consists of the original $1,000 principal plus $2,500 in total interest ($100 for each year the money was invested). However, $1,000 invested at 10% compounded annually has a balance of $10,834.71, which is $7,334.71 more! This difference is the result of interest being converted to principal and thus earning even more interest for you. A close examination of the first two years reveals the following:

- In the first year, both investments have a $1,100 balance. With simple interest, the $100 of interest remains as accrued interest and is not placed into the account while the principal remains at $1,000. With compound interest, the $100 of interest is converted to principal, resulting in a $1,100 principal for the second year.

- In the second year, the simple interest account still has a $1,000 principal, which earns another $100 of interest ($1,000 × 10% = $100). The account now has $1,000 of principal plus $200 in accrued interest. The compound interest account earns 10% on the new principal of $1,100, or $110 of interest ($\$1,100 \times 10\% = \$110$). This interest is placed into the account at the end of year two, making the principal $1,210 for the third year . . . and so on.

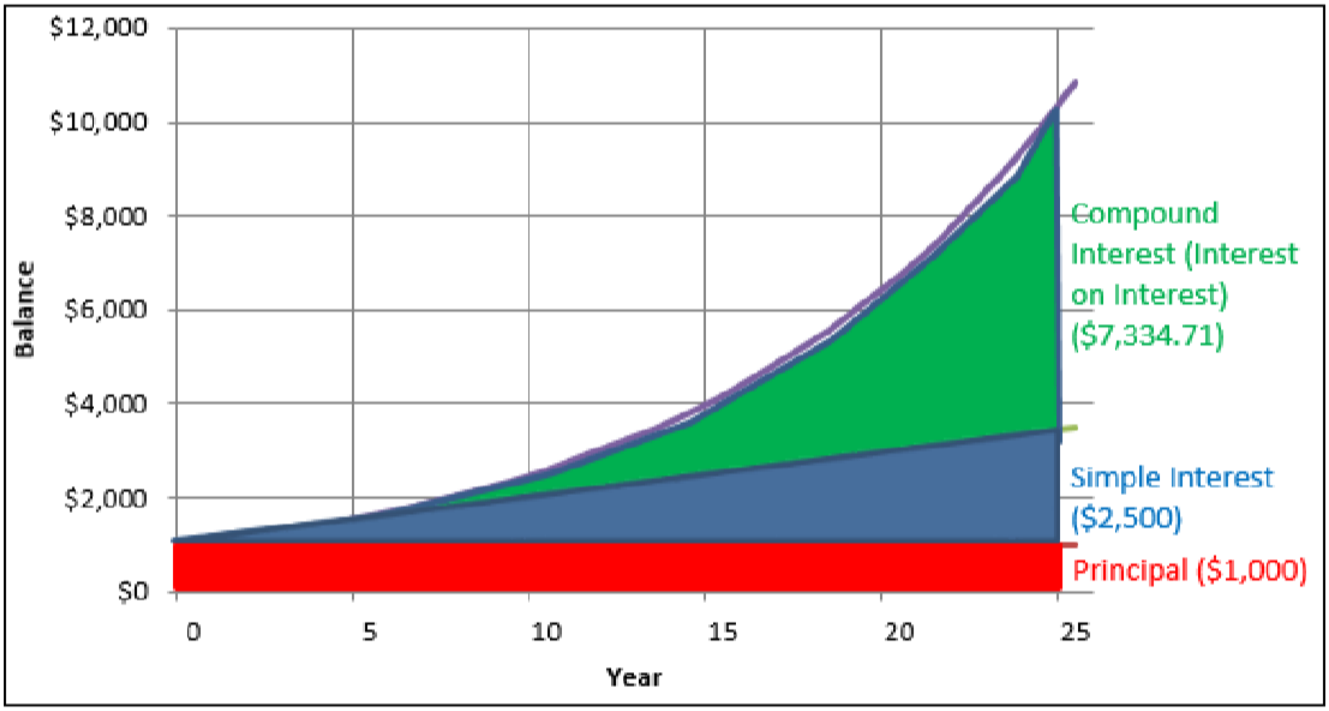


Figure $9.1.1$

At 10% Simple Interest				At 10% Compound Interest		
End of Year	**Principal**	**Accrued Interest**	**Balance**	**Principal**	**Accrued Interest**	**Balance**
Start						$1,000.00
1	$1,000.00	$100.00	$1,100.00	$1,000.00	$100.00	$1,100.00
2	$1,000.00	$200.00	$1,200.00	$1,100.00	$110.00	$1,210.00
3	$1,000.00	$300.00	$1,300.00	$1,210.00	$121.00	$1,331.00
4	$1,000.00	$400.00	$1,400.00	$1,331.00	$133.10	$1,464.10
5	$1,000.00	$500.00	$1,500.00	$1,464.10	$146.41	$1,610.51
	.....	.....	.....	.....	.....	.....
10	$1,000.00	$1,000.00	$2,000.00	$2,357.95	$235.79	$2,593.74
	.....	.....	.....	.....	.....	.....
15	$1,000.00	$1500.00	$2,500.00	$3,797.50	$379.75	$4,177.25
	.....	.....	.....	.....	.....	.....
20	$1,000.00	$2000.00	$3,000.00	$6,115.91	$611.59	$6,727.50

At 10% Simple Interest				At 10% Compound Interest		
	.....	.....	.....	.....	.....	.....
25	$1,000.00	$2500.00	$3,500.00	$9.849.73	$984.98	$10,834.71

Ultimately, as the principal in the compound interest account increases, the amount of interest earned also increases. Observe that the difference in the balance between simple interest and compound interest in the first five years is slight, but the gap widens over time.

Compound Interest Rates

If an equal amount of principal invested for 25 years earns an interest rate of 9% compounded annually versus 3% compounded annually, would the interest accumulated by the 9% investment be three times as large?

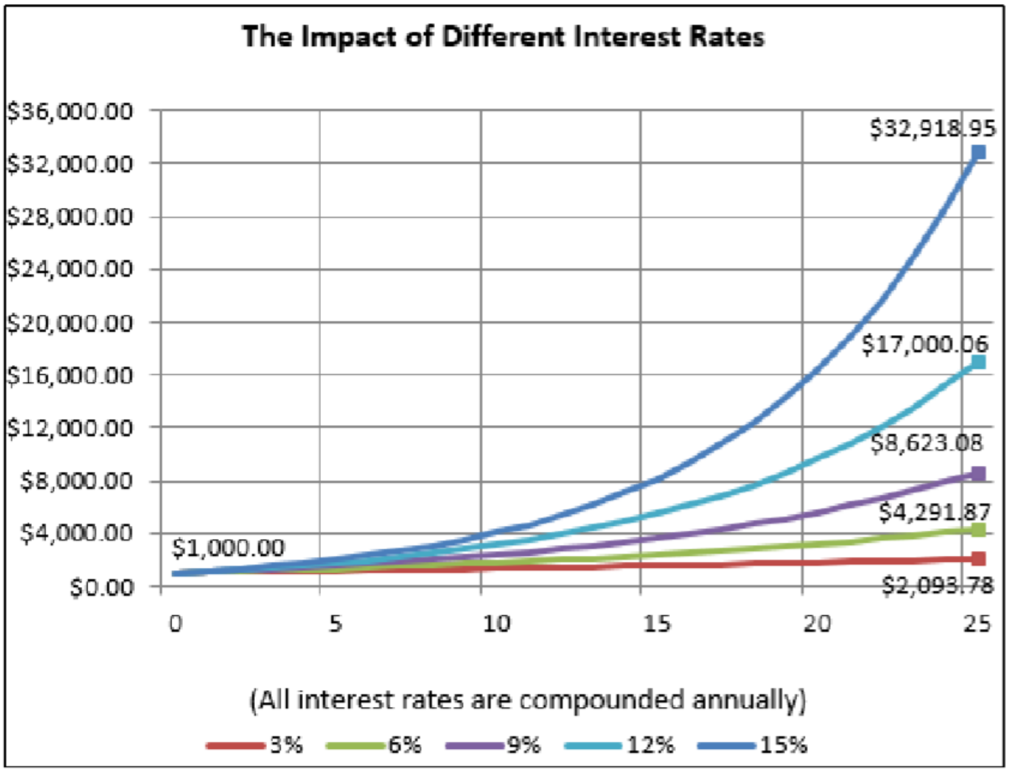


Figure $9.1.2$

In a *simple interest* environment, this would be true since the principal never changes: in each of the 25 years the amounts of simple interest would be constant, and the amount for the 9% investment would be three times larger than for the 3% investment. However, in *compound interest* each subsequent year's interest is compounded on an increasingly larger principal. This chart shows $1,000 invested at five different annually compounded rates. Observe the following after 25 years:

- The investment at 9% compounded annually has a balance of $8,623.08 compared to $2,093.78 for the 3% compounded annually. Therefore, the interest earned is $7,623.08 versus $1,093.78, reflecting a ratio approximating 7 : 1.
- Comparing the 15% interest rate versus the 6% interest rate, many people imagine the final balances would be in the same ratio as the ratio of the rates, 15% : 6%, which simplifies to 2½ : 1. However, in the chart, the final balances are $32,918.95 versus $4,291.87, reflecting a ratio approximating 7⅔ : 1!

These examples illustrate that in compound interest scenarios, the ratio of interest earned is not directly proportionate to the numerical ratio of the interest rates. Higher interest rates result in higher interest amounts, therefore yielding more principal upon which future interest is earned. Over a long time period, this growth in the balance is exponential.

How Often Interest Compounds

The examples so far have involved compound interest that has been compounded annually—the accrued interest is being converted to principal at the end of every year. But this is not the only option. Interest can be converted to principal at any frequency, including daily, weekly, monthly, quarterly (every three months), or semi-annually (every six months). Under any of these options the principal increases more frequently, which in turn results in more interest being earned.

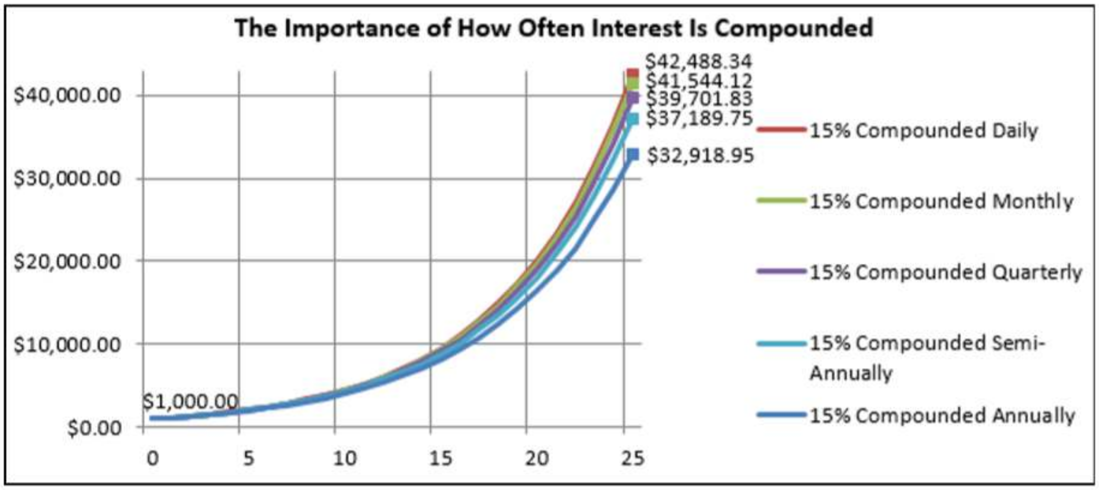


Figure $9.1.3$

The chart in the figure above illustrates the impact of different compounding frequencies. In the case of monthly versus annual compounding, the monthly compound increases the account balance every month whereas the annual compound increases the balance once every 12 months. To understand why the monthly compound earns more interest, note that after each month the principal increases and therefore earns more interest. Each of the 12 increases is small on its own, but the cumulative effect is large: by having the interest deposited every month instead of every year, over the course of 25 years an additional $\$41,544.12 - \$32,918.95 = \$8,625.17$ in interest is earned!

Paths To Success

The table below summarizes the desirable characteristics when either investing or borrowing.

Investing	Characteristic	Borrowing
Compound	**Type of Interest**	**Simple**
You want your interest to earn you more interest.		You don't want interest being converted to principal.
⇑ The higher the rate, the more you earn.	**Interest Rate**	⇓ The lower the rate, the less you pay.
⇑ The more often it compounds, the more principal that can earn interest. A daily compound would be best!	**Compounding**	⇓ The less often it compounds, the less principal that can earn interest. An annual compound would be best!

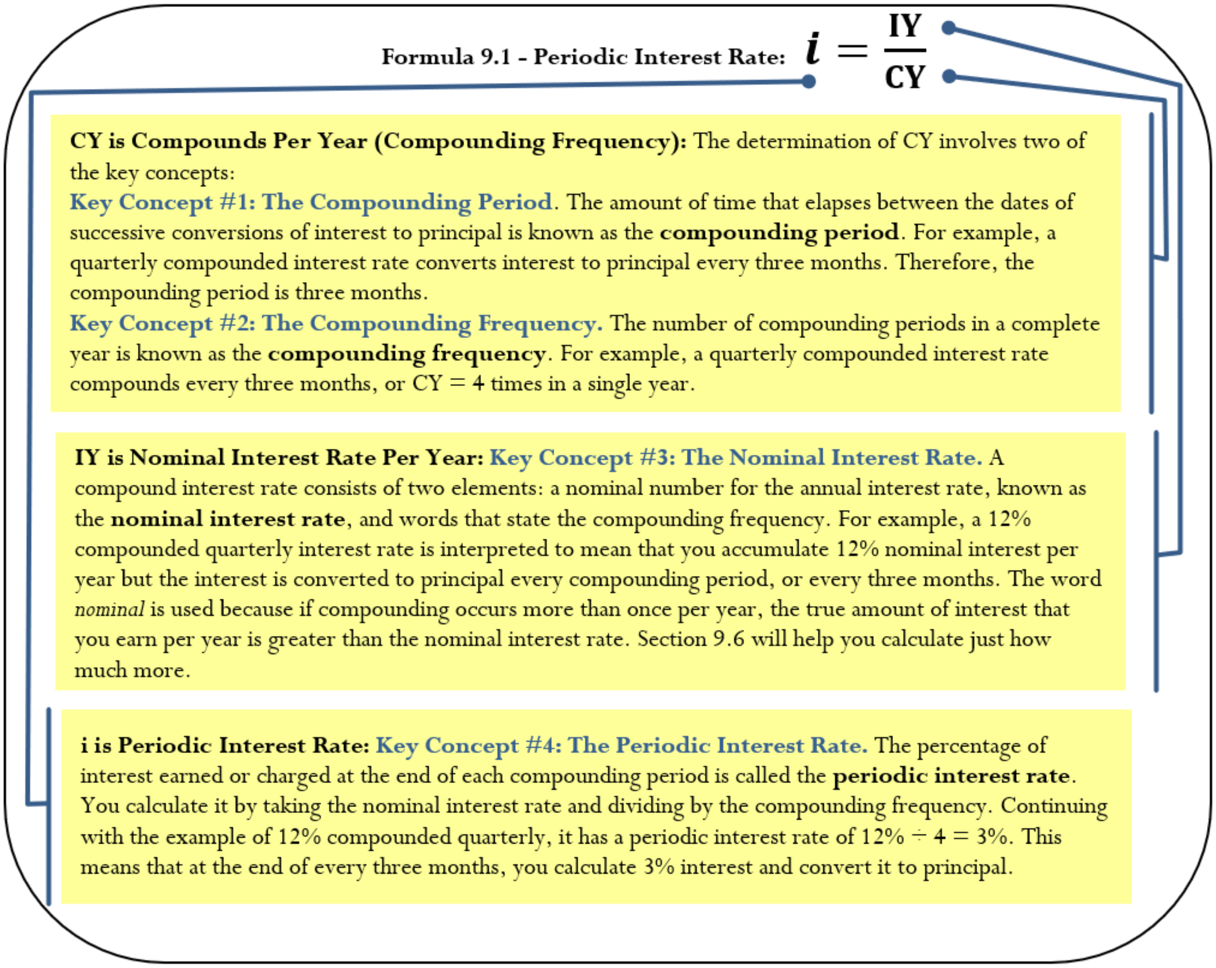
Investing	Characteristic	Borrowing
⇑ As the principal continually grows, you earn more interest. Longer time frames are desirable.	**Time (or Term)**	⇓ You don't want the principal to grow and earn more interest. Short time frames are desirable.

Calculating the Periodic Interest Rate

The first step in learning about investing or borrowing under compound interest is to understand the interest rate used in converting interest to principal. You commonly need to convert the posted interest rate to find the exact rate of interest earned or charged in any given time period.

The Formula

Formula 9.1 involves four key concepts, which are explained next, to do with conversion of the interest rate.



Formula 9.1 - Periodic Interest Rate: $i = \frac{IY}{CY}$

CY is Compounds Per Year (Compounding Frequency): The determination of CY involves two of the key concepts:
Key Concept #1: The Compounding Period. The amount of time that elapses between the dates of successive conversions of interest to principal is known as the **compounding period**. For example, a quarterly compounded interest rate converts interest to principal every three months. Therefore, the compounding period is three months.
Key Concept #2: The Compounding Frequency. The number of compounding periods in a complete year is known as the **compounding frequency**. For example, a quarterly compounded interest rate compounds every three months, or CY = 4 times in a single year.

IY is Nominal Interest Rate Per Year: Key Concept #3: The Nominal Interest Rate. A compound interest rate consists of two elements: a nominal number for the annual interest rate, known as the **nominal interest rate**, and words that state the compounding frequency. For example, a 12% compounded quarterly interest rate is interpreted to mean that you accumulate 12% nominal interest per year but the interest is converted to principal every compounding period, or every three months. The word *nominal* is used because if compounding occurs more than once per year, the true amount of interest that you earn per year is greater than the nominal interest rate. Section 9.6 will help you calculate just how much more.

i is Periodic Interest Rate: Key Concept #4: The Periodic Interest Rate. The percentage of interest earned or charged at the end of each compounding period is called the **periodic interest rate**. You calculate it by taking the nominal interest rate and dividing by the compounding frequency. Continuing with the example of 12% compounded quarterly, it has a periodic interest rate of 12% ÷ 4 = 3%. This means that at the end of every three months, you calculate 3% interest and convert it to principal.

Formula 9.1

How It Works

To solve any question involving the periodic interest rate, follow these steps:

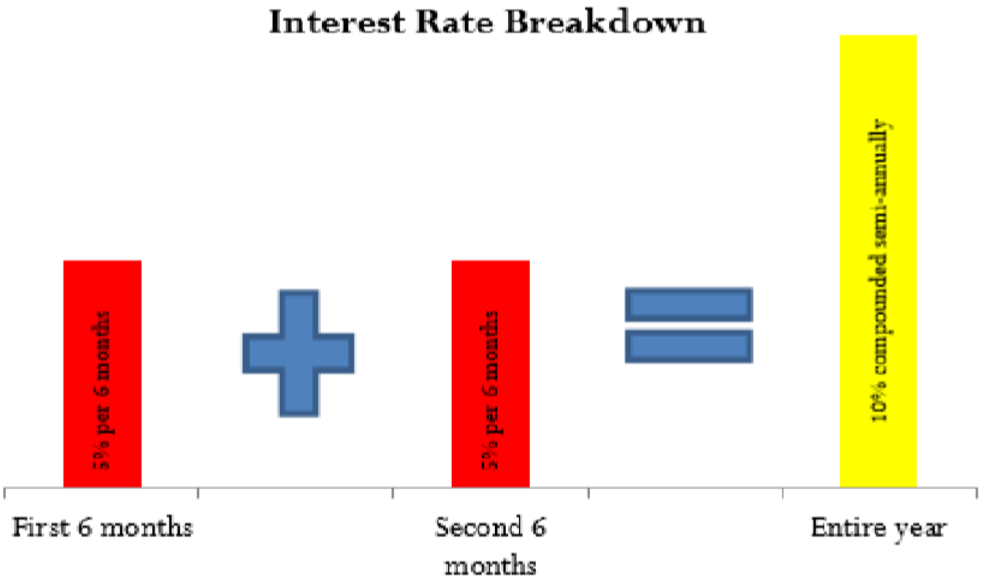


Figure $9.1.4$

Step 1: Identify two of the three key variables—the nominal interest rate (IY), compounding frequency (CY), or periodic interest rate (i).

Step 2: Substitute into Formula 9.1, rearranging if needed, and solve for the unknown variable. If you calculate the compound frequency (CY), you must convert the number back into the compounding words associated with the frequency. For example, CY = 2 means twice per year and is stated as "semi-annually."

An example involving the calculation of the periodic interest rate for "10% compounded semi-annually" illustrates these steps.

Step 1: The wording "semi-annually" means the compounding period is every six months. One year contains two such compounding periods, making the compounding frequency twice per year, or CY = 2. The nominal annual interest rate is 10%, or IY = 10%.

Step 2: Applying Formula 9.1, calculate the periodic interest rate as $i = 10\% \div 2 = 5\%$. Figure $9.1.4$ illustrates that every six months 5% interest is converted to principal. You could look at the nominal interest rate for the year, 10%, as the total of the periodic interest rates for each of the two periods. But remember, unlike with simple interest, the actual interest amount here will increase from one period to the next.

Important Notes

When you express a compound interest rate, you must always state the words for the compounding frequency along with the nominal annual number. For example, you must say "10% compounded semi-annually" and not just "10%." In the absence of an explicit compounding frequency, the number is interpreted by default to mean "compounded annually" except if an industry standard dictates otherwise. For example, in the case of mortgages in Canada, the default is semi-annual compounding, so when you hear of a 10% mortgage, you should assume it to be 10% compounded semi-annually. But in most industries "10%" with no words generally means "10% compounded annually."

Things To Watch Out For

It is common to confuse the compounding period and the compounding frequency. The table below shows the relationship between compounding periods and frequencies. Remember that to calculate the periodic interest rate you need the compounding frequency, not the compounding period.

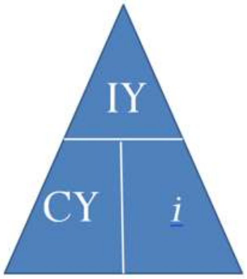
Common Compounds	Compounding Period	Compounding Frequency (CY)
Annually	Every year	1
Semi-annually	Every 6 months	2

Common Compounds	Compounding Period	Compounding Frequency (CY)
Quarterly	Every 3 months	4
Monthly	Every 1 month	12
Weekly	Every 1 week	52*
Daily	Every day	365*

*Note that although there are not exactly 52 weeks in a year and there are 366 days in a leap year, it is common practice to use these values for compounding frequency.

Paths To Success

For ease of manipulation, you can remember the rearrangement of Formula 9.1 using the triangle technique once again. Recall that as long as you know any two pieces of the puzzle, you can solve for the third.



? Exercise 9.1.1: Give It Some Thought

a. If you are investing money, what interest characteristics should you seek out?
b. If you are borrowing money, what interest characteristics should you seek out?
c. Of 10% compounded semi-annually or 10% compounded monthly, which earns more interest?
d. If you invested a principal at the same compounded interest rate for 20 years, would the amount of interest earned in the second 10 years be less than twice, exactly twice, or more than twice the interest earned during the first 10 years?
e. John has an interest rate of 4% compounded annually while Dwayne is earning 8% compounded annually. If both people placed the same principal into their investments for 25 years, would Dwayne have twice, less than twice, or more than twice as much money as John?
f. Of 10% simple interest or 10% interest compounded annually, which earns more interest over time?
g. Of the periodic interest rate or the nominal interest rate, which is always at least as large as the other?

Answer

a. High interest rate, high compounding frequency, long time frames, compound interest.
b. Low interest rate, low compounding frequency, short time frames, simple interest.
c. 10% monthly, as demonstrated in Figure 9.1.3.
d. More than twice, as demonstrated in Figure 9.1.2.
e. More than twice, as demonstrated in Figure 9.1.2.
f. 10% compounded annually, as demonstrated in Figure 9.1.1.
g. The nominal interest rate.

✓ Example 9.1.1: The Periodic Interest Rate

Calculate the periodic interest rate for the following nominal interest rates:

a. 9% compounded monthly
b. 6% compounded quarterly

Solution

For each question, calculate the periodic interest rate (i).

What You Already Know

Step 1:

For each question, use the following nominal interest rate and compounding frequency:

a. IY = 9%; CY = monthly = 12 times per year
b. IY = 6%; CY = quarterly = 4 times per year

How You Will Get There

Step 2:

For each question apply Formula 9.1.

Perform

a. $i = \frac{9\%}{12} = 0.75\%$ per month
b. $i = \frac{6\%}{4} = 1.5\%$ per quarter

Present

a. Nine percent compounded monthly is equal to a periodic interest rate of 0.75% per month. This means that interest is converted to principal 12 times throughout the year at the rate of 0.75% each time.
b. Six percent compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. This means that interest is converted to principal 4 times (every three months) throughout the year at the rate of 1.5% each time.

✓ Example $9.1.2$: The Nominal Interest Rate

Calculate the nominal interest rate for the following periodic interest rates:

a. $0.58\overline{3}\%$ per month
b. 0.05% per day

Solution

For each question, calculate the nominal interest rate (IY).

What You Already Know

Step 1:

For each question, use the following periodic interest rate and compounding frequency:

a. $i = 0.58\overline{3}\%$; CY = monthly = 12 times per year
b. $i = 0.05\%$; CY = daily = 365 times per year

How You Will Get There

Step 2:

For each question, apply Formula 9.1 and rearrange for IY.

Perform

a. $0.58\overline{3}\% = \frac{1Y}{12}; IY = 0.58\overline{3}\% \times 12 = 7\%$ compounded monthly
b. $0.05\% = \frac{IY}{365}; IY = 0.05\% \times 365 = 18.25\%$ compounded daily

Present

a. A periodic interest rate of $0.58\overline{3}\%$ per month is equal to a nominal interest rate of 7% compounded monthly.
b. A periodic interest rate of 0.05% per day is equal to a nominal interest rate of 18.25% compounded daily.

✓ Example $9.1.3$: Compounds per Year

Calculate the compounding frequency for the following nominal and periodic interest rates:

a. nominal interest rate = 6%, periodic interest rate = 3%
b. nominal interest rate = 9%, periodic interest rate = 2.25%

Solution

For each question, calculate the compounding frequency (CY) and convert the calculated number to words.

What You Already Know

Step 1:

For each question, use the following nominal and periodic interest rates:

a. IY = 6%; i = 3%
b. IY = 9%; i = 2.25%

How You Will Get There

Step 2:

For each question, apply Formula 9.1 and rearrange for CY.

Perform

a. $3\% = \dfrac{6\%}{CY}; CY = \dfrac{6\%}{3\%} = 2$ compounds per year = semi-annually
b. $2.25\% = \dfrac{9\%}{CY}; CY = \dfrac{9\%}{2.25\%} = 4$ compounds per year = quarterly

Present

a. For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency must be twice per year, which means a compounding period of every six months, or semi-annually.
b. For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency must be four times per year, which means a compounded period of every three months, or quarterly.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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9.2: Determining the Future Value

Your company has an employee assistance plan through which employees can borrow funds at 12% compounded semi-annually, much like a loan from a bank, then repay the money at their convenience. An employee who borrowed $4,000 two years ago has been unable to repay the loan until today. As the human resources manager in charge of the assistance plan, you must tell him the exact amount he needs to pay to return his balance to zero. How do you do this?

Now that you know how to calculate the periodic interest rate, you can compute compound interest. This chapter focuses on single amounts, also called lump-sum amounts. This knowledge will form the basis for you to work with compound interest on a series of payments, which will be covered in Chapter 11 through Chapter 15. Investing the time now to understand these fundamentals will reap dividends as you proceed in your studies of business mathematics.

Future Value Calculations with No Variable Changes

The simplest future value scenario for compound interest is for all of the variables to remain unchanged throughout the entire transaction. To understand the derivation of the formula, continue with the opening scenario. If the money was borrowed two years ago, then the employee will owe two years of compound interest in addition to the original principal of $4,000. That means PV = $4,000. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i=\dfrac{12\%}{2}=6\%$. Therefore, after the first six months, your employee has 6% interest converted to principal. This a future value, or FV, calculated as follows:

Principal after one compounding period (six months) = Principal plus interest

$$\begin{aligned}FV&=PV+i(PV)\\&=\$4,000+0.06(\$4,000)\\&=\$4,000+\$240=\$4,240\end{aligned}$$

Now proceed to the next six months. The future value after two compounding periods (one year) is calculated in the same way. Note that the equation $FV=PV+i(PV)$ can be factored and rewritten as $FV=PV(1+i)$.

FV (after two compounding periods) = $PV(1+i)=\$4,240(1+0.06)=\$4,240(1.06)\ =\$4,494.40$

Since the $PV=\$4,240$ is a result of the previous calculation where $PV(1+i)=\$4,240$, the following algebraic substitution is possible:

$$FV=PV(1+i)(1+i)=\$4,000(1.06)(1.06)=\$4,240(1.06)=\$4,494.40$$

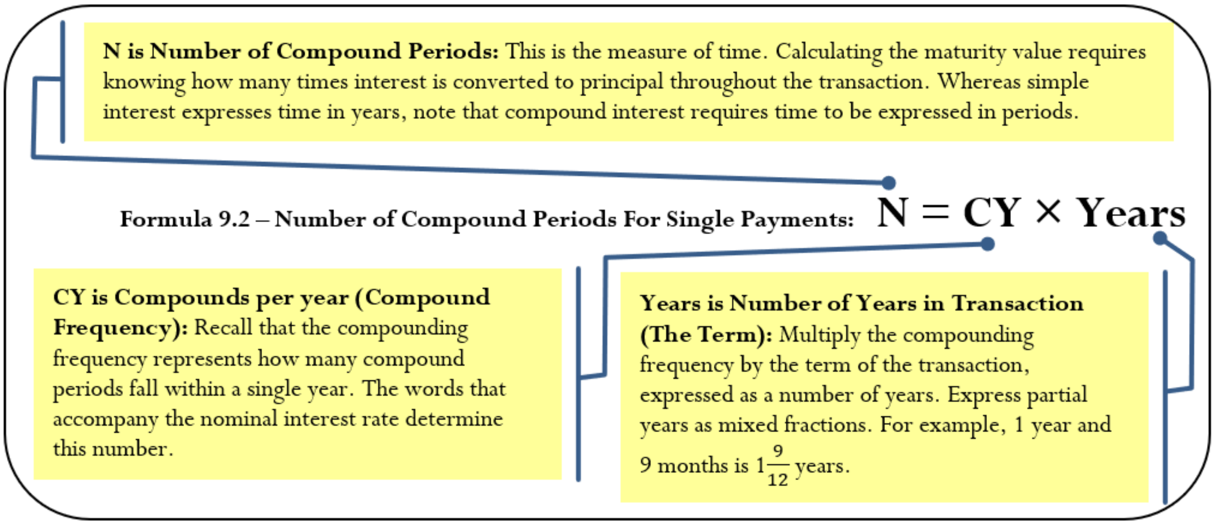
Applying exponent rules from Section 2.4 and simplifying it algebraically, you get:

$$FV=PV\,(1+i)\,(1+i)=PV\,(1+i)^2$$

Do you notice a pattern? With one compounding period, the formula has only one $(1+i)$. With two compounding periods involved, it has two factors of $(1+i)$. Each successive compounding period multiplies a further $(1+i)$ onto the equation. This makes the exponent on the $(1+i)$ exactly equal to the number of times that interest is converted to principal during the transaction.

The Formula

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 9.2 to determine the number of compound periods involved in the transaction



N is Number of Compound Periods: This is the measure of time. Calculating the maturity value requires knowing how many times interest is converted to principal throughout the transaction. Whereas simple interest expresses time in years, note that compound interest requires time to be expressed in periods.

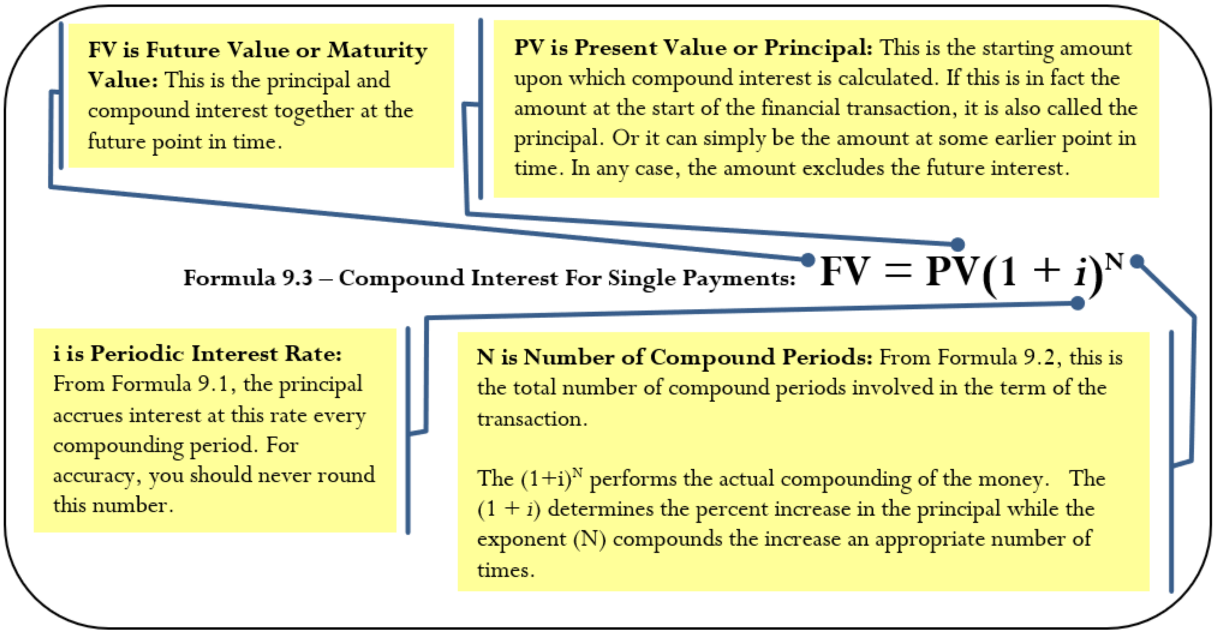
Formula 9.2 – Number of Compound Periods For Single Payments: $N = CY \times Years$

CY is Compounds per year (Compound Frequency): Recall that the compounding frequency represents how many compound periods fall within a single year. The words that accompany the nominal interest rate determine this number.

Years is Number of Years in Transaction (The Term): Multiply the compounding frequency by the term of the transaction, expressed as a number of years. Express partial years as mixed fractions. For example, 1 year and 9 months is $1\frac{9}{12}$ years.

Formula 9.2

Once you know N, substitute it into Formula 9.3, which finds the amount of principal and interest together at the end of the transaction, or the maturity value.



FV is Future Value or Maturity Value: This is the principal and compound interest together at the future point in time.

PV is Present Value or Principal: This is the starting amount upon which compound interest is calculated. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time. In any case, the amount excludes the future interest.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1+i)^N$

i is Periodic Interest Rate: From Formula 9.1, the principal accrues interest at this rate every compounding period. For accuracy, you should never round this number.

N is Number of Compound Periods: From Formula 9.2, this is the total number of compound periods involved in the term of the transaction.

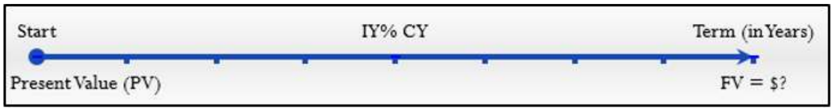
The $(1+i)^N$ performs the actual compounding of the money. The $(1+i)$ determines the percent increase in the principal while the exponent (N) compounds the increase an appropriate number of times.

Formula 9.3

How It Works

Follow these steps to calculate the future value of a single payment:

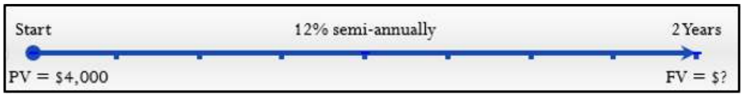
Step 1: Read and understand the problem. If necessary, draw a timeline similar to the one here identifying the present value, the nominal interest rate, the compounding, and the term.



Step 2: Calculate the periodic interest rate (i) from Formula 9.1.

Step 3: Calculate the total number of compound periods (N) from Formula 9.2.

Step 4: Solve Formula 9.3. Revisit the employee who had \$4,000 outstanding for two years with interest at 12% compounded semi-annually.



Step 1: Calculate the amount of the loan after two years (FV). Observe that PV = \$4,000, IY = 12%, CY = 2 (every six months or twice per year), and Years = 2.

Step 2: According to Formula 9.1, $i=\frac{12\%}{2}=6\%$. Thus, interest at a rate of 6% is converted to principal at the end of each compounding period of six months.

Step 3: Applying Formula 9.2, $N=CY\times \text{ Years }=2\times 2=4$. Interest is converted to principal four times over the course of the two-year term occurring at the 6, 12, 18, and 24 month marks.

Step 4: Calculate the maturity value using Formula 9.3:

$$FV=\$4,000(1+0.06)^4=\$5,049.91$$

To pay off the loan the employee owes \$5,049.91.

Important Notes

Calculating the Interest Amount:

In any situation of lump-sum compound interest, you can isolate the interest amount using an adapted Formula 8.3:

$$I=S-P \text{ becomes } I=FV-PV$$

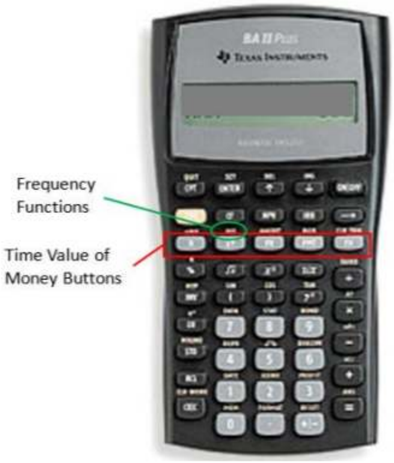
Using your employee's \$4,000 loan with a future repayment of \$5,049.91, the interest paid is calculated as

$$I=\$5,049.91-\$4,000.00=\$1,049.91$$

This formula applies only to compound interest situations involving lump-sum amounts. If regular payments are involved, this is called an annuity, for which a modified version of the interest formula will be introduced in Chapter 11.

The BAII Plus Calculator:

Your BAII Plus calculator is a business calculator pre-programmed with compound interest formulas. These functions are called the “time value of money” buttons. The compound interest buttons are found in two areas of the calculator, as shown in the photo.



1. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row’s buttons are different in color from the rest of the buttons on the keypad. The table on the next page relates each button to formula symbols used in this text along with what each button represents and any data entry

requirements.To enter any information into any one of these buttons, key in the data first and then press the appropriate button. For example, if you want to enter N = 34, then key in 34 followed by pressing N.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
N	N	The number of compounding periods (from Formula 9.2)	An integer or decimal number; no negatives
I/Y	IY	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	PV	Present value or principal	An integer or decimal number
PMT	PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations	An integer or decimal number
FV	FV	Future value or maturity value	An integer or decimal number

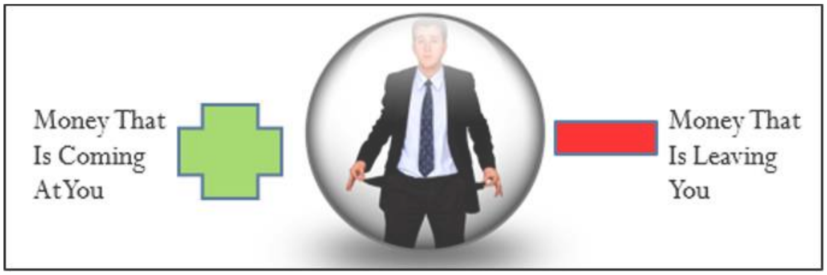
2. Since frequency and interest rates are related, as shown in Formula 9.1, the frequency function is logically placed above the I/Y button and it is labelled P/Y. This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing 2nd P/Y to find the entry fields shown in the table below, which you can scroll through using the i and h arrow buttons. To exit the P/Y window, press 2nd Quit.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
P/Y	PY	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y	A positive, nonzero number only
C/Y	CY	Compounds per year (compounding frequency)	A positive, nonzero number only

The table relates each window variable to the formula symbols along with what each button represents and its data entry requirements. To enter information into these variables, key in the data first and then press Enter. For example, to enter a compounding frequency of 4, press 4 while C/Y is on your screen and then press Enter. Most commonly the P/Y and C/Y are the same number, as demonstrated in later chapters. Therefore, the calculator's built-in shortcut feature automatically copies any value entered into the P/Y variable to the C/Y variable. If the P/Y and C/Y are not the same, you can scroll down and re-enter the C/Y as needed.

BAII Plus Cash Flow Sign Convention:

An investment and a loan are very different. An investment earns interest and the principal increases over time. A loan is charged interest but is usually paid off through payments, resulting in the principal decreasing over time. Notice that nowhere on the calculator is there a button to enter this critical piece of information. How does the calculator distinguish between the two? You must apply a principle called the cash flow sign convention to the PV, PMT, and FV buttons:



1. If you have money leaving your possession and going somewhere else (such as being put into an investment or making a payment against a loan), you must enter the number as a NEGATIVE number.
2. If you have money coming into your possession and you are receiving it (such as borrowing money from the bank or having an investment mature and pay out to you), you must enter the number as a POSITIVE number.

When doing financial calculations it is important to “be somebody” in the transaction. In any compound interest scenario, two parties are always involved—somebody is investing and somebody is borrowing. From this standpoint, think about whether the money leaves you or comes at you. This will help you place the correct sign in front of the PV, PMT, and FV when using your calculator. Who you are will not change the numbers of the transaction, just the cash flow sign convention.

1. If you borrow money from your friend and then pay it back, the initial loan is received by you and hence entered as a positive PV for you. As you pay back the loan, money leaves you and therefore the FV is negative for you.
2. Taking the other side of the coin and being your friend, the loaning of money is a negative PV for him. When you repay the loan, your friend receives it and therefore results in a positive FV for him.

Notice that the PV and FV always have opposite signs. Investments must always mature, providing a payback to the investor. Loans must always be repaid.

BAII Plus Memory:

Your calculator has permanent memory. Once you enter data into any of the time value buttons it is permanently stored until

- You override it by entering another piece of data and pressing the button;
- You clear the memory of the time value buttons by pressing 2nd CLR TVM before proceeding with another question; or
- The reset button on the back of the calculator is pressed.

This permanent storage is an advantage in multistep compound interest problems. If any piece of information remains constant from step to step, you need to enter it only once.

BAII Plus Keying in a Question:

Seven variables are involved with compound interest. To solve any compound interest question, you must key in six of them. To solve for the missing variable, press CPT followed by the variable. For example, if solving for future value, press CPT FV.

Returning to the employee loan example where a $4,000 loan was taken for two years at 12% interest compounded semi-annually, recall that N = 4, I/Y = 12, PV = $4,000, and CY = 2. Assuming the role of the employee, you would key in the problem in the following sequence:

- 2nd CLR TVM (clear the memory; this is not required, but it reduces the chance of using stale data)
- 4 N (total number of compounding periods)
- 12 I/Y (the nominal interest rate)
- 4000 PV (a positive since the employee received the money from the company)
- 0 PMT (you will not use this button until you get to annuities in Chapter 11)
- 2nd P/Y (to open the frequency window)
- 2 Enter (the P/Y and C/Y are the same when working with lump-sum amounts; by entering the C/Y here, you simultaneously set both variables to the same number)
- 2nd Quit (to close the window)
- CPT FV

Answer: -5,049.91 (it is negative since the employee owes this money)

Things To Watch Out For

The most common error in the application of Formula 9.3 is to substitute the nominal interest rate for the periodic interest rate. Hence, for 12% compounded semi-annually you might inadvertently use a mistaken value of i = 0.12 instead of the appropriate i = 0.06. Formula 9.3 encompasses Formula 9.1 and Formula 9.2. If you write Formula 9.3 without requiring any substitution it appears as follows:

$$FV = PV\left(1+\frac{IY}{CY}\right)^{(CY\times\text{Years})}$$

Although you could use this equation instead of the one presented in Formula 9.3, most students find it best to use the sequence of three formulas. This requires the systematic approach involved in steps 2 to 4 of the process:

1. Calculate i.
2. Calculate N.
3. Calculate FV.

Use the phrase “**iN** the **F**uture” to remember this process.

Paths To Success

When you compute solutions on the BAII Plus calculator, one of the most common error messages displayed is "Error 5." This error indicates that the cash flow sign convention has been used in a manner that is financially impossible. Some examples of these financial impossibilities include loans with no repayment or investments that never pay out. In these cases, the PV and FV have been incorrectly set to the same cash flow sign.

? Exercise 9.2.1: Give It Some Thought

Formula 9.3, $FV = PV(1+i)^N$, places the number of compound periods into the exponent.

Answer

1. The 8% compounded monthly investment realizes 60 compound periods of interest over the five years, while the 8% compounded annually investment realizes only five compound periods. With interest going in much more often, the principal is increasingly larger and therefore earns more interest.
2. Each compound period increases the existing principal by the periodic rate through multiplication. Since the periodic interest rate in each compound period multiplies onto the existing principal amount, you apply exponent rules for adding exponents with the same base. The total number of increases in principal equals the number of compound periods in the transaction. See pages xxx–xxx for a full explanation.

✓ Example 9.2.1: Making an Investment

If you invested $5,000 for 10 years at 9% compounded quarterly, how much money would you have?

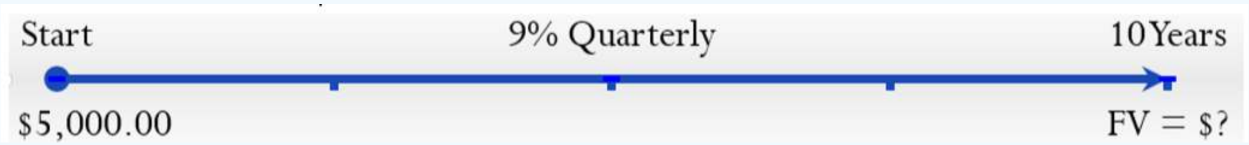
Solution

Calculate the principal and the interest together, which is called the maturity value (FV).

What You Already Know

Step 1:

The principal, interest rate, and term, as illustrated in the timeline, are known.



How You Will Get There

Step 2:

Calculate the periodic interest by applying Formula 9.1.

Step 3:

Calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the maturity value by applying Formula 9.3.

Perform

Step 2:

$$i=\frac{9\%}{4}=2.25\%=0.0225$$

Step 3:

$$N=4\times 10=40$$

Step 4:

$$FV=\$5,000(1+0.0225)^{40}=\$12,175.94$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
40	9	-5,000	0	Answer: $ 12,175.94	4	4

After 10 years, the principal grows to $12,175.94, which includes your $5,000 principal and $7,175.94 of compound interest.

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? Formula 9.3 produces the correct final answer only when all variables remain unchanged. To illustrate this situation, assume your company modified its employee assistance plan one year after the money was borrowed, changing the interest rate in the second year from 12% compounded semi-annually to 12% compounded quarterly. Now how do you calculate the future value?

When any variable changes, you must break the timeline into separate time fragments at the point of the change. This timeline format is similar to those used in Section 8.1, involving variable simple interest rates. The timeline illustrates the employee's new scenario.

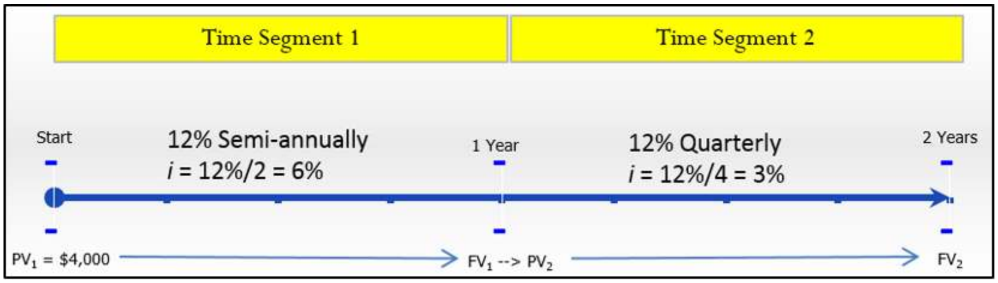


Figure 9.2.3

Solving for the unknown FV on the right of the timeline means that you must start at the left side of the timeline. To arrive at the solution, you need to work from left to right one time segment at a time using Formula 9.3. As noted on the timeline, at the one-year point the future value of the first time segment then becomes the present value for the second time segment since the interest is not just accrued but actually placed into the account.

How It Works

Follow these steps when variables change in calculations of future value based on lump-sum compound interest:

Step 1: Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using Formula 9.1.

Step 3: For each time segment, calculate the total number of compound periods (N) using Formula 9.2.

Step 4: Starting with the present value in the first time segment (starting on the left), solve Formula 9.3.

Step 5: Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.

Step 6: Using Formula 9.3, calculate the future value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the final future value from the final time segment.

In the employee's new situation, he has borrowed $4,000 for two years with 12% compounded semi-annually in the first year and 12% compounded quarterly in the second year.

Step 1: Figure $9.2.3$ shows a timeline. In time segment one, $PV_1 = \$4,000$, $IY = 12\%$, $CY = 2$, and the time segment is one year long. In time segment two, the only change is $CY = 4$.

Step 2: In the first time segment, the periodic interest rate is $i_1 = 12\%/2 = 6\%$. In the second time segment, the periodic interest rate is $i_2 = 12\%/4 = 3\%$.

Step 3: The first time segment is one year long; therefore, $N_1 = 2 \times 1 = 2$. The second time segment is also one year long; therefore, $N_2 = 4 \times 1 = 4$.

Step 4: Apply Formula 9.3 to the first time segment:

$$FV_1 = PV\,(1+i_1)^{N1} = \$4,000(1+0.06)^2 = \$4,494.40$$

Step 5: Let $FV_1 = \$4,494.40 = PV_2$.

Step 6: Apply Formula 9.3 to the second time segment:

$$FV_2 = PV_2\,(1+i_2)^{N2} = \$4,494.40(1+0.03)^4 = \$4,494.40 \times 1.03^4 = \$5,058.49$$

Step 7: You reach the end of the timeline. The employee needs to repay $5,058.49 to clear the loan.

To perform the above steps on the calculator:

Segment	N	I/Y	PV	PMT	FV	C/Y	P/Y
1	2	12	4000	0	Answer: -4,494.40	2	2
2	4	√	4494.40	√	Answer: -5,058.49	4	4

Note: A ✓ that the value is already entered and does not need to be re-keyed into the calculator.

Important Notes

The BAII Plus Calculator:

Transforming the future value from one time segment into the present value of the next time segment does not require re-entering the computed value. Instead, apply the following technique:

1. Load the calculator with all known compound interest variables for the first time segment.
2. Compute the future value at the end of the segment.

3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the ± key, and then store the unrounded number back into the present value button by pressing PV. Change the N, I/Y, and C/Y as required for the next segment.
4. Return to step 2 for each time segment until you have completed all time segments.

Things To Watch Out For

When you draw timelines, it is critical to recognize that any change in any variable requires a new time segment. This applies to changes in principal, the nominal interest rate, or the compounding frequency.

Paths To Success

Compound interest on a single payment is linked to the concept of percent change in Section 3.1. Every time interest is compounded and added to the principal, the periodic rate is the percent change in the principal. To use the percent change function on the calculator, assign $\text{Old} = PV$, $\text{New} = FV, \Delta\% = i$, and $\#PD = N$. The following two-step sequence shows the percent change function applied to the working example of the employee's new loan situation.

Segment	OLD	NEW	%CH	#PD
1	4000	Answer: $4,494.40	6	2
2	4494.40	Answer: $5,058.49	3	4

This is an alternative way to use the calculator to compute the future value of lump-sum amounts using compound interest.

? Exercise $9.2.2$: Give It Some Thought

a. Which of the following investments involving the same principal results in the highest maturity value? Assume equal terms at each interest rate.
   i. 8% compounded annually followed by 6% compounded semi-annually
   ii. 8% compounded semi-annually followed by 6% compounded semi-annually
   iii. 8% compounded monthly followed by 6% compounded quarterly
   iv. 8% compounded annually followed by 6% compounded quarterly
b. How many time segments are involved in an investment where the successive interest rates are 9% compounded quarterly, 9% compounded monthly, 8% compounded monthly, and 9% compounded quarterly?

Answer

a. (iii) results in the largest maturity value. It has the highest compounding frequency in the first time segment and is tied for the highest compounding frequency in the second time segment.
b. Four time segments are involved. Each change of interest rate requires a new time segment.

✓ Example $9.2.2$: Delaying a Facility Upgrade

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of $48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first 1½ years, 7% compounded semi-annually for the following 2½ years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

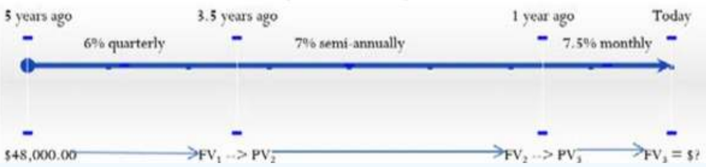
Solution

Take the original quote and move it into the future with the price increases. You can view this as a single lump sum with multiple successive interest rates. The amount of money needed today is the maturity amount (FV).

What You Already Know

Step 1:

The timeline below shows the original quote from five years ago until today.



The principal, terms, and interest rates are known: PV_1 = $48,000

First time segment: IY = 6%; CY = quarterly = 4 Term = 1½ years

Second time segment: IY = 7%; CY = semi-annually = 2 Term = 2½ years

Third time segment: IY = 7.5%; CY = monthly = 12 Term = 1 year

How You Will Get There

Step 2:

For each time segment, calculate the periodic interest rate by applying Formula 9.1.

Step 3:

For each time segment, calculate the number of compound periods by applying Formula 9.2.

Step 4/5:

Calculate the future value of the first time segment using Formula 9.3. Let $FV_1 = PV_2$.

Step 6/5:

Calculate the future value of the second time segment using Formula 9.3. Let $FV_2 = PV_3$. Repeat

Step 6/7:

Calculate the future value of the third time segment using Formula 9.3. FV_3 is the final future value amount.

Perform

Step 2:

First segment: $i = \frac{6\%}{4} = 1.5\%$

Second segment: $i = \frac{7\%}{2} = 3.5\%$

Third segment: $i = \frac{7.5\%}{12} = 0.625\%$

Step 3:

First segment: $N = 4 \times 1\frac{1}{2} = 6$

Second segment: $N = 2 \times 2\frac{1}{2} = 5$

Third segment: $N = 12 \times 1 = 12$

Step 4/5:

First segment: $FV_1 = \$48,000 \times (1+0.015)^6 = \$52,485.27667$

Step 6/5:

Second segment: $FV_2 = \$52,485.27667 \times (1+0.035)^5 = \$62,336.04435$

Step 6/7:

Third segment: $FV_3 = \$62,336.04435 \times (1+0.00625)^{12} = \$67,175.35$

Calculator Instructions

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	6	6	48000	0	Answer: -52,485.27677	4	4
2	5	7	-52,485.27677	√	Answer: -62,336.04435	2	2
3	12	7.5	-62,336.04435	√	Answer: -67,175.35	12	12

Coast Appliances requires $67,175.35 to perform the upgrade today. This consists of $48,000 from the original quote plus $19,175.35 in price increases.

Paths To Success

When you calculate the future value of a single payment for which only the interest rate fluctuates, it is possible to find the maturity amount in a single multiplication:

$$FV = PV \times (1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_n)^{N_n}$$

where n represents the time segment number. Note that the technique in Example $9.2.2$ calculates each of these multiplications one step at a time, whereas this adaptation solves all time segments simultaneously. In the example above, you can calculate the same maturity value as follows:

$$FV = \$48,000 \times (1.015)^6 \times (1.035)^5 \times (1.0625)^{12} = \$67,175.35$$

✓ Example $9.2.3$: Making an Additional Contribution

Two years ago Lorelei placed $2,000 into an investment earning 6% compounded monthly. Today she makes a deposit to the investment in the amount of $1,500. What is the maturity value of her investment three years from now?

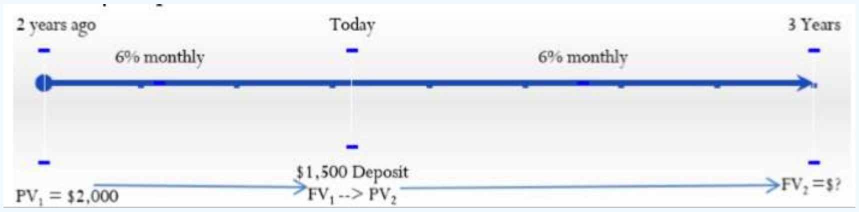
Solution

Take the original investment and move it into the future with the additional contribution. The amount of money three years from today is the maturity amount (FV).

What You Already Know

Step 1:

The timeline for this investment is below.



First time segment: PV_1 = $2,000, IY = 6%, CY = 12, Years = 2

Second time segment: Deposit = $1,500, IY = 6%, CY = 12, Years = 3

How You Will Get There

Step 2:

For each time segment, calculate the periodic interest rate by applying Formula 9.1.

Step 3:

For each time segment, calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the future value of the first time segment using Formula 9.3.

Step 5:

Let $FV_1 = PV_2$. Increase PV_2 by the additional contribution amount of \$1,500.

Step 6:

Calculate the future value of the second time segment using Formula 9.3.

Step 7:

FV_2 is the final future value amount.

Perform

Step 2:

First segment: $i = \dfrac{6\%}{12} = 0.5\%$

Second segment: $i = \dfrac{6\%}{12} = 0.5\%$

Step 3:

First segment: $N = 12 \times 2 = 24$

Second segment: $N = 12 \times 2 = 24$

Step 4:

First segment: $FV_1 = \$2,000 \times (1 + 0.005)^{24} = \$2,254.319552$

Step 5:

First segment: $\$2,254.319552 + \$1,500 = \$3,754.319552 = PV_2$

Step 6:

Second segment: $FV_2 = \$3,754.319552 \times (1 + 0.005)^{36} = \$4,492.72$

Calculator Instructions

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	24	6	-2000	0	Answer: 2,254.319552	12	12
2	36	√	-3,754.319552	√	Answer: 4,492.721092	√	√

Three years from now Lorelei will have \$4,492.72. This represents \$3,500 of principal and \$992.72 of compound interest.

✓ Example 9.2.4: What Do You Still Owe?

Ruth purchased her \$5,465 diamond ring four years ago through a payment plan offered by the jeweller. This plan did not require any regular payments within the four years; however, the balance including interest must be paid in full by the end of the fourth year to avoid a large financial penalty. During the four-year period, a quarterly compounded variable interest rate was charged. Initially, the interest was 8.75% before increasing to 9.25% after two years and then 9.75% after 3¼ years. She made two payments of \$2,000 each, the first nine months after the purchase and the second after 3¼ years. What is the balance that she must pay at the end of the four years so that she doesn't incur any financial penalties?

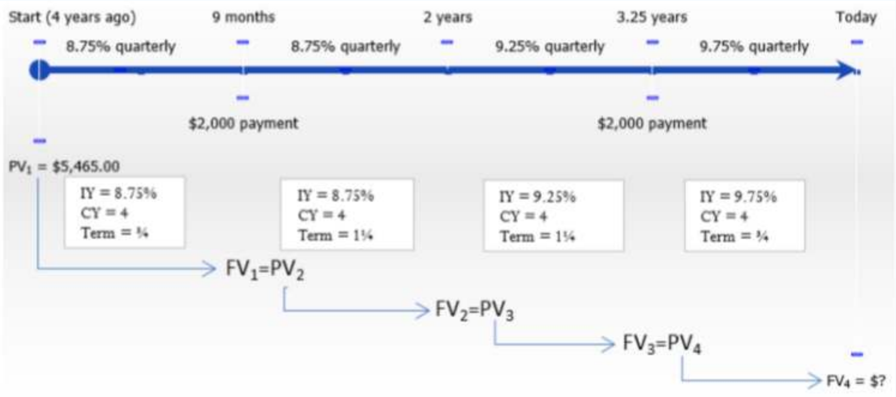
Solution

Take Ruth's initial purchase and charge it interest over the course of the four years while applying her payments to the principal at the appropriate points. Calculate the balance owing after four years; this is the future value (FV).

What You Already Know

Step 1:

The timeline is below.


Start (4 years ago)
9 months
2 years
3.25 years
Today
8.75% quarterly
8.75% quarterly
9.25% quarterly
9.75% quarterly
$2,000 payment
$2,000 payment
PV₁ = $5,465.00
IY = 8.75% CY = 4 Term = ¾
IY = 8.75% CY = 4 Term = 1¼
IY = 9.25% CY = 4 Term = 1¼
IY = 9.75% CY = 4 Term = ¾
FV₁=PV₂
FV₂=PV₃
FV₃=PV₄
FV₄ = $?

First time segment: PV_1 = $5,465, IY = 8.75%, CY = 4, Years = ¾

Second time segment: Payment = $2,000, IY = 8.75%, CY = 4, Years = 1¼

Third time segment: IY = 9.25%, CY = 4, Years = 1¼

Fourth time segment: Payment = $2,000, IY = 9.75%, CY = 4, Years = ¾

How You Will Get There

Step 2:

For each time segment, calculate the periodic interest rate by applying Formula 9.1.

Step 3:

For each time segment, calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the future value of the first time segment using Formula 9.3.

Step 5:

Let $FV_1 = PV_2$. Decrease PV_2 by the $2,000 payment.

Step 6:

Calculate the future value of the 2nd time segment using Formula 9.3.

Repeat Step 5: Let $FV_2 = PV_3$.

Repeat Step 6: Calculate the future value of the third time segment using Formula 9.3.

Repeat Step 5: Let $FV_3 = PV_4$. Decrease PV_4 by the $2,000 payment.

Repeat Step 6: Calculate the future value of the fourth time segment using Formula 9.3.

Step 7:

FV_4 is the final future value amount.

Perform

Step 2:

First segment: $i = \frac{8.75\%}{4} = 2.1875\%$

Second segment: $i = \frac{8.75\%}{4} = 2.1875\%$

Third segment: $i = \frac{9.25\%}{4} = 2.3125\%$

Fourth segment: $i = \frac{9.75\%}{4} = 2.4375\%$

Step 3:

First segment: $N = 4 \times \frac{3}{4} = 3$

Second segment: $N = 4 \times 1\frac{1}{4} = 5$

Third segment: $N = 4 \times 1\frac{1}{4} = 5$

Fourth segment: $N = 4 \times \frac{3}{4} = 3$

Step 4:

First segment: $FV_1 = \$5,465 \times (1 + 0.021875)^3 = \$5,465 \times 1.021875^3 = \$5,831.543094$

Step 5:

First segment: $\$5,831.543094 - \$2,000.00 = \$3,831.543094 = PV_2$

Step 6:

Second segment: $FV_2 = \$3,831.543094 \times (1 + 0.021875)^5 = \$4,269.358126$

Step 5-6:

Third segment: $FV_3 = \$4,269.358126 \times (1 + 0.023125)^5 = \$4,786.36782$

Step 5:

Third segment: $\$4,786.36782 - \$2,000.00 = \$2,786.36782 = PV_4$

Step 6:

Fourth segment: $FV_4 = \$2,786.36782 \times (1 + 0.024375)^3 = \$2,995.13$

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	3	8.75	5465	0	Answer: -5,831.543094	4	4
2	5	√	3831.543094	√	Answer: -4,269.358126	√	√
3	√	9.25	4269.358126	√	Answer: -4,786.36782	√	√
4	3	9.75	2786.36782	√	Answer: -2,995.13	√	√

At the end of the four years, Ruth still owes $2,995.13 to pay off her ring.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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9.3: Determining the Present Value

Should you pay your bills early? If so, what amount should be paid? From a strictly financial perspective, if you are going to pay a bill earlier than its due date, the amount needs to be reduced somehow. If not, then why pay it early?

To illustrate, assume that you just received a $3,000 bonus from your employer. Stopping at your mailbox, you pick up a large stack of envelopes that include a financial statement for your $3,000 purchase of furniture from The Brick on its three-year, no-interest, no-payments plan. Should you use your bonus to extinguish this debt? The choices are that you can either pay $3,000 today or $3,000 three years from now.

- If you pay the bill today, The Brick can then take your money and invest it themselves for the next three years. At 2.75% compounded semi-annually, The Brick earns $256.17. At the end of your agreement, The Brick then has both your $3,000 payment and the additional interest of $256.17! You might as well have just paid The Brick $3,256.17 for your furniture!
- If you invest your bonus instead and pay your bill when it is due, The Brick receives its $3,000 and you have the $256.17 of interest left over in your bank account. Clearly, this is the financially smart choice.

For The Brick to be financially equitable in its dealings with you, it must reduce any early payment by an amount such that with interest the value of your payment accumulates to the debt amount upon the maturity of the agreement. This means that The Brick should be willing to accept a payment of $2,763.99 today as payment in full for your $3,000 bill. If The Brick then invests that payment at 2.75% compounded semi-annually, it grows to $3,000 when your payment becomes due in three years.

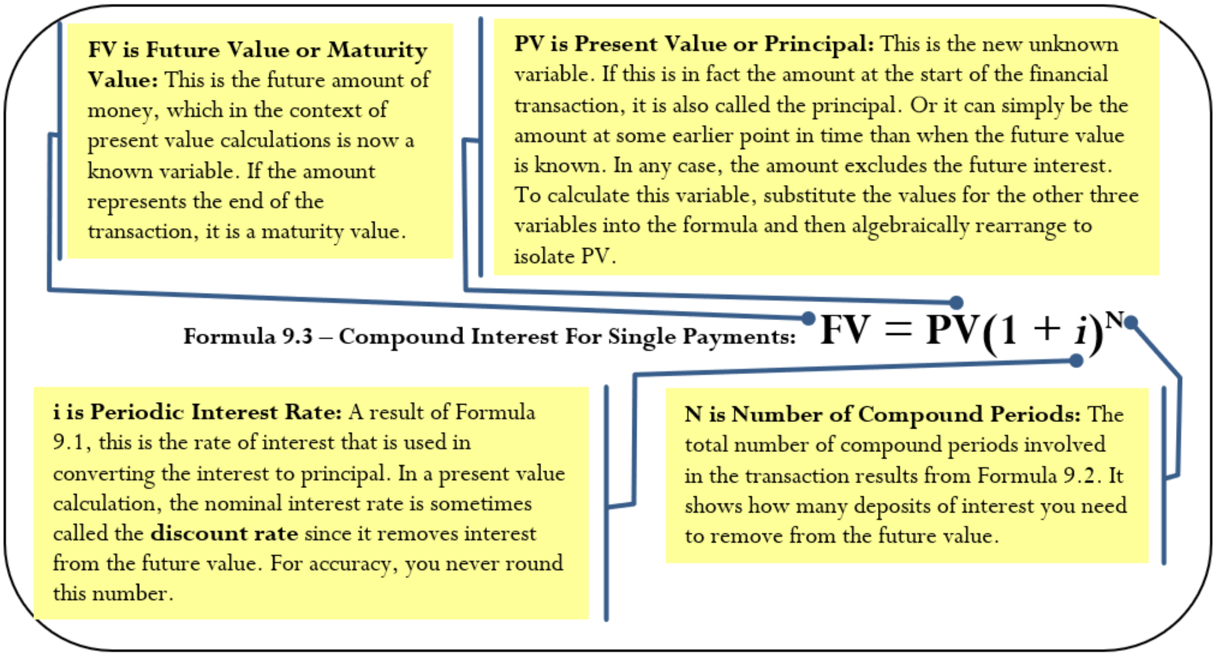
Whether you are paying bills personally or professionally, it is important to understand present value. The amount of interest to be removed from a future value needs to reflect both how far in advance the payment occurs and an equitable interest rate. In this section, you calculate the present value of a future lump-sum amount under both conditions of constant variables and changing variables.

Present Value Calculations with No Variable Changes

As in your calculations of future value, the simplest scenario for present value is for all the variables to remain unchanged throughout the entire transaction. This still involves compound interest for a single payment or lump-sum amount and thus does not require a new formula.

The Formula

Solving for present value requires you to use Formula 9.3 once again. The only difference is that the unknown variable has changed from FV to PV.



FV is Future Value or Maturity Value: This is the future amount of money, which in the context of present value calculations is now a known variable. If the amount represents the end of the transaction, it is a maturity value.

PV is Present Value or Principal: This is the new unknown variable. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time than when the future value is known. In any case, the amount excludes the future interest. To calculate this variable, substitute the values for the other three variables into the formula and then algebraically rearrange to isolate PV.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1+i)^N$

i is Periodic Interest Rate: A result of Formula 9.1, this is the rate of interest that is used in converting the interest to principal. In a present value calculation, the nominal interest rate is sometimes called the **discount rate** since it removes interest from the future value. For accuracy, you never round this number.

N is Number of Compound Periods: The total number of compound periods involved in the transaction results from Formula 9.2. It shows how many deposits of interest you need to remove from the future value.

How It Works

Follow these steps to calculate the present value of a single payment:

Step 1: Read and understand the problem. If necessary, draw a timeline identifying the future value, the nominal interest rate, the compounding, and the term.

Step 2: Determine the compounding frequency (CY) if it is not already known, and calculate the periodic interest rate (i) by applying Formula 9.1.

Step 3: Calculate the number of compound periods (N) by applying Formula 9.2.

Step 4: Substitute into Formula 9.3, rearranging algebraically to solve for the present value.

Revisiting that furniture you bought on The Brick's three-year, no-interest and no-payments plan, if the amount owing three years from now is $3,000 and prevailing market interest rates are 2.75% compounded semi-annually, what should The Brick be willing to accept as full payment today?

Step 1: The value of the payment today (PV) is required. The future value (FV) is $3,000. The nominal interest rate is $IY = 2.75\%$, and the compounding frequency of semi-annually is $CY = 2$. The term is to pay it three years early.

Step 2: The periodic interest rate is $i = 2.75\%/2 = 1.375\%$.

Step 3: The number of compounds is $N = 3 \times 2 = 6$.

Step 4: Applying Formula 9.3, $\$3,000 = PV(1+0.01375)^6 \text{ or } PV = \frac{\$3.000}{1.01375^6} = \$2,763.99$.

If The Brick will accept $2,763.99 as full payment, then pay your bill today. If not, keep your money, invest it yourself, and then pay the $3,000 three years from now while retaining all of the interest earned.

Important Notes

You use the financial calculator in the exact same manner as described in Section 9.2. The only difference is that the unknown variable is PV instead of FV. You must still load the other six variables into the calculator and apply the cash flow sign convention carefully.

Paths To Success

Did you notice the following?

- **Future Value** This calculation takes the present value and multiplies it by the interest factor. This increases the single payment by the interest earned.
- **Present Value** This calculation takes the future value and divides it by the interest factor (rearranging Formula 9.3 for PV produces $\frac{FV}{(1+t)^N} = PV$). This removes the interest and decreases the single payment.

✓ Example 9.3.1: Achieving a Savings Goal

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is $38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

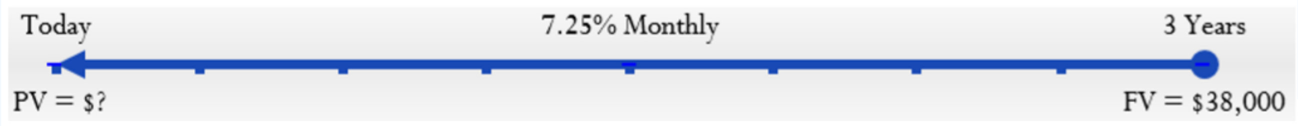
Solution

You aim to calculate the amount of principal that Castillo's must put aside today such that it can grow with interest to the desired savings goal. The principal today is the present value (PV).

What You Already Know

Step 1:

The maturity value, interest rate, and term are known: FV = $38,000; IY = 7.25%; CY = monthly = 12 times per year; Term = 3 years



How You Will Get There

Step 2:

Calculate the periodic interest by applying Formula 9.1.

Step 3:

Calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the present value by substituting into Formula 9.3 and then rearranging for PV.

Perform

Step 2:

$$i=\frac{7.25\%}{12}=0.6041\overline{6}\%=0.006041\overline{6}$$

Step 3:

$$N=12\times 3=36$$

Step 4:

$$\begin{gathered}\$38,000=PV(1+0.006041\overline{6})^{36}\\ PV=\frac{\$38,000}{(1+0.0060416)^{36}}=\$30,592.06\end{gathered}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
36	7.25	Answer: $30,592.06	0	38000	12	12

If Castillo’s Warehouse places $30,592.06 into the investment, it will earn enough interest to grow to $38,000 three years from now to purchase the forklift.

Present Value Calculations with Variable Changes

Addressing variable changes in present value calculations follows the same techniques as future value calculations. You must break the timeline into separate time segments, each of which involves its own calculations.

Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using Formula 9.3, rearranging for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.

How It Works

Follow these steps to calculate a present value involving variable changes in single payment compound interest:

Step 1: Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the segment’s length in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using Formula 9.1.

Step 3: For each time segment, calculate the total number of compound periods (N) using Formula 9.2.

Step 4: Starting with the future value in the first time segment on the right, solve Formula 9.3.

Step 5: Let the present value calculated in the previous step become the future value for the next time segment to the left. If the principal changes, adjust the new future value accordingly.

Step 6: Using Formula 9.3, calculate the present value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the present value from the leftmost time segment.

Important Notes

To use your calculator efficiently in working through multiple time segments, follow a procedure similar to that for future value:

1. Load the calculator with all the known compound interest variables for the first time segment on the right.
2. Compute the present value at the beginning of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the ± key, then store the unrounded number back into the future value button by pressing FV. Change the N, I/Y, and C/Y as required for the next segment.

Return to step 2 for each time segment until you have completed all time segments.

✓ Example 9.3.2: A Variable Rate Investment

Sebastien needs to have $9,200 saved up three years from now. The investment he is considering pays 7% compounded semi-annually, 8% compounded quarterly, and 9% compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?

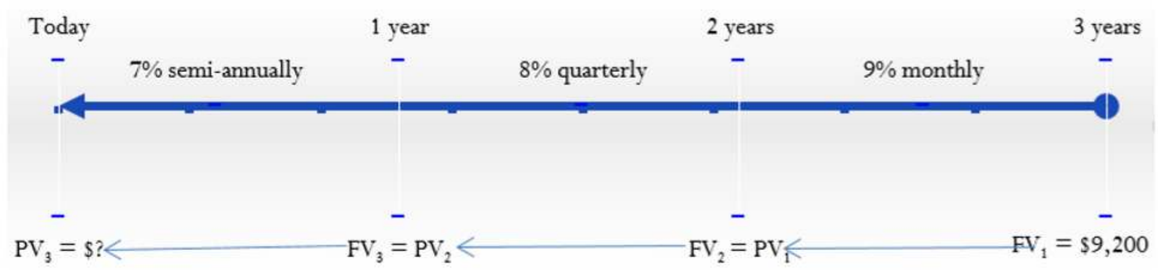
Solution

Take the maturity amount and bring it back to today by removing the interest. Notice you are dealing with a single lump sum with multiple successive interest rates. The amount of money to be invested today is the principal (PV).

What You Already Know

Step 1:

The maturity amount, terms, and interest rates are on the timeline:



$FV_1 = \$9,200$

Starting from the right end of the timeline and working backwards:

First time segment: IY = 9%; CY = monthly = 12 Term = 1 year

Second time segment: IY = 8%; CY = quarterly = 4 Term = 1 year

Third time segment: IY = 7%; CY = semi-annually = 2 Term = 1 year

How You Will Get There

Step 2:

For each time segment, calculate the periodic interest rate by applying Formula 9.1.

Step 3:

For each time segment, calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the present value of the first time segment using Formula 9.3 and rearrange for PV_1.

Step 5:

Assign $FV_2 = PV_1$.

Step 6:

Reapply Formula 9.3 and isolate PV_2 for the second time segment.

Step 7:

Assign $FV_3 = PV_2$. Reapply Formula 9.3 and isolate PV_3 for the third time segment.

Perform

Step 2:

First segment: $i = \frac{9\%}{12} = 0.75\%$

Second segment: $i = \frac{8\%}{4} = 2\%$

Third segment: $i = \frac{7\%}{2} = 3.5\%$

Step 3:

First segment: $N = 12 \times 1 = 12$

Second segment: $N = 4 \times 1 = 4$

Third segment: $N = 2 \times 1 = 2$

Step 4:

$$\begin{aligned}\$9,200 &= PV_1 \times (1+0.0075)^{12} \\ PV_1 &= \frac{\$9,200}{1.0075^{12}} \\ &= \$8,410.991026\end{aligned}$$

Step 5:

$$FV_2 = \$8,410.991026$$

Step 6:

$$\begin{aligned}\$8,410.991026 &= PV_2 \times (1+0.02)^4 \\ PV_2 &= \frac{\$8,410.91026}{1.02^4} \\ &= \$7,770.455587\end{aligned}$$

Step 7:

$$\begin{aligned}&\mathrm{FV}_3 = \$7,770.455587 \\ &\$7,770.455587 = \mathrm{PV}_3 \times (1+0.035)^2 \\ &\mathrm{PV}_3 = \frac{\$7,770.455587}{1.035^2} = \$7,253.80\end{aligned}$$

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	12	9	Answer: -$8,410.991026	0	9200	12	12

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
2	4	8	Answer: $7,770.455587	√	8410.991026	4	4
3	2	7	Answer: -$7,253.803437	√	7770.455587	2	2

Sebastien needs to place $7,253.80 into the investment today to have $9,200 three years from now.

Paths To Success

When you calculate the present value of a single payment for which only the interest rate fluctuates, it is possible to find the principal amount in a single division:

$$PV = \frac{FV}{(1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_n)^{N_n}}$$

where n represents the time segment number. Note that the technique in Example $9.3.2$ resolves each of these divisions one step at a time, whereas this technique solves them all simultaneously. You can calculate the same principal as follows:

$$PV = \frac{\$9,200}{(1.0075)^{12} \times (1.02)^4 \times (1.035)^2} = \$7,253.80$$

✓ Example $9.3.3$: Paying a Debt Early

Birchcreek Construction has three payments left on a debt obligation in the amounts of $5,500, $10,250, and $8,000 due 1.5 years, 2 years, and 3.5 years from today. Prevailing interest rates are projected to be 8% compounded quarterly in the first year and 8.25% compounded semi-annually thereafter. If Birchcreek wants to settle the debt today, what amount should the creditor be willing to accept?

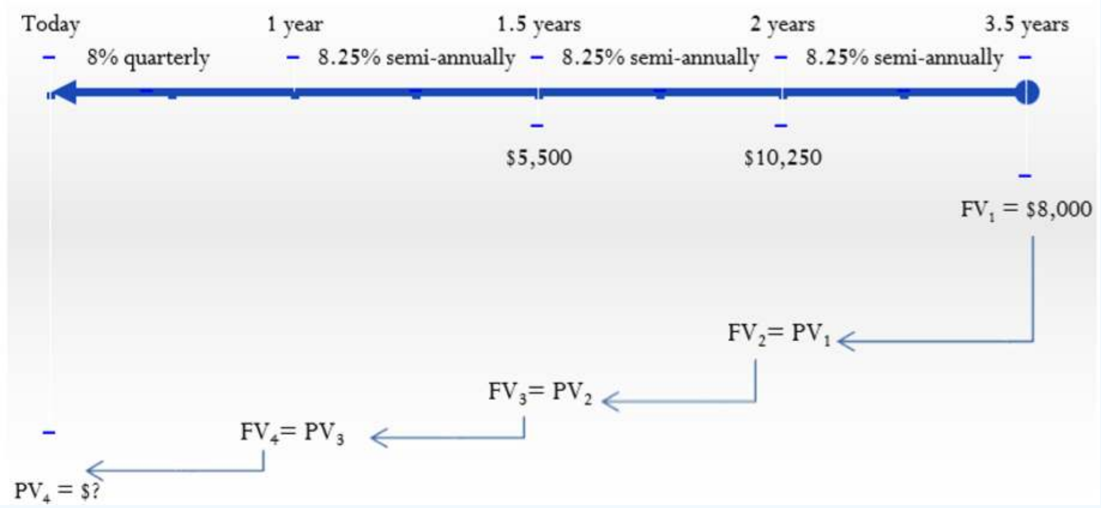
Solution

Take the three lump-sum payments in the future and remove the interest back to today to find the fair amount that Birchcreek Construction should pay. This is the present value (PV).

What You Already Know

Step 1:

With multiple amounts and interest rates, the timeline displays the changing variables. There are four time segments. The IY, CY, and Term are identified for each.


Today
1 year
1.5 years
2 years
3.5 years
8% quarterly
8.25% semi-annually
8.25% semi-annually
8.25% semi-annually
$5,500
$10,250
FV₁ = $8,000
FV₂= PV₁
FV₃= PV₂
FV₄= PV₃
PV₄ = $?

How You Will Get There

Step 2:

For each segment, calculate the periodic interest rate by applying Formula 9.1.

Step 3:

For each segment, calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the present value of the first time segment using Formula 9.3 and rearrange for PV_1.

Step 5:

Assign $FV_2 = PV_1$ and increase by the additional lump-sum payment.

Step 6:

Reapply Formula 9.3 and isolate PV_2 for the second time segment:.

Step 7:

Assign $FV_3 = PV_2$ and increase by the additional lump-sum payment. Reapply Formula 9.3 and isolate PV_3 for the third time segment.

Repeat Step 7: Assign $FV_4 = PV_3$. Reapply Formula 9.3 and isolate PV_4 for the final time segment.

Perform

Step 2:

First segment: $i = \dfrac{8.25\%}{2} = 4.125\%$

Second segment: $i = \dfrac{8.25\%}{2} = 4.125\%$

Third segment: $i = \dfrac{8.25\%}{2} = 4.125\%$

Fourth segment: $i = \dfrac{8\%}{4} = 2\%$

Step 3:

First segment: $N = 2 \times 1.5 = 3$

Second segment: $N = 2 \times \dfrac{1}{2} = 1$

Third segment: $N = 2 \times \dfrac{1}{2} = 1$

Fourth segment: $N = 4 \times 1 = 4$

Step 4:

$$\begin{aligned} \$8,000 &= PV_1 \times (1+0.04125)^3 \\ PV_1 &= \frac{\$8,000}{1.04125^3} \\ &= \$7,086.388265 \end{aligned}$$

Step 5:

$$FV_2 = \$7,086.388265 + \$10,250.00 = \$17,336.38826$$

Step 6:

$$\begin{aligned} \$17,336.38826 &= PV_2 \times (1+0.04125)^1 \\ PV_2 &= \frac{\$17,336.3826}{1.04125} = \$16,649.59257 \end{aligned}$$

Step 7:

$$\begin{aligned} FV_3 &= \$16,649.59257 + \$5,500.00 = \$22,149.59257 \\ \$22,149.59257 &= PV_3 \times (1+0.04125)^1 \\ PV_3 &= \frac{\$22,149.59257}{1.04125} = \$21,272.11772 \end{aligned}$$

Repeat Step 7:

$$\begin{aligned} FV_4 &= \$21,272.11772 \\ \$21,272.11772 &= PV_4 \times (1+0.02)^4 \\ PV_4 &= \frac{\$21,272.11772}{1.02^4} = \$19,652.15 \end{aligned}$$

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	3	8.25	Answer: -$7,086.388265	0	8000	2	2
2	1	√	Answer: -$16,649.59257	√	17336.38826	√	√
3	√	√	Answer: -$21,272.11772	√	22149.59257	√	√
4	4	8	Answer: -$19,652.14865	√	21272.11772	√	√

Using prevailing market rates, the fair amount that Birchcreek Construction owes today is $19,652.15.

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9.4: Equivalent Payments

Unforeseen events and circumstances can force you to rearrange your financial commitments. When this happens, the new deal has to be fair to all parties concerned.

Suppose you and your co-workers rely on your company's annual Christmas bonuses. However, the CEO just announced that because of tough economic conditions no one will get a bonus this year. You already earmarked that money to pay a $5,000 debt due next week. You want to be financially responsible, but you cannot possibly make the payment. Before picking up the phone to call your creditor, you need to determine what course of action you should pursue.

You need to make alternative arrangements that leave your creditor in the same financial position as the original agreement did. You saw in the previous section that if you were going to make an early payment, the payment should be reduced by an agreed-upon discount rate. In this case, though, you are going to make a late payment, so you must grant the creditor interest. Thus, if you propose paying the debt six months late and your creditor agrees to 9% compounded monthly as a fair rate, then you owe $5,229.26 (applying Formula 9.3).

This concept applies to all aspects of your personal and professional life. Except for gifts, personal debts to friends or family members should bear some interest. Everyone should be financially fair to each other. A business must be willing to work with its clients in the event they need to alter an agreement. A company that is inflexible tends to find itself writing off bad debt or pursuing unpleasant and sometimes expensive legal action.

This section explores the concept of equivalent payment streams. This involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice. You then apply the concept of present value to loans and loan payments.

Fundamental Concepts

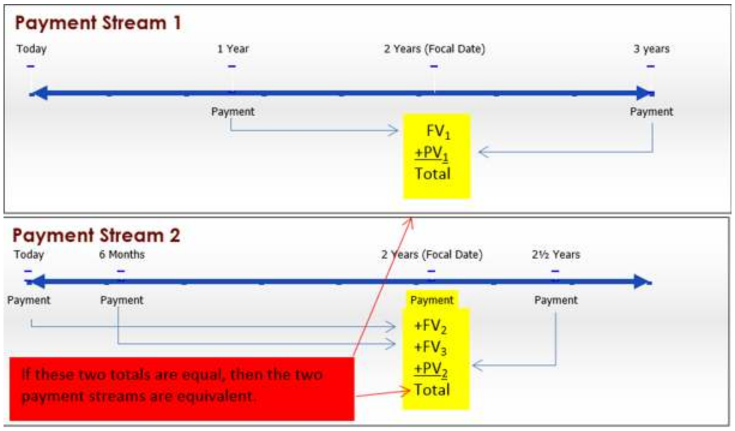
The Fundamental Concept of Time Value of Money

All numbers in an equation must be expressed in the same units, such as kilometers or meters. In the case of money, you treat the time at which you are considering the value as the "unit" that needs to be the same. Because of interest, a dollar today is not the same as a dollar a year ago or a dollar a year from now, so you cannot mix dollars from different times within the same equation; you must bring all monies forward or back to a common point in time, called the focal date.

The fundamental concept of time value of money states that all monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined, including the following:

1. Simple mathematics such as addition or subtraction.
2. Deciding whether to adopt alternative financial agreements.
3. Determining if payment streams are equal.

The Fundamental Concept of Equivalency



The fundamental concept of equivalency states that two or more payment streams are equal to each other if they have the same economic value on the same focal date. As illustrated in the figure, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date. Note that the monies involved in each payment stream can be summed on the focal date because of the fundamental concept of time value of money.

Equivalent Payments

You make alternative payment streams equivalent to each other by applying a prevailing interest rate that allows for the following:

- Any late payments to be charged interest through future value calculations (Section 9.2)
- Any early payments to have interest deducted through present value calculations (Section 9.3)

The Formula

The good news is that you need no new formulas. Depending on the structure of the payment streams needing to be equated, apply Formula 9.3 as is or rearrange the formula for PV. As illustrated in the previous figure, Payment Stream 1 involves calculating one future value and one present value to move the money to the focal date. Payment Stream 2 involves calculating two future values and one present value. Once all money is moved to the same focal date, you work with the equality between the values of the two payment streams, solving for any unknown variable or variables.

How It Works

Follow these steps to solve an equivalent payment question:

Step 1: Draw as many timelines as needed to illustrate each of the original and proposed agreements. Clearly indicate dates, payment amounts, and the interest rate(s).

Step 2: Choose a focal date to which all money will be moved.

Step 3: Calculate all needed periodic interest rates using Formula 9.1.

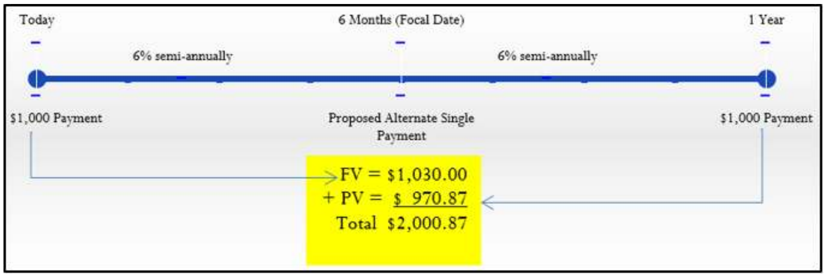
Step 4: Calculate N for each payment using Formula 9.2.

Step 5: Perform the appropriate time value calculation using Formula 9.3.

Step 6: Equate the values of the original and proposed agreements on the focal date and solve for any unknowns.

Assume you owe $1,000 today and $1,000 one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the proposed payment stream) is equivalent to the two payments (the original payment stream)?

Step 1: The timeline illustrates the scenario.



Step 2: Apply the fundamental concept of time value of money, moving all of the money to the same date. Since the proposed payment is for six months from now, you choose a focal date of six months.

Step 3: Note that semi-annual compounding means $CY = 2$. From Formula 9.1, $I = 6\%/2 = 3\%$.

Step 4: Formula 9.2 produces $N = 2 \times \frac{1}{2} = 1$ compound for both payments (each moving a $\frac{1}{2}$ year).

Step 5: Moving today's payment of $1,000 six months into the future, you have $FV = \$1,000(1+0.03)^1 = \$1,030.00$. Moving the future payment of $1,000 six months earlier, you have $\$1,000 = PV(1+0.03)^1$ or $PV = \$970.87$.

Step 6: Now that money has been moved to the same date you can sum the two totals to determine the equivalent payment, which is $\$1,030.00 + \$970.87 = \$2,000.87$. Note that this is financially fair to both parties. For making your $1,000 payment six months late, the creditor is charging you $30 of interest. Also, for making your second $1,000 payment six months early, you are receiving a benefit of $29.13. This leaves both parties compensated equitably: Neither party is financially better or worse off because of the change in the deal.

Paths To Success

When you make two (or more) payment streams equal to each other, two tips will make the procedure easier:

1. **Proper Timelines** Timelines help you see what to do. If you draw two or more timelines, align them vertically, ensuring that all corresponding dates are in the same columns. This allows you to see which payments need to be future valued and which need to be present valued to express them in terms of the chosen focal date.
2. **Locate the Focal Date at an Unknown** In one sense, it does not matter what focal date you choose because two values that are equal when moved to one date in common will still be equal when both are moved together to another date. But you should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable. Then when you calculate the root of the equation, it will already be in the right amount on the right date. You will not require further calculations to move the money to its correct date. For example, in the scenario above you had to determine the amount of a payment that was to be made six months from now. By setting the focal date at six months—the date of the unknown you were trying to find—you avoided any extra conversions.

✓ Example $9.4.1$: Replacing a Payment Stream with a Single Payment

Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of $4,500 due today and another $6,300 due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

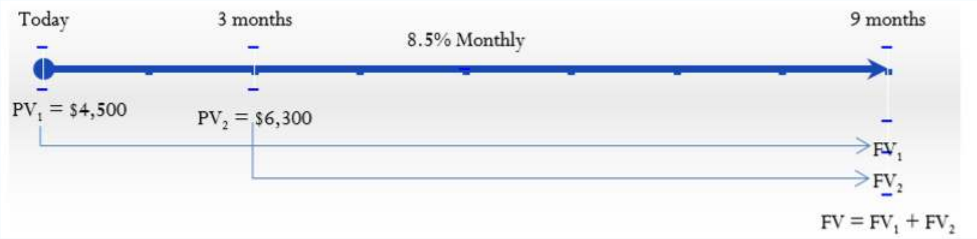
Solution

Calculate and sum the future value of each of the missed payments at the nine-month point with interest, or FV for both.

What You Already Know

Step 1:

The amount of the missed payments, interest rate, and term are illustrated on the timeline.



PV_1 (today) = $4,500; PV_2 (3 months from now) = $6,300; IY = 8.5%; CY = 12

Step 2:

Due date for all = 9 months from today. This is your focal date.

How You Will Get There

Step 3:

Calculate the periodic interest rate by applying Formula 9.1.

Step 4:

For each payment, calculate the number of compound periods (N) by applying Formula 9.2.

Step 5:

For each payment, calculate the maturity value (FV_1 and FV_2 by applying Formula 9.3.

Step 6:

Calculate the total payment by summing the two payments, (FV_1 and FV_2, together.

Perform

Step 3:

$$i=\frac{8.5\%}{12}=0.708\overline{3}\%=0.00708\overline{3}$$

Step 4:

The first payment moves nine months into the future, or 9/12 of a year. N = 12 × 9/12 = 9.

The second payment moves six months into the future, or 6/12 of a year. N = 12 × 6/12 = 6.

Step 5:

First payment: $FV_1=\$4,500(1+0.00708\overline{3})^9=\$4,795.138902$

Second payment: $FV_2=\$6,300(1+0.00708\overline{3})^6=\$6,572.536425$

Step 6:

$$FV=\$4,795.138902+\$6,572.536425=\$11,367.68$$

Calculator Instructions

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	9	8.5	4500	0	Answer: -$ 4,795.138902	12	12
2	6	✓	6300	✓	Answer: -$ 6,572.526425	✓	✓

With interest, the two payments total $11,367.68. This is the $10,800 of the original principal plus $567.68 in interest for making the late payments.

✓ Example 9.4.2: Replacing a Payment Stream with Multiple Payments

You have three debts to the same creditor: $3,000 due today, $2,500 due in 2¼ years, and $4,250 due in 3 years 11 months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: $3,500 in nine months and a second payment due in two years. You agree upon an interest rate of 9.84% compounded monthly. What is the amount of the second payment?

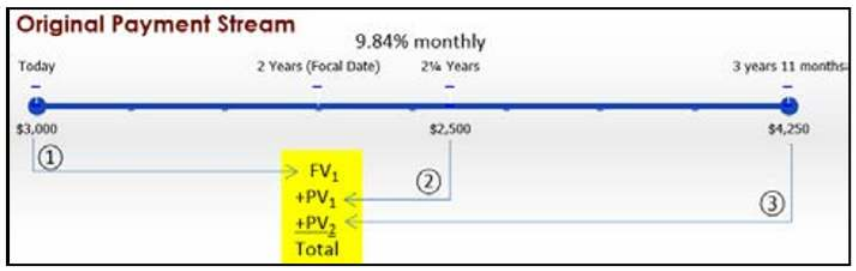
Solution

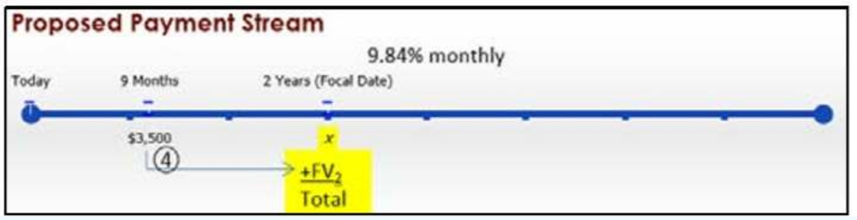
Determine the amount of the second payment that is due two years from today. Apply the fundamental concept of time value of money, moving all money from the original and proposed payment streams to a focal date positioned at the unknown payment. Once all money is moved to this focal date, apply the fundamental concept of equivalence, solving for the unknown payment, or x.

What You Already Know

Step 1:

With two payment streams and multiple amounts all on different dates, visualize two timelines. The payment amounts, interest rate, and due dates for both payment streams are known.


Original Payment Stream
9.84% monthly
Today
2 Years (Focal Date)
2¼ Years
3 years 11 months
$3,000
$2,500
$4,250
①
②
③
FV₁
+PV₁
+PV₂
Total


Proposed Payment Stream
9.84% monthly
Today
9 Months
2 Years (Focal Date)
$3,500
x
④
+FV₂
Total

IY = 9.84%; CY = monthly = 12

How You Will Get There

Step 2:

A focal date of Year 2 has been selected.

Step 3:

Calculate the periodic interest rate by applying Formula 9.1.

Step 4:

For each payment, calculate the number of compound periods by applying Formula 9.2.

Step 5:

For each payment, apply Formula 9.3. Note that all payments before the two year focal date require you to calculate future values, while all payments after the two-year focal date require you to calculate present values.

Step 6:

Set the value of the original payment stream equal to the proposed payment stream. Solve for the unknown payment, x, using algebra.

$FV_1 + PV_1 + PV_2 = x + FV_2$

Perform

Step 3:

$$i = \frac{9.84\%}{12} = 0.82\% \text{ or } 0.0082$$

Steps 4 and 5:

Using the circled number references from the timelines:

① $N = 12 \times 2 = 24$; $FV_1 = \$3,000(1+0.0082)^{24} = \$3,649.571607$

② $N = 12 \times ¼ = 3$; $\$2,500 = PV_1(1+0.0082)^3$; $PV_1 = \dfrac{\$2,500}{1.0082^3} = \$2,439.494983$

③ $N = 12 \times 1\frac{11}{12} = 23$; $\$4,250 = PV_2(1+0.0082)^{23}$; $PV_2 = \dfrac{\$4,250}{1.0082^{23}} = \$3,522.207915$

④ $N = 12 \times 1\frac{1}{4} = 15$; $FV_2 = \$3,500(1+0.0082)^{15} = \$3,956.110749$

Step 6:

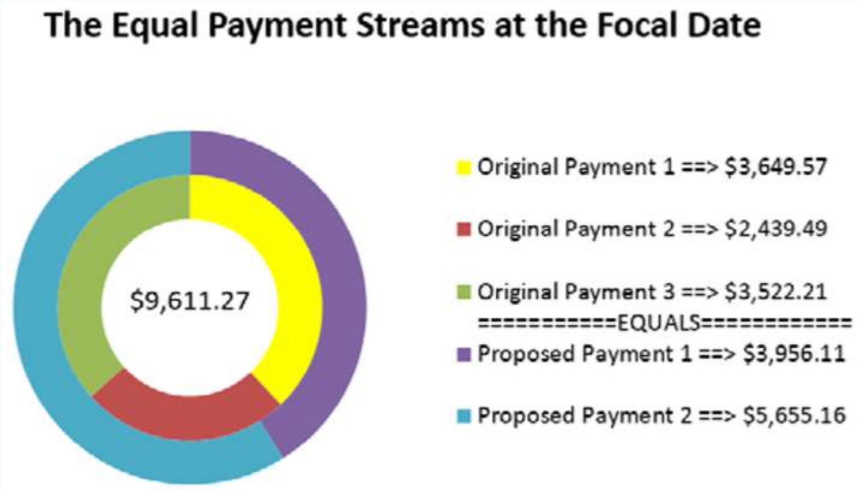
$$\begin{aligned} \$3,649.571607 + \$2,439.494983 + \$3,522.207915 &= x + \$3,956.110749 \\ \$9,611.274505 &= x + \$3,956.110749 \end{aligned}$$

$$\$5,655.16 = x$$

Calculator Instructions

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
①	24	9.84	3000	0	Answer: -$ 3,649.571607	12	12
②	3	√	Answer: -$ 2,439.494983	√	2500	√	√
③	23	√	Answer: -$ 3,522.207915	√	4250	√	√
④	15	√	3500	√	Answer: -$ 3,956.110749	√	√

The second payment is $5,655.16. This makes the original and proposed agreements equivalent to each other.

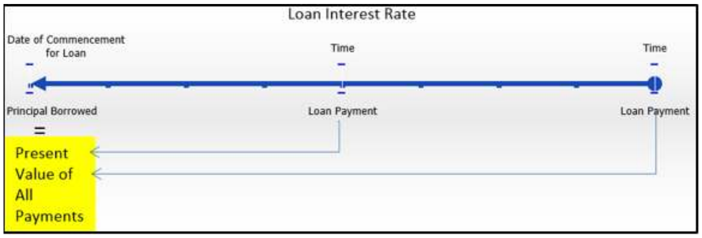


Working with Loans and Payments

The fundamental concepts of time value of money and equivalency will help you understand better how loans operate. When a loan is set up, the number of payments and the repayment amounts are established. Since the loan acquires interest throughout the transaction, this means future payments must pay back both principal and interest. Ultimately, all loan payments together need to fully reimburse the lender for the full amount of the principal.

Present Value Principle for Loans

The present value principle for loans states that the present value of all payments on a loan is equal to the principal that was borrowed. As illustrated in the timeline, taking all future payments and removing the interest, the payments must total to the original principal amount on the initial date of the loan. The interest rate used in this calculation is the interest rate for the loan itself.



This relationship can be expressed as follows:

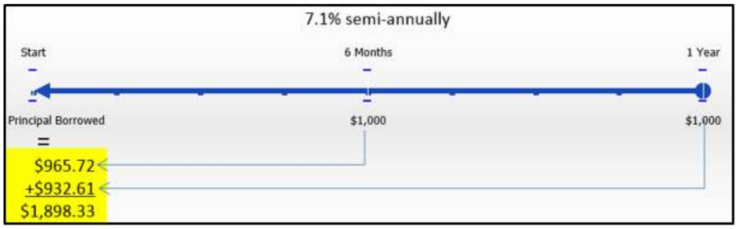
$$\text{Principal Borrowed} = \text{Present Value of All Payments}$$

How It Works

When you work with loans, the most common unknown variables are either the principal borrowed or the amount(s) of any unknown payment(s). You follow the same steps as for equivalent payments, discussed earlier in this section.

For example, assume a loan requires two equal payments of $1,000 semi-annually and you want to know the original principal of the loan. The loan's interest rate is 7.1% compounded semi-annually.

Step 1:The timeline is illustrated here.



Step 2: According to the present value principle for loans, when you bring the two payments back to the starting date of the loan (the focal date), then the sum of the present values equals the initial amount borrowed.

Step 3: The IY = 7.1% and CY = 2, therefore $i = 7.1\%/2 = 3.55\%$.

Step 4: The first payment needs to come back a $\frac{1}{2}$ year, or $N = 2 \times \frac{1}{2} = 1$. The second payment needs to come back 1 year, or $N = 2 \times 1 = 2$.

Step 5: Applying Formula 9.3, for the first payment $\$1,000 = PV(1+0.0355)^1$ or $PV = \$965.717044$. For the second payment, $\$1,000 = PV(1+0.0355)^2$ or $PV = \$932.609410$.

Step 6: The amount borrowed is the sum of the two present values, or $\$965.717044 + \$932.609410 = \$1,898.33$. At 7.1% semi-annually, two payments of $1,000 semi-annually pay off a loan of $1,898.33.

Paths To Success

You can still use the time value of money buttons on your calculator to perform calculations involving unknown variables. Recall from the algebra discussion in Section 2.4 that any algebraic term consists of both a numerical and a literal coefficient. To move an unknown variable through time, enter the numerical coefficient of the variable into the calculator and compute its new value at the focal date. This new value on its focal date must then have the literal coefficient written after it before you proceed with further operations.

For example, if you want to perform a present value calculation on a future value of $2y$, enter the numerical coefficient "2" as your FV and compute the PV. Suppose the PV is 1.634. The literal coefficient has not disappeared. You just could not enter the letter on the calculator. Therefore, copying the literal coefficient to the present value yourself, you see that $PV = 1.634y$. Example $9.4.3$ applies this concept.

Exercise $9.4.1$: Give It Some Thought

a. The principal on a loan is $10,000 and two payments are required. If the first payment is $5,000, what statement could you make about the second payment?
b. Consider two loans. The first requires four $2,500 quarterly payments. The second loan requires four $2,500 monthly payments. Assuming equal interest rates, what statement could you make about the principals of these two loans?

Answer

a. The principal is less than $8,000 since all four payments must have the interest removed.
b. The second payment is more than $5,000 since it must pay back the interest as well as the remaining principal.

c. The principal on the second loan is larger since the payments are made for only four months afterwards, so they involve relatively little interest. The first loan's payments extend for one year, so it requires the removal of larger sums of interest to arrive at the principal

Example $9.4.3$: Determining Two Unknown Payments in a Loan

A $24,000 loan at 7% compounded semi-annually requires three payments at 1½ years, 3 years, and 4 years. The first payment is $3,000 and the second payment is three times as large as the final payment. Calculate the values of the second and third payments on the loan.

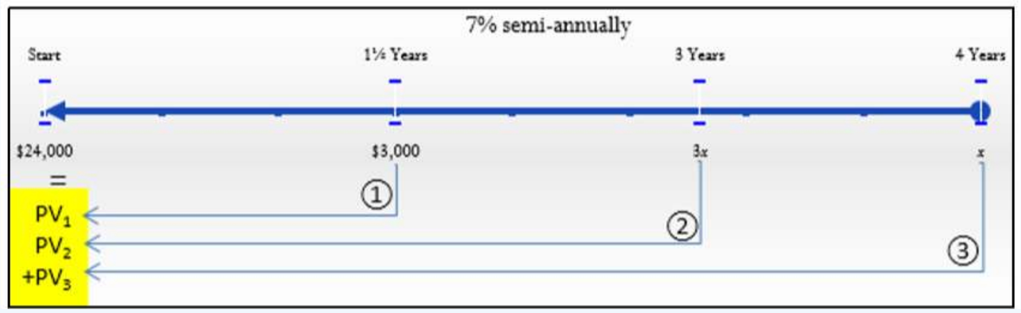
Solution

Find the amount of the second and third payments. Since the second payment is three times the size of the third payment, solve for the third payment, or x, and then calculate the second payment afterwards.

What You Already Know

Step 1:

The principal, payment amounts, interest rate, and dates are known, as indicated on the timeline.



IY = 7%; CY = 2 Note: If the final payment is x and the second payment is three times as large, then it equals $3x$.

How You Will Get There

Step 2:

The focal date is the start of the loan.

Step 3:

Calculate the periodic interest rate by applying Formula 9.1.

Step 4:

For each payment, calculate the number of compound periods by applying Formula 9.2.

Step 5:

For each payment apply Formula 9.3.

Step 6:

Using the present value principle for loans, equate the payments and the principal, then solve for the unknown variable x.

$\text{Principal } = PV_1 + PV_2 + PV_3$

Once you know x, calculate the size of the second payment = $3x$.

Perform

Step 3:

$$i = \frac{7\%}{2} = 3.5\% = 0.035$$

Steps 4 and 5:

Using the circled number references in the timeline:

① $N=2\times 11/2=3; \$3,000=PV_1(1+0.035)^3; PV_1=\dfrac{\$3,000}{1.035^3}=\$2,705.828117$

② $N=2\times 3=6; 3x=PV_2(1+0.035)^6; PV_2=\dfrac{3x}{1.035^6}=2.440501x$

③ $N=2\times 4=8; x=PV_3(1+0.035)^8; PV_3=\dfrac{x}{1.035^8}=0.759411x$

Step 6:

$$\begin{aligned}\$24,000&=\$2,705.828117+2.440501x+0.759411x\\ \$24,000&=\$2,705.828117+3.199913x\\ \$21,294.17188&=3.199913x\\ \$6,654.61&=x\end{aligned}$$

Solving for the $3x$ payment: $3(\$6,654.61)=\$19,963.83$

Calculator Instructions

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
①	3	7	Answer: $ 2,705.828117	0	-3000	2	2
②	6	√	Answer: 2.440501	√	2500	√	√
③	8	√	Answer: 0.759411	√	-1	√	√

The second payment, located three years from the start of the loan, is $19,963.83, which is three times the size of the final payment of $6,654.61 one year later.

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9.5: Determining the Interest Rate

Whether you are borrowing or investing, it is extremely important to know what compound interest rate you are being charged or earning. Unfortunately, most consumers live for today, paying attention just to the “don’t pay for one year” clause in the deals they are offered. They do not stop to consider they might be paying too much for credit.

For example, assume you are just about to sign the purchase papers for a new 55” LED HD 3D television. The retail sales clerk turns to you, saying, “Well, I’ve run the numbers and they show you could pay $4,000 today or if you take advantage of our ‘don’t pay for one year’ offer, you will owe us $4,925.76 one year from now.” You urgently want to get your new TV home, but you had better think twice about a financial decision this important. What interest rate did the sales clerk use in determining your $4,925.76 payment? Is this interest rate fair? Could you finance the TV for less elsewhere?

This section shows how to calculate the nominal interest rate on single payments when you know both the future value and the present value.

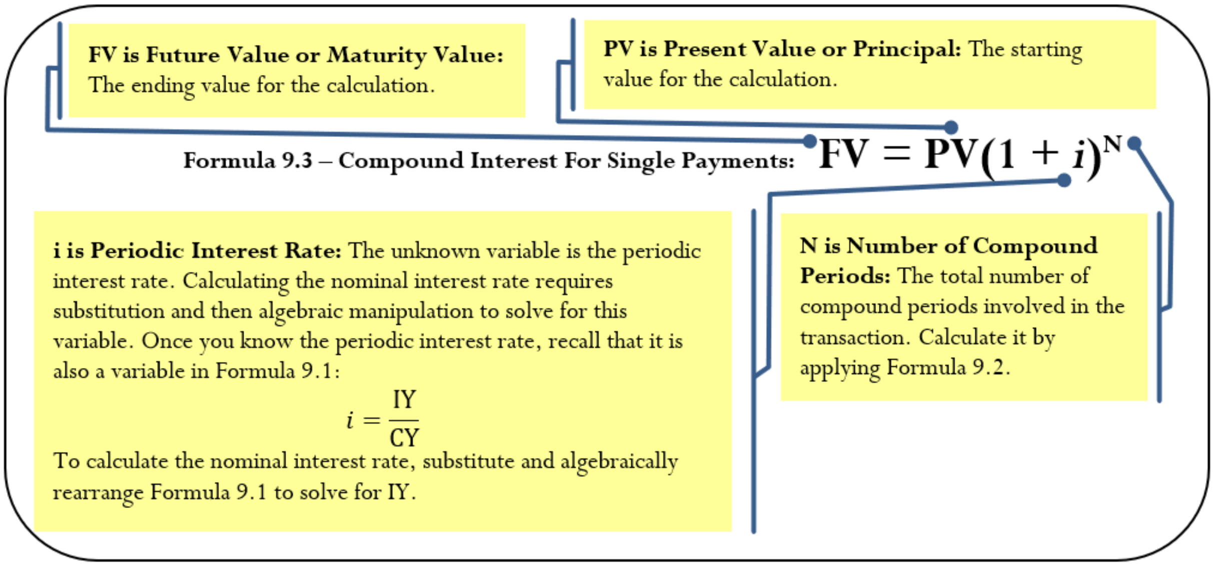
The Nominal Interest Rate

You need to calculate the nominal interest rate under many circumstances including (but not limited to) the following:

- Determining the interest rate on a single payment loan
- Understanding what interest rate is needed to achieve a future savings goal
- Calculating the interest rate that generated a specific amount of interest
- Finding a fixed interest rate that is equivalent to a variable interest rate

The Formula

Calculating the nominal interest rate requires you to use Formula 9.3 once again. The only difference is that the unknown variable has changed from FV to IY. Note that IY is not directly part of Formula 9.3, but you are able to calculate its value after determining the periodic interest rate, or i.



FV is Future Value or Maturity Value: The ending value for the calculation.

PV is Present Value or Principal: The starting value for the calculation.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1+i)^N$

i is Periodic Interest Rate: The unknown variable is the periodic interest rate. Calculating the nominal interest rate requires substitution and then algebraic manipulation to solve for this variable. Once you know the periodic interest rate, recall that it is also a variable in Formula 9.1:

$$i=\frac{IY}{CY}$$

To calculate the nominal interest rate, substitute and algebraically rearrange Formula 9.1 to solve for IY.

N is Number of Compound Periods: The total number of compound periods involved in the transaction. Calculate it by applying Formula 9.2.

How It Works

Follow these steps to solve for the nominal interest rate on a single payment:

Step 1: Draw a timeline to help you visualize the question. Of utmost importance is identifying the values of PV and FV, the number of years involved, and the compounding for the interest rate.

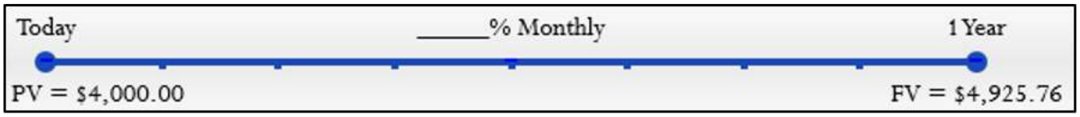
Step 2: Calculate the number of compounds, N, using Formula 9.2.

Step 3: Substitute known variables into Formula 9.3, rearrange and solve for the periodic interest rate, i.

Step 4: Substitute the periodic interest rate and the compounding frequency into Formula 9.1, rearrange, and solve for the nominal interest rate, IY. Ensure that the solution is expressed with the appropriate compounding words.

Revisiting the section opener, your $4,000 TV cost $4,925.76 one year later. What monthly compounded interest rate was being used?

Step 1: The timeline below illustrates the scenario and identifies the values.



The compounding is monthly, making $CY = 12$.

Step 2: The term is one year, so $N = 1 \times 12 = 12$.

Step 3: Substituting into Formula 9.3, $\$4,925.76 = \$4,000(1+i)^{12}$. Rearranging and solving for i results in $i = 0.0175$.

Step 4: Substituting into Formula 9.1, $0.0175 = \frac{IY}{12}$. Rearranging and solving for IY calculates $IY = 0.21$, or 21%. Thus, the sales clerk used an interest rate of 21% compounded monthly.

Important Notes

Handling Decimals in Interest Rate Calculations

When you calculate interest rates, the solution rarely works out to a terminating decimal number. Since most advertised or posted interest rates commonly involve no more than a few decimals, why is this the case? Recall that when an interest dollar amount is calculated, in most circumstances this amount must be rounded off to two decimals. In single payments, a future value is always the present value plus the rounded interest amount. This results in a future value that is an imprecise number that may be up to a half penny away from its true value. When this imprecise number is used for calculating any interest rate, the result is that nonterminating decimals show up in the solutions. To express the final solution for these nonterminating decimals, you need to apply two rounding rules:

- **Rule 1: A Clear Marginal Effect** Use this rule when it is fairly obvious how to round the interest rate. The dollar amounts used in calculating the interest rate are rounded by no more than a half penny. Therefore, the calculated interest rate should be extremely close to its true value. For example, if you calculate an IY of 7.999884%, notice this value would have a marginal difference of only 0.000116% from a rounded value of 8%. Most likely the correct rate is 8% and not 7.9999%. However, if you calculate an IY of 7.920094%, rounding to 8% would produce a difference of 0.070006%, which is quite substantial. Applying marginal rounding, the most likely correct rate is 7.92% and not 7.9201%, since the marginal impact of the rounding is only 0.000094%.
- **Rule 2: An Unclear Marginal Effect** Use this rule when it is not fairly obvious how to round the interest rate. For example, if the calculated IY = 7.924863%, there is no clear choice of how to round the rate. In these cases or when in doubt, apply the standard rule established for this book of rounding to four decimals. Hence, IY = 7.9249% in this example.

It is important to stress that the above recommendations for rounding apply to final solutions. If the calculated interest rate is to be used in further calculations, then you should carry forward the unrounded interest rate.

Your BAII Plus Calculator

Solving for the nominal interest rate requires the computation of I/Y on the BAII Plus calculator. This requires you to enter all six of the other variables, including N, PV, PMT (which is zero), FV, and both values in the P/Y window (P/Y and C/Y) following the procedures established in Section 9.2. Ensure proper application of the cash flow sign convention to PV and FV, according to which one number must be negative while the other is positive.

Things To Watch Out For

Pay careful attention to what the situation requires you to calculate—the nominal interest rate or the periodic interest rate. The most common mistake when calculating the interest rate is to confuse these two rates. If you need the periodic interest rate, then you must rearrange and solve Formula 9.3 for i, and no further calculations are required after step 3. If you need the nominal interest rate, first calculate the periodic interest rate (i) but then substitute it into Formula 9.1 and rearrange to solve for IY, thus completing step 4 of the process.

? Exercise $9.5.1$: Give It Some Thought

Round off the following calculated values of IY to the appropriate decimals.

a. 4.5679998%
b. 12.000138%
c. 6.8499984%
d. 8.0200121%
e. 7.1224998%

Answer

a. 4.568%
b. 12%
c. 6.85%
d. 8.02%
e. 7.1225%

✓

When Sandra borrowed $7,100 from Sanchez, she agreed to reimburse him $8,615.19 three years from now including interest compounded quarterly. What interest rate is being charged?

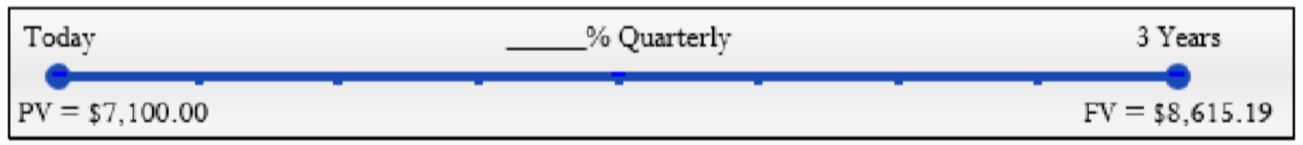
Solution

Find the nominal quarterly compounded rate of interest (IY).

What You Already Know

Step 1:

The present value, future value, term, and compounding are known, as illustrated in the timeline.



CY = quarterly = 4 Term = 3 years

How You Will Get There

Step 2:

Calculate N using Formula 9.2.

Step 3:

Substitute into Formula 9.3 and rearrange for i.

Step 4:

Substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

$N=4\times 3=12$

Step 3:

$$\begin{aligned}
\$8,615.10 &= \$7,100(1+i)^{12}\\
1.213394 &= (1+i)^{12}\\
1.213394^{\frac{1}{12}} &= 1+i\\
1.016249 &= 1+i
\end{aligned}$$

$$0.016249 = i$$

Step 4:

$$\begin{aligned} 0.016249 &= \frac{IY}{4} \\ IY &= 0.064996 = 0.065 \text{ or } 6.5\% \end{aligned}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
12	Answer: 6.499709	-7100	0	8615.10	4	4

Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.

Paths To Success

When a series of calculations involving the nominal interest rate must be performed, many people find it helpful first to rearrange Formula 9.3 algebraically for i, thus bypassing a long series of manipulations. The rearranged formula appears as follows:

$$i = \left[\left(\frac{FV}{PV}\right)^{\frac{1}{N}} - 1\right]$$

This rearrangement calculates the periodic interest rate. If the nominal interest rate is required, you can combine Formula 9.3 and Formula 9.1 together:

$$IY = \left[\left(\frac{FV}{PV}\right)^{\frac{1}{N}} - 1\right] \times CY$$

✓ Example 9.5.2: Known Interest Amount

Five years ago, Taryn placed $15,000 into an RRSP that earned $6,799.42 of interest compounded monthly. What was the nominal interest rate for the investment?

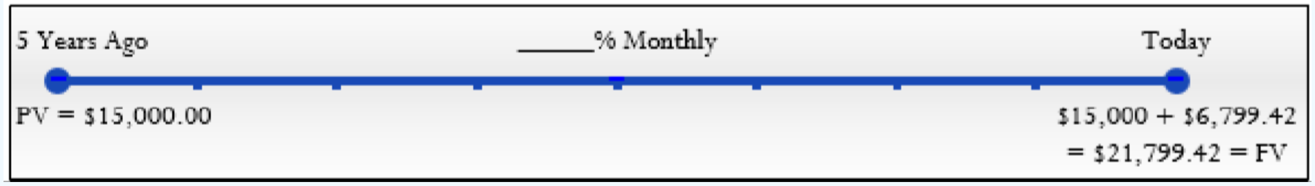
Solution

Find the nominal monthly compounded rate of interest (IY).

What You Already Know

Step 1:

The present value, interest earned, term, and compounding are known, as illustrated in the timeline.



Use Formula 9.3 to arrive at the FV in the figure. CY = monthly = 12 Term = 5 years

How You Will Get There

Step 2:

Calculate N using Formula 9.2.

Step 3:

Substitute into Formula 9.3 and rearrange for i.

Step 4:

Substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

$N=12\times 5=60$

Step 3:

$$\begin{aligned}\$21,799.42&=\$15,000(1+i)^{60}\\1.453294&=(1+i)^{60}\\1.453294^{\frac{1}{60}}&=1+i\\1.00625&=1+i\\0.00625&=i\end{aligned}$$

Step 4:

$$\begin{aligned}0.00625&=\frac{IY}{12}\\IY&=0.075\text{ or }7.5\%\end{aligned}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	Answer: 7.500003	-15000	0	21799.42	12	12

Taryn's investment in his RRSP earned 7.5% compounded monthly over the five years.

Converting Variable Interest Rates to a Fixed Interest Rate

When you deal with a series of variable interest rates it is extremely difficult to determine their overall effect. This also makes it hard to choose wisely between different series. For example, assume that you could place your money into an investment earning interest rates of 2%, 2.5%, 3%, 3.5%, and 4.5% over the course of five years, or alternatively you could invest in a plan earning 1%, 1.5%, 1.75%, 3.5%, and 7% (all rates compounded semi-annually). Which plan is better? The decision is unclear. But you can make it clear by converting the variable rates on each investment option into an equivalent fixed interest rate.

How It Works

Follow these steps to convert variable interest rates to their equivalent fixed interest rates:

Step 1: Draw a timeline for the variable interest rate. Identify key elements including any known PV or FV, interest rates, compounding, and terms.

Step 2: For each time segment, calculate i and N using Formula 9.1 and Formula 9.2, respectively.

Step 3: One of three situations will occur, depending on what variables are known:

1. PV **Is Known** Calculate the future value at the end of the transaction using Formula 9.3 and solving for FV in each time segment, working left to right across the timeline.
2. FV **Is Known** Calculate the present value at the beginning of the transaction using Formula 9.3 and rearranging to solve for PV in each time segment, working right to left across the timeline.
3. **Neither** PV **nor** FV **Is Known** Pick an arbitrary number for PV ($10,000 is recommended) and use Formula 9.3 in each time segment to solve for the future value at the end of the transaction, working left to right across the timeline.

Step 4: Determine the compounding required on the fixed interest rate (CY) and use Formula 9.2 to calculate a new value for N to reflect the entire term of the transaction.

Step 5: Rearrange and solve Formula 9.3 for i using the N from step 4 along with the starting PV and ending FV for the entire timeline.

Step 6: Rearrange and solve Formula 9.1 for IY.

Example $9.5.3$: Interest Rate under Variable Rate Conditions

Continue working with the two investment options mentioned previously. The choices are to place your money into a fiveyear investment earning semi-annually compounded interest rates of either:

a. 2%, 2.5%, 3%, 3.5%, and 4.5%
b. %, 1.5%, 1.75%, 3.5%, and 7%

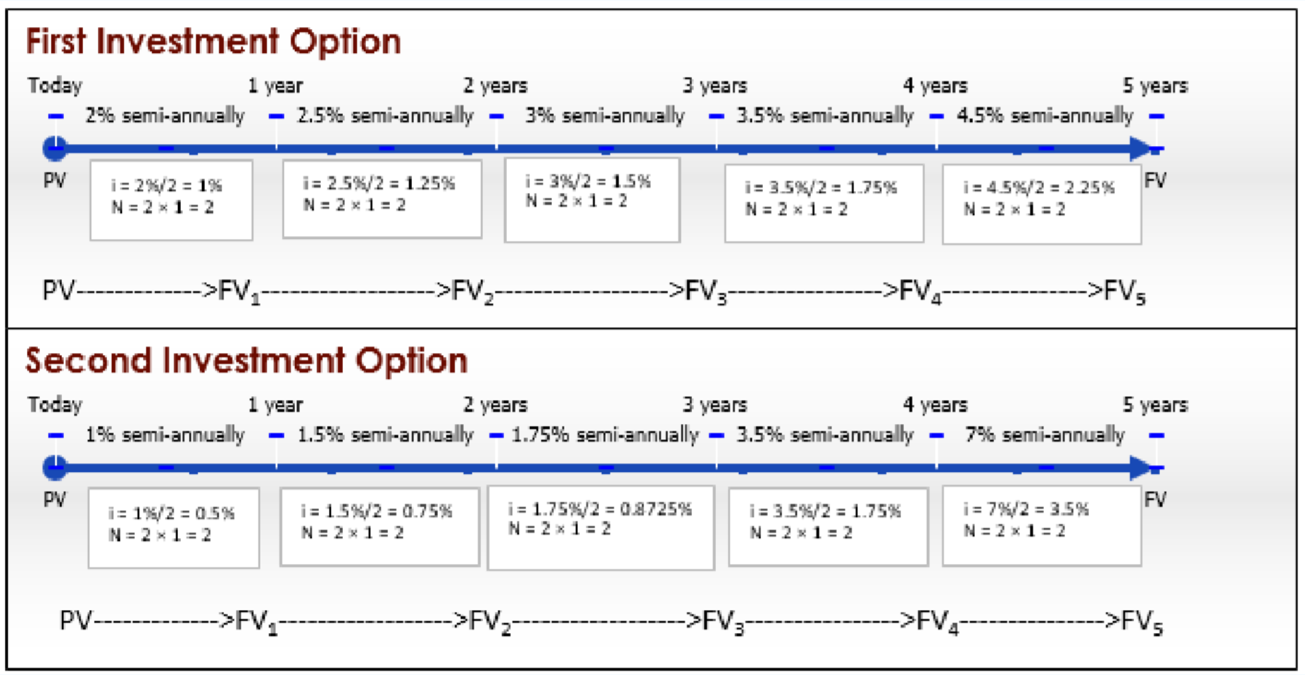
Calculate the equivalent semi-annual fixed interest rate for each plan and recommend an investment.

Solution

What You Already Know

Step 1:

Draw a timeline for each investment option, as illustrated below.


First Investment Option
Today
1 year
2 years
3 years
4 years
5 years
2% semi-annually
2.5% semi-annually
3% semi-annually
3.5% semi-annually
4.5% semi-annually
PV
FV
i = 2%/2 = 1%
N = 2 × 1 = 2
i = 2.5%/2 = 1.25%
N = 2 × 1 = 2
i = 3%/2 = 1.5%
N = 2 × 1 = 2
i = 3.5%/2 = 1.75%
N = 2 × 1 = 2
i = 4.5%/2 = 2.25%
N = 2 × 1 = 2
PV------------->FV1------------------>FV2------------------>FV3----------------->FV4---------------->FV5
Second Investment Option
Today
1 year
2 years
3 years
4 years
5 years
1% semi-annually
1.5% semi-annually
1.75% semi-annually
3.5% semi-annually
7% semi-annually
PV
FV
i = 1%/2 = 0.5%
N = 2 × 1 = 2
i = 1.5%/2 = 0.75%
N = 2 × 1 = 2
i = 1.75%/2 = 0.8725%
N = 2 × 1 = 2
i = 3.5%/2 = 1.75%
N = 2 × 1 = 2
i = 7%/2 = 3.5%
N = 2 × 1 = 2
PV------------->FV1------------------>FV2------------------>FV3----------------->FV4---------------->FV5

How You Will Get There

Step 2:

For each time segment, calculate the i and N using Formula 9.1 and Formula 9.2. All interest rates are semi-annually compounded with CY = 2.

Step 3:

There is no value for PV or FV. Choose an arbitrary value of PV = $10,000 and solve for FV using Formula 9.3. Since only the interest rate fluctuates, solve in one calculation:

$$FV_5 = PV \times (1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_5)^{N_5}$$

Step 4:

For each investment, calculate a new value of N to reflect the entire five-year term using Formula 9.2.

Step 5:

For each investment, substitute into Formula 9.3 and rearrange for i

Step 6:

For each investment, substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

All periodic interest rate (i) and N calculations are found in the timeline figure above.

Step 3:

First Investment: $FV_5 = \$10,000(1+0.01)^2(1+0.0125)^2(1+0.015)^2(1+0.0175)^2(1+0.0225)^2 = \$11,661.65972$

Second Investment: $FV_5 = \$10,000(1+0.005)^2(1+0.0075)^2(1+0.00875)^2(1+0.0175)^2(1+0.035)^2 = \$11,570.14666$

Step 4:

First Investment: $N = 2 \times 5 = 10$

Second Investment: $N = 2 \times 5 = 10$

Step 5:

First Investment:

$$\begin{aligned}
&\$11,661.65972 = \$10,000(1+i)^{10} \\
&1.166165 = (1+i)^{10} \\
&1.166165^{\frac{1}{10}} = 1+i \\
&1.001549 = 1+i \\
&0.001549 = i
\end{aligned}$$

Second Investment:

$$\begin{aligned}
&\$11,570.14666 = \$10,000(1+i)^{10} \\
&1.157014 = (1+i)^{10} \\
&1.157015^{\frac{1}{10}} = 1+i \\
&1.014691 = 1+i \\
&0.014691 = i
\end{aligned}$$

Step 6:

First Investment:

$$\begin{aligned}
0.001549 &= \frac{IY}{2} \\
IY &= 0.030982 \text{ or } 3.0982\%
\end{aligned}$$

Second Investment:

$$\begin{aligned}
0.014691 &= \frac{IY}{2} \\
\text{IY } &= 0.029382 \text{ or } 2.9382\%
\end{aligned}$$

Calculator Instructions

Investment #1

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	2	2	-10000	0	Answer: 10,201	2	2
2	√	2.5	-10201	√	Answer: 10,457.61891	√	√
3	√	3	-10457.61891	√	Answer: 10,773.70044	√	√
4	√	3.5	-10773.70044	√	Answer: 11,154.0794	√	√

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
5	√	4.5	-11154.0794	√	Answer: 11661.65972	√	√
All	10	Answer: 0.030982	-10000	√	11661.66	√	√

Investment #2

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	60	1	-10000	0	Answer: 10,100.25	2	2
2	√	1.5	-10100.25	√	Answer: 10,252.32189	√	√
3	√	1.75	-10252.32189	√	Answer: 10,432.52247	√	√
4	√	3.5	-10432.52247	√	Answer: 10,800.85571	√	√
5	√	7	-10800.85571	√	Answer: 11,570.14666	√	√
All	10	Answer: 0.029382	-10000	√	11570.15	√	√

The variable interest rates on the first investment option are equivalent to a fixed interest rate of 3.0982% compounded semi-annually. For the second option, the rates are equivalent to 2.9382% compounded semi-annually. Therefore, recommend the first investment since its rate is higher by 3.0982% − 2.9382% = 0.16% compounded semi-annually.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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9.6: Equivalent and Effective Interest Rates

How can you compare interest rates posted with different compounding? For example, let's say you are considering the purchase of a new home, so for the past few weeks you have been shopping around for financing. You have spoken with many banks as well as onsite mortgage brokers in the show homes. With semi-annual compounding, the lowest rate you have come across is 6.6%. In visiting another show home, you encounter a mortgage broker offering a mortgage for 6.57%. In the fine print, it indicates the rate is compounded quarterly. You remember from your business math class that the compounding is an important component of an interest rate and wonder which one you should choose—6.6% compounded semi-annually or 6.57% compounded quarterly.

When considering interest rates on loans, you clearly want the best rate. If all of your possible loans are compounded in the same manner, selecting the best interest rate is a matter of picking the lowest number. However, when interest rates are compounded differently the lowest number may in fact not be your best choice. For investments, on the other hand, you want to earn the most interest. However, the highest nominal rate may not be as good as it appears depending on the compounding.

To compare interest rates fairly and select the best, they all have to be expressed with equal compounding. This section explains the concept of an effective interest rate, and you will learn to convert interest rates from one compounding frequency to a different frequency.

Effective Interest Rates

If you place $1,000 into an investment earning 10% compounded semi-annually, how much will you have in your account after one year? Less than $1,100, exactly $1,100, or more than $1,100? If you say more than $1,100, you are absolutely correct. In other words, the 10% nominal rate does not fully reflect the true amount of interest that the investment earns annually, which depends on how often the principal increases. This raises two questions:

1. What exact amount of interest is earned annually?
2. What annual interest rate is truly being earned?

The effective interest rate is the true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency. In other words, the amount of interest accrued at the effective interest rate once in an entire year exactly equals the amount of interest accrued at the periodic interest rate successively compounded the stated number of times in a year. To calculate the effective interest rate, you must convert the compounding on the nominal interest rate into an annual compound.

The Formula

To see how the formula develops, take a $1,000 investment at 10% compounded semi-annually through a full year.

Start with $PV = \$1,000, IY = 10\%$, and $CY = 2$ (semi-annually). Therefore, $i = 10\%/2 = 5\%$. Using Formula 9.3, after the first six-month compounding period ($N = 1$) the investment is worth

$$FV = \$1,000(1+0.05)^1 = \$1,050$$

With this new principal of $PV = \$1,050$, after the next six-month compounding period the investment becomes

$$FV = \$1,050(1+0.05)^1 = \$1,102.50$$

Therefore, after one year the investment has earned $102.50 of interest. Notice that this answer involves compounding the periodic interest twice, according to the compounding frequency of the interest rate. What annually compounded interest rate would produce the same result? Try 10.25% compounded annually. With $PV = \$1,000, IY = 10.25\%$, and $CY = 1$, then $i = 10.25\%/1 = 10.25\%$. Thus, after a year

$$FV = \$1,000(1+0.1025)^1 = \$1,102.50$$

This alternative yields the same amount of interest, $102.50. In other words, 10.25% compounded annually produces the same result as 10% compounded semi-annually. Hence, the effective interest rate on the investment is 10.25%, and this is what the investment truly earns annually.

Generalizing from the example, you calculate the future value and interest amount for the rate of 10% compounded semi-annually using the formulas

$$FV = PV(1+i)(1+i) \quad \text{and} \quad I = FV - PV$$

Notice that the periodic interest is compounded by a factor of $(1+i)$ a number of times equaling the compounding frequency ($CY = 2$). You then rewrite the future value formula:

$$FV = PV(1+i)^{CY}$$

Substituting this into the interest amount formula, $I = FV - PV$, results in

$$I = PV(1+i)^{CY} - PV$$

Factor and rearrange this formula:

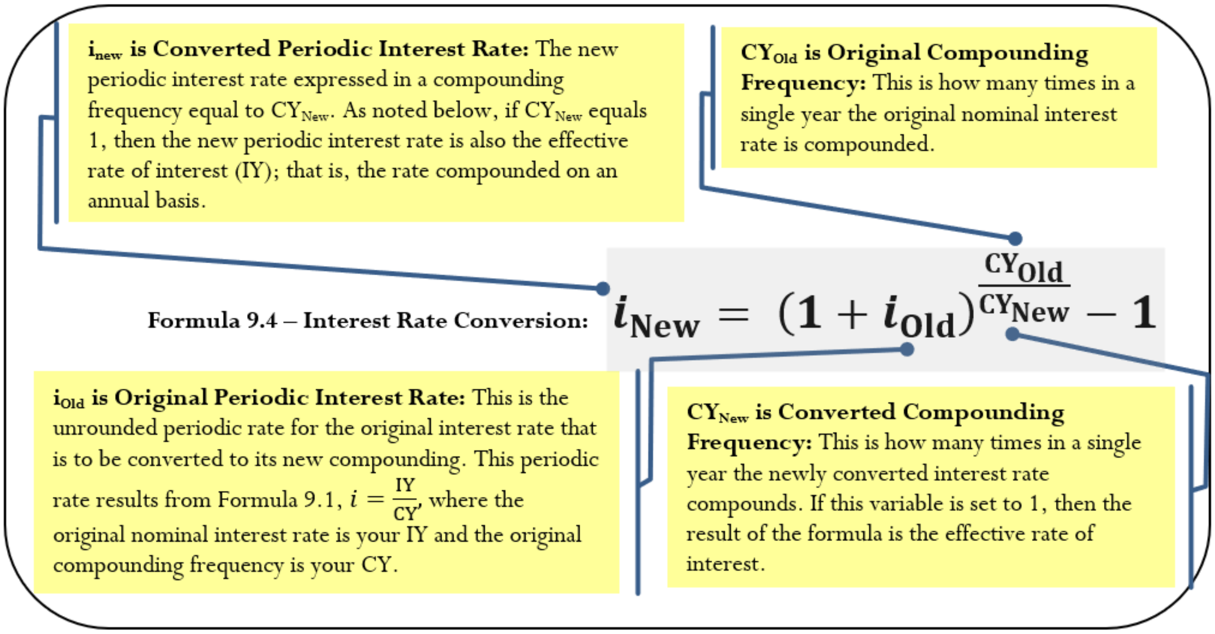
$$I = PV\left((1+i)^{CY} - 1\right)$$

$$\frac{I}{PV} = (1+i)^{CY} - 1$$

On the left-hand side, the interest amount divided by the present value results in the interest rate:

$$\text{rate } = (1+i)^{CY} - 1$$

Formula 9.4 expresses this equation in terms of the variables for time value of money. It further adapts to any conversion between different compounding frequencies.



i_{new} is Converted Periodic Interest Rate: The new periodic interest rate expressed in a compounding frequency equal to CY_{New}. As noted below, if CY_{New} equals 1, then the new periodic interest rate is also the effective rate of interest (IY); that is, the rate compounded on an annual basis.

CY_{Old} is Original Compounding Frequency: This is how many times in a single year the original nominal interest rate is compounded.

Formula 9.4 – Interest Rate Conversion: $i_{New} = (1 + i_{Old})^{\frac{CY_{Old}}{CY_{New}}} - 1$

i_{Old} is Original Periodic Interest Rate: This is the unrounded periodic rate for the original interest rate that is to be converted to its new compounding. This periodic rate results from Formula 9.1, $i = \frac{IY}{CY}$, where the original nominal interest rate is your IY and the original compounding frequency is your CY.

CY_{New} is Converted Compounding Frequency: This is how many times in a single year the newly converted interest rate compounds. If this variable is set to 1, then the result of the formula is the effective rate of interest.

Formula 9.4

How It Works

Follow these steps to calculate effective interest rates:

Step 1: Identify the known variables including the original nominal interest rate (IY) and original compounding frequency ($CY_{Old} = 1$). Set the $CY_{New} = 1$.

Step 2: Apply Formula 9.1 to calculate the periodic interest rate (i_{Old}) for the original interest rate.

Step 3: Apply Formula 9.4 to convert to the effective interest rate. With a compounding frequency of 1, this makes $i_{New} = IY$ compounded annually.

Revisiting the opening scenario, comparing the interest rates of 6.6% compounded semi-annually and 6.57% compounded quarterly requires you to express both rates in the same units. Therefore, you could convert both nominal interest rates to effective rates.

	6.6% compounded semi-annually	6.57% compounded quarterly

	6.6% compounded semi-annually	6.57% compounded quarterly
Step 1	$IY = 6.6$	$IY = 6.57$
Step 2	$i_{Old} = 6.6$	$i_{Old} = 6.57$
Step 3	$i_{New} = (1+0.033)^{2/1} - 1 = 6.7089$	$i_{New} = (1+0.016425)^{4/1} - 1 = 6.7336$

The rate of 6.6% compounded semi-annually is effectively charging 6.7089%, while the rate of 6.57% compounded quarterly is effectively charging 6.7336%. The better mortgage rate is 6.6% compounded semi-annually, as it results in annually lower interest charges.

Important Notes

The Texas Instruments BAII Plus calculator has a built-in effective interest rate converter called ICONV located on the second shelf above the number 2 key. To access it, press 2nd ICONV. You access three input variables using your ↑ or ↓scroll buttons. Use this function to solve for any of the three variables, not just the effective rate.

Variable	Description	Algebraic Symbol
NOM	Nominal Interest Rate	IY
EFF	Effective Interest Rate	i_{New} (annually compounded)
C/Y	Compound Frequency	CY

To use this function, enter two of the three variables by keying in each piece of data and pressing ENTER to store it. When you are ready to solve for the unknown variable, scroll to bring it up on your display and press CPT. For example, use this sequence to find the effective rate equivalent to the nominal rate of 6.6% compounded semi-annually:

$$\text{2nd ICONV}, 6.6 \text{ Enter } \uparrow, 2 \text{ Enter } \uparrow, \text{CPT}$$

$$\text{Answer: } 6.7089$$

Paths To Success

Annually compounded interest rates can be used to quickly answer a common question: "How long does it take for my money to double?" The Rule of 72 is a rule of thumb stating that 72% divided by the annually compounded rate of return closely approximates the number of years required for money to double. Written algebraically this is

$$\text{Approximate Years } = \frac{72\%}{IY \text{ annually}}$$

For example, money invested at 9% compounded annually takes approximately $72 \div 9\% = 8$ years to double (the actual time is 8 years and 15 days). Alternatively, money invested at 3% compounded annually takes approximately $72 \div 3\% = 24$ years to double (the actual time is 23 years and 164 days). Note how close the approximations are to the actual times.

? Exercise 9.6.1: Give It Some Thought

i.

b. borrowing?

- If one investment takes 36 years to double while another takes 18 years to double, which has the higher effective rate?

Answer

a. The effective rate is equal to or higher than the nominal rate.
b. i. 9%. since more interest is earned
   ii. 8%, since less interest is paid
c. Eighteen years, since a higher rate takes less time to double

Example 9.6.1: Understanding Your Investment

If your investment earns 5.5% compounded monthly, what is the effective rate of interest? According to the Rule of 72, approximately how long will it take your investment to double at this effective rate?

Solution

Calculate the effective rate of interest (i_{New}). Once known, apply the Rule of 72 to approximate the doubling time.

What You Already Know

Step 1:

The original nominal interest rate and compounding along with the new compounding are known: $IY = 5.5$ = monthly = 12; CY_{New} = 1

How You Will Get There

Step 2:

Apply Formula 9.1 to the original interest rate.

Step 3:

Apply Formula 9.4, where $i_{New} = IY$ annually.

Step 4:

To answer approximately how long it will take for the money to double, apply the Rule of 72.

Perform

Step 2:

$$i_{Old} = \frac{5.5\%}{12} = 0.458\overline{3}\%$$

Step 3:

$$i_{New} = (1 + 0.00458\overline{3})^{\frac{12}{1}} - 1 = 0.056408 = 5.6408\%$$

Step 4:

$$\text{Approximate Years } = \frac{72\%}{5.6408\%} = 12.76$$

Calculator Instructions

2nd ICONV

NOM	C/Y	EFF
5.5	12	Answer: 5.640786

You are effectively earning 5.6408% interest per year. At this rate of interest, it takes approximately 12¾ years, or 12 years and 9 months, for the principal to double.

Example 9.6.2: Your Car Loan

As you search for a car loan, all banks have quoted you monthly compounded rates (which are typical for car loans), with the lowest being 8.4%. At your last stop, the credit union agent says that by taking out a car loan with them, you would effectively be charged 8.65%. Should you go with the bank loan or the credit union loan?

Solution

Since it is normal for a car loan to be compounded monthly, convert the effective rate to a monthly rate (IY) so that it matches all the other quotes.

What You Already Know

Step 1:

The effective rate and the compounding are as follows: $i_{New} = 8.65$ = monthly = 12; CY_{New} = 1

How You Will Get There

(Note: In this case, the i_{New} is known, so the process is reversed to arrive at the IY. Thus, steps 2 and 3 are performed in the opposite order.)

Step 2:

Substitute into Formula 9.4, rearrange, and solve for i_{Old}.

Step 3:

Substitute into Formula 9.1, rearrange, and solve for IY.

Perform

Step 2:

$$\begin{aligned}0.0865 &= (1+i_{Old})^{12\div1}-1\\1.0865 &= (1+i_{Old})^{12}\\1.0865^{1/12} &= 1+i_{Old}\\1.006937 &= 1+i_{\text{Old}}\\0.006937 &= i_{\text{Old}}\end{aligned}$$

Step 3:

$$\begin{aligned}0.006937 &= \frac{IY}{12}\\IY &= 0.083249 \text{ or } 8.3249\%\end{aligned}$$

Calculator Instructions

2nd ICONV

NOM	C/Y	EFF
Answer: 8.324896	12	8.65

The offer of 8.65% effectively from the credit union is equivalent to 8.3249% compounded monthly. If the lowest rate from the banks is 8.4% compounded monthly, the credit union offer is the better choice.

Equivalent Interest Rates

At times you must convert a nominal interest rate to another nominal interest rate that is not an effective rate. For example, in the opening scenario of this section your mortgage rates were all quoted semi-annually except for one monthly rate. One way to compare these rates was to make them all effective rates. An alternative is to take the "oddball" rate and convert it to match the compounding of all the other rates. This brings up the concept of equivalent interest rates, which are interest rates with different compounding that produce the same effective rate and therefore are equal to each other. After one year, two equivalent rates have the same future value.

How It Works

To convert nominal interest rates you need no new formula. Instead, you make minor changes to the effective interest rate procedure and add an extra step. Follow these steps to calculate any equivalent interest rate:

Step 1: For the nominal interest rate that you are converting, identify the nominal interest rate (IY) and compounding frequency (CY_{Old}). Also identify the new compounding frequency ($CY_{Old} = 1$).

Step 2: Apply Formula 9.1 to calculate the original periodic interest rate ($i_{Old} = 1$).

Step 3: Apply Formula 9.4 to calculate the new periodic interest rate (i_{New}).

Step 4: Apply Formula 9.1 using i_{New} and CY_{New},rearrange, and solve for the new converted nominal interest rate (IY).

Once again revisiting the mortgage rates from the section opener, compare the 6.6% compounded semi-annually rate to the 6.57% compounded quarterly rate by converting one compounding to another. It is arbitrary which interest rate you convert. In this case, choose to convert the 6.57% compounded quarterly rate to the equivalent nominal rate compounded semi-annually.

Step 1: The original nominal interest rate $IY = 6.57\%$ and the $CY_{Old} = \text{ quarterly } = 4$. Convert to $CY_{New} = \text{ semi-annually } = 2$.

Step 2: Applying Formula 9.1, $i_{\text{old}} = 6.57\%/4 = 1.6425\%$.

Step 3: Applying Formula 9.4, $i_{\text{New}} = (1+0.016425)^{4\div 2} - 1 = 0.033119$

Step 6: Applying Formula 9.1, $0.033119 = \dfrac{IY}{2}$ or $IY = 0.06624$.

Thus, 6.57% compounded quarterly is equivalent to 6.624% compounded semi-annually. Pick the mortgage rate of 6.6% compounded semi-annually since it is the lowest rate available. Of course, this is the same decision you reached earlier.

Important Notes

Converting nominal rates on the BAII Plus calculator takes two steps:

Step 1: Convert the original nominal rate and compounding to an effective rate. Input NOM and C/Y, then compute the EFF.

Step 2: Input the new C/Y and compute the NOM.

For the mortgage rate example above, use this sequence:

$$\text{2nd ICONV}, 6.57 \text{ Enter } \uparrow, 4 \text{ Enter } \uparrow, \text{CPT} \downarrow, 2 \text{ Enter } \downarrow, \text{CPT}$$

$$\text{Answer: } 6.623956$$

A good way to memorize this technique is to remember to "go up then down the ladder." To convert a nominal rate to an effective rate, you press "↑" twice. To convert an effective rate back to a nominal rate, you press "↓" twice. Hence, you go up and down the ladder!

Things To Watch Out For

When converting interest rates, the most common source of error lies in confusing the two values of the compounding frequency, or CY. When working through the steps, clearly distinguish between the old compounding (CY_{Old}) that you want to convert from and the new compounding (CY_{New}) that you want to convert to. A little extra time spent on double-checking these values helps avoid mistakes.

? Exercise 9.6.2: Give It Some Thought

a. If you convert a quarterly rate into a monthly rate, is the converted nominal rate higher or lower?

Answer

a. Higher. The interest is placed into the account less frequently, so more interest needs to go in each time.
b. Lower. The interest is placed into the account more frequently, so less interest is needed each time.

✓ Example 9.6.3: Which Investment to Choose?

You are looking at three different investments bearing interest rates of 7.75% compounded semi-annually, 7.7% compounded quarterly, and 7.76% compounded semi-annually. Which investment offers the highest interest rate?

Solution

Notice that two of the three interest rates are compounded semi-annually while only one is compounded quarterly. Although you could convert all three to effective rates (requiring three calculations), it is easier to convert the quarterly compounded rate to a semi-annually compounded rate. Then all rates are compounded semi-annually and are therefore comparable.

What You Already Know

Step 1:

The original interest rate and compounding are known: IY = 7.7%; CY_{Old} = quarterly = 4; CY_{New} = semi-annually = 2

How You Will Get There

Step 2:

Apply Formula 9.1 to the original interest rate.

Step 3:

Convert the original interest rate to its new periodic rate using Formula 9.4.

Step 4:

Substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

$$i_{Old} = \frac{7.7\%}{4} = 1.925\%$$

Step 3:

$$i_{New} = (1 + 0.01925)^{4 \div 2} - 1 = 0.038870$$

Step 4:

$$\begin{aligned} 0.038870 &= \frac{IY}{2} \\ IY &= 0.077741 \text{ or } 7.7741\% \end{aligned}$$

Calculator Instructions

2nd ICONV

Steps	NOM	C/Y	EFF
1-3	7.7	4	Answer: 7.925204
4-6	Answer: 7.774112	2	√

The quarterly compounded rate of 7.7% is equivalent to 7.7741% compounded semi-annually. In comparison to the semi-annually compounded rates of 7.75% and 7.76%, the 7.7% quarterly rate is the highest interest rate for the investment.

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9.7: Determining the Number of Compounds

How long will it take to reach a financial goal? At a casual get-together at your house, a close friend discusses saving for a 14-day vacation to the Blue Bay Grand Esmeralda Resort in the Mayan Riviera of Mexico upon graduation. The estimated cost from Travelocity.ca is $1,998.94 including fares and taxes. He has already saved $1,775 into a fund earning 8% compounded quarterly. Assuming the costs remain the same and he makes no further contributions, can you tell him how soon he will be basking in the sun on the beaches of Mexico?

This section shows you how to calculate the time frame for single payment compound interest transactions. You can apply this knowledge to any personal financial goal. Or in your career, if you work at a mid-size to large company, you might need to invest monies with the objective of using the funds upon maturity to pursue capital projects or even product development opportunities. So knowing the time frame for the investment to grow large enough will allow you to schedule the targeted projects.

Introductory scenarios examine situations where the number of compounding periods works out to be an integer. Then you will tackle more challenging scenarios involving time frame computations with non-integer compounding periods.

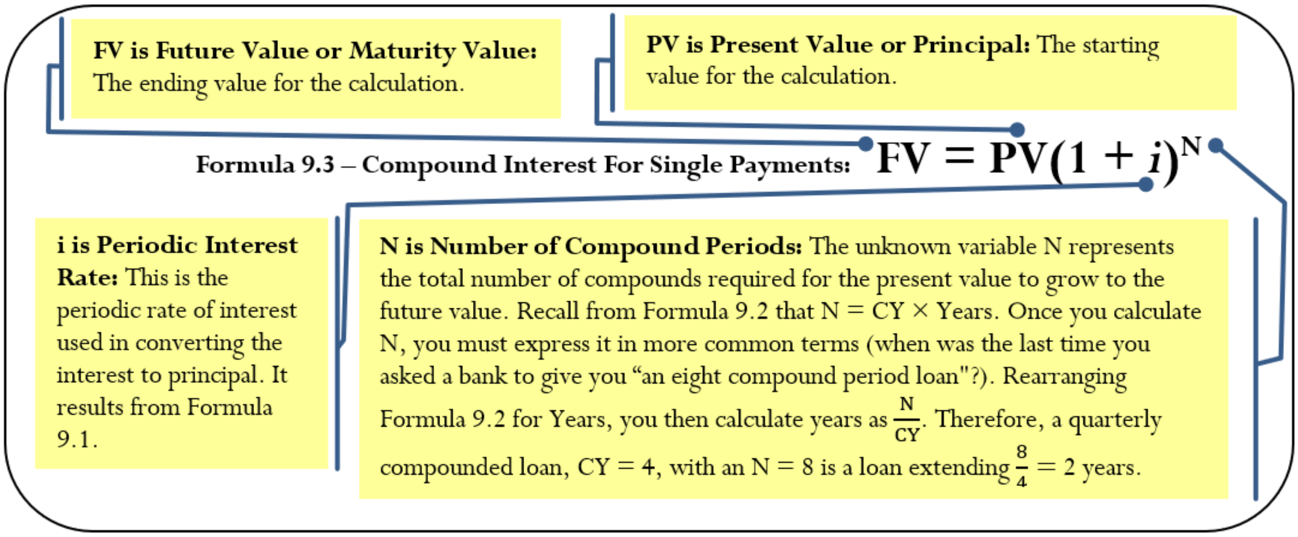
Integer Compounding Periods

Some applications of solving for the number of compounding periods include the following:

- Determining the time frame to meet a financial goal
- Calculating the time period elapsing between a present and future value
- Evaluating the performance of financial investments

The Formula

To solve for the number of compounds you need Formula 9.3 one more time. The only difference from your previous uses of this formula is that the unknown variable changes from FV to N, which requires you to substitute and rearrange the formula algebraically.



FV is Future Value or Maturity Value: The ending value for the calculation.

PV is Present Value or Principal: The starting value for the calculation.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1+i)^N$

i is Periodic Interest Rate: This is the periodic rate of interest used in converting the interest to principal. It results from Formula 9.1.

N is Number of Compound Periods: The unknown variable N represents the total number of compounds required for the present value to grow to the future value. Recall from Formula 9.2 that $N = CY \times \text{Years}$. Once you calculate N, you must express it in more common terms (when was the last time you asked a bank to give you “an eight compound period loan"?). Rearranging Formula 9.2 for Years, you then calculate years as $\frac{N}{CY}$. Therefore, a quarterly compounded loan, $CY = 4$, with an $N = 8$ is a loan extending $\frac{8}{4} = 2$ years.

How It Works

Follow these steps to compute the number of compounding periods (and ultimately the time frame):

Step 1: Draw a timeline to visualize the question. Most important at this step is to identify PV, FV, and the nominal interest rate (both IY and CY).

Step 2: Solve for the periodic interest rate (i) using Formula 9.1.

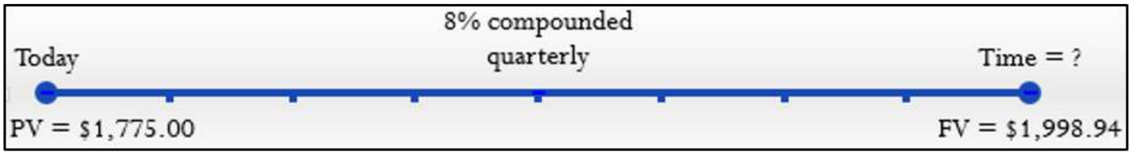
Step 3: Substitute into Formula 9.3, rearrange, and solve for N. Note that the value of N represents the number of compounding periods. For example, if the compounding is quarterly, a value of $N = 9$ is nine quarters.

Step 4: Take the value of N and convert it back to a more commonly expressed format such as years and months. When the number of compounding periods calculated in step 3 works out to an integer, handling step 4 involves applying the rearranged Formula 9.2 and solving for $\text{Years}=\frac{N}{CY}$.

1. If the Years is an integer, you are done.
2. If the Years is a non-integer, the whole number portion (the part in front of the decimal) represents the number of years. As needed, take the decimal number portion (the part after the decimal point) and multiply it by 12 to convert it to months. For example, if you have $\text{Years}=8.25$ then you have 8 years plus $0.25\times 12=3$ months, or 8 years and 3 months.

Revisiting the opening scenario, your friend has saved $1,775 and needs it to become $1,998.94 at 8% compounded quarterly. How long will it take?

Step 1: The timeline illustrates this scenario. Note that $IY=8\%$ and $CY=\text{quarterly}=4$.



Step 2: The periodic interest rate is $i=8\%/4=2\%$.

Step 3: Applying Formula 9.3, you have $\$1,998.94=\$1,775(1+0.02)^N$ or $N=6$ (details of the algebra can be found in subsequent examples).

Step 4: Applying the rearranged Formula 9.2, $\text{Years}=\frac{6}{4}=1.5$. Your friend will be headed to the Mayan Riviera in $1\frac{1}{2}$ years. If you prefer to express this in months, it is 1 year plus $0.5\times 12=6$ months, or 1 year and 6 months.

Important Notes

You can use your financial calculator in the exact same manner as in Section 9.2 except that your unknown variable will be N instead of FV. As always, you must load the other six variables and obey the cash flow sign convention. Once you compute N, you will have to complete step 4 manually to express the solution in its more common format of years and months.

Things To Watch Out For

N rarely works out to an integer, and that causes a lot of grief. When the values of both the PV and FV are known, usually one of these numbers is rounded. Because of this rounding, the calculation of N almost always produces decimals, even if those decimals are very small. For example, in calculating $N=6$ it is possible to come up with a value of 5.999963 or 6.000018. How can you tell if those decimals result from a rounded value (meaning that N is really an integer) or if those decimals are significant and need to be dealt with as they stand (as you will see discussed below when we consider non-integer compounding periods)?

You determine which way to go by mentally rounding your calculated N to the third decimal place. If this results in an integer, then the decimals are due to the rounding of FV and from then on you can treat the N as an integer. If a non-integer is still present, then the decimals are meaningful and you will have to address a non-integer compounding period (as discussed later in this section). For example, 5.999963 when mentally rounded to three decimals is 6.000. Therefore, you consider the N to be 6 and not 5.999963. In contrast, 5.985996 when mentally rounded to three decimals is 5.986, which is a non-integer. This means the decimals are significant, so you will not be able to round off to the closest integer.

Paths To Success

In Section 9.6, the Rule of 72 allowed you to estimate the time it takes in years for money to double when you know the effective rate. When you work with any time frame that involves the doubling of money, this rule can allow you to quickly check whether your calculated value of N is reasonable.

? Exercise 9.7.1: Give It Some Thought

a. Listed are some values of N. Determine if you should treat the N like an integer or a noninteger.
   i. 9.000345
   ii. 5.993129
   iii. 6.010102

    iv. 11.999643
    v. 10.000987
    vi. 3.999999

b. Of $1,000 growing to $2,000 or $2,000 growing to $3,000, which will take longer at the same interest rate?

Answer

a. i. 9.000 integer
    ii. 5.993 non-integer
    iii. 6.010 non-integer
    iv. 12.000 integer
    v. 10.001 non-integer
    vi. 4.000 integer

b. $1,000 growing to $2,000 will take longer since it involves a doubling of money, while $2,000 growing to $3,000 increases the amount by less than a doubling.

Example $9.7.1$: Integer Compounding Period Investment

Jenning Holdings invested $43,000 at 6.65% compounded quarterly. A report from the finance department shows the investment is currently valued at $67,113.46. How long has the money been invested?

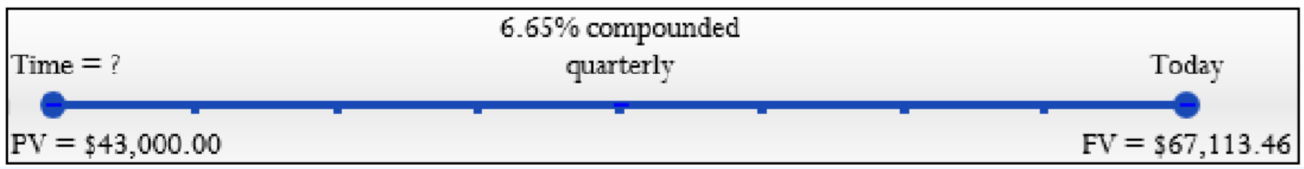
Solution

Determine the amount of time that the principal has been invested. This requires calculating the number of compounding periods (N).

What You Already Know

Step 1:

The principal, future value, and interest rate are known, as illustrated in the timeline.



IY = 6.65%; CY =4

How You Will Get There

Step 2:

Apply Formula 9.1.

Step 3:

Substitute into Formula 9.3, rearrange, and solve for N. Recall from Section 2.6 that to solve an equation for an unknown exponent you take the logarithm of both sides. You will also need to apply the general rule for logarithm of a power $\ln(x^y) = y(\ln x)$.

Step 4:

Apply Formula 9.2, rearrange, and solve for Years.

Perform

Step 2:

$$i = \frac{6.65\%}{4} = 1.6625\%$$

Step 3:

$$\begin{aligned}\$67,113.46 &= \$43,000(1+0.016625)^N \\ 1.560778 &= 1.016625^N \\ \ln(1.560778) &= \ln\left(1.016625^N\right) = N \times \ln(1.016625) \\ 0.445184 &= N \times 0.016488 \\ N &= 26.999996 \text{ or } 27 \text{ quarterly compounds}\end{aligned}$$

Step 4:

$$\begin{aligned}27 &= 4 \times \text{Years} \\ \text{Years} &= 6.75\end{aligned}$$

which is 6 years plus 0.75 × 12 = 9 months

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 26.999996	6.65	-43000	0	67113.46	4	4

Jenning Holdings has had the money invested for six years and nine months.

Non-integer Compounding Periods

When the number of compounding periods does not work out to an integer, the method of calculating N does not change. However, what changes is the method by which you convert N and express it in more common terms (step 4 of the process). Typically, the non-integer involves a number of years, months, and days.

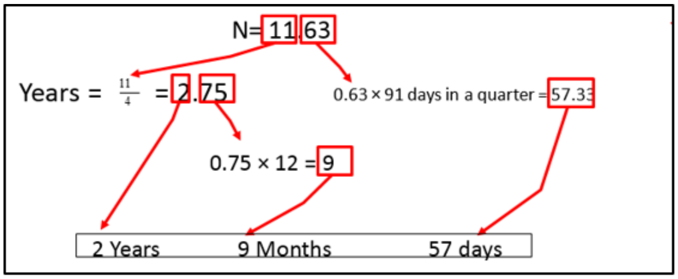
As summarized in the table below, to convert the compounding period into the correct number of days you can make the following assumptions:

Compounding Period	# of Days in the Period
Annual	365
Semi-annual	182*
Quarter	91*
Month	30*
Week	7
Daily	1

How It Works

You still use the same four steps to solve for the number of compounding periods when N works out to a non-integer as you did for integers. However, in step 4 you have to alter how you convert the N to a common expression. Here is how to convert the value:

1. Separate the integer from the decimal for your value of N.
2. With the integer portion, apply the same technique used with an integer N to calculate the number of years and months.
3. With the decimal portion, multiply by the number of days in the period to determine the number of days and round off the answer to the nearest day (treating any decimals as a result of a rounded interest amount included in the future value).


N= 11.63
Years = $\frac{11}{4}$ = 2.75
0.63 × 91 days in a quarter = 57.33
0.75 × 12 = 9
2 Years
9 Months
57 days

The figure to the right illustrates this process for an $N = 11.63$ with quarterly compounding, or $CY = 4$. Thus, an $N = 11.63$ quarterly compounds converts to a time frame of 2 years, 9 months, and 57 days. Note that without further information, it is impossible to simplify the 57 days into months and days since it is unclear whether the months involved have 28, 29, 30, or 31 days.

Example $9.7.2$: Saving for Postsecondary Education

Tabitha estimates that she will need $20,000 for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up $8,500, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually?

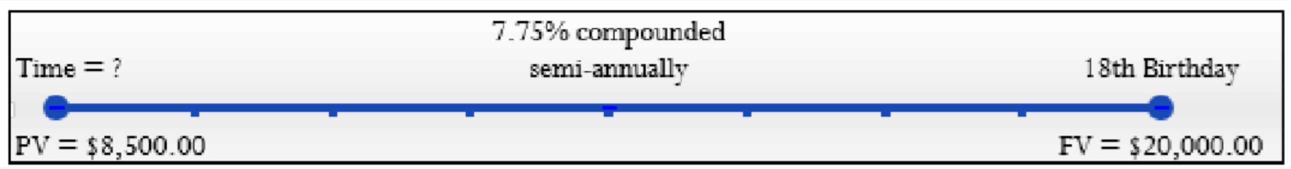
Solution

Determine the amount of time in advance of the daughter's 18th birthday that the money needs to be invested. The number of compounding periods (N) must be calculated.

What You Already Know

Step 1:

The principal, future value, and interest rate are known, as illustrated in the timeline.



IY = 7.75%; CY =2

How You Will Get There

Step 2:

Apply Formula 9.1.

Step 3:

Substitute into Formula 9.3, rearrange, and use the rule for logarithm of a power to solve for N.

Step 4:

Apply Formula 9.2, rearrange, and solve for Years.

Step 2:

$$i = \frac{7.75\%}{2} = 3.875\%$$

Step 3:

$$\begin{aligned} \$20,000 &= \$8,500(1+0.03875)^N \\ 2.352941 &= 1.03875^N \\ \ln(2.352941) &= N \times \ln(1.03875) \\ 0.855666 &= N \times 0.038018 \\ N &= 22.506828 \text{ semi-annual compounds} \end{aligned}$$

Step 4:

Take the integer:

$$\begin{aligned} 22 &= 2 \times \text{Years} \\ \text{Years} &= 11 \end{aligned}$$

Take the decimal: semi-annual compounding according is 182 days, so 0.506828 × 182 days = 92 days

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 22.506828	7.75	-8500	0	20000	2	2

If Tabitha invests the $8,500 11 years and 92 days before her daughter's 18th birthday, it will grow to $20,000.

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9.S: Summary

Key Concepts Summary

9.1: Compound Interest Fundamentals (What Can a Dollar Buy?)

- How compounding works
- How to calculate the periodic interest rate

9.2: Determining the Future Value (I Want to Pay Later)

- The basics of taking a single payment and moving it to a future date
- Moving single payments to the future when variables change

9.3: Determining the Present Value (I Want to Pay Earlier)

- The basics of taking a single payment and moving it to an earlier date
- Moving single payments to the past when variables change

9.4: Equivalent Payments (Let’s Change the Deal)

- The concept of equivalent payments
- The fundamental concept of time value of money
- The fundamental concept of equivalency
- Applying single payment concepts to loans and payments

9.5: Determining the Interest Rate (Is a Lot of Interest Accumulating or Just a Little?)

1. Solving for the nominal interest rate
2. How to convert a variable interest rate into its equivalent fixed interest rate

9.6: Equivalent and Effective Interest Rates (How Do I Compare Different Rates?)

- The concept of effective rates
- Taking any nominal interest rate and finding its equivalent nominal interest rate

9.7: Determining the Number of Compounds (How Far Away Is That?)

- Figuring out the term when N is an integer
- Figuring out the term when N is a non-integer

The Language of Business Mathematics

compound interest

A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

compounding frequency

The number of compounding periods in a complete year.

compounding period

The amount of time that elapses between the dates of successive conversions of interest to principal.

discount rate

An interest rate used to remove interest from a future value.

effective interest rate

The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.

equivalent payment streams

Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.

equivalent interest rates

Interest rates with different compounding that produce the same effective rate and therefore are equal to each other.

focal date

A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.

fundamental concept of equivalency

Two or more payment streams are equal to each other if they have the same economic value on the same focal date.

fundamental concept of time value of money

All monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined.

nominal interest rate

A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.

periodic interest rate

The percentage of interest earned or charged at the end of each compounding period.

present value principle for loans

The present value of all payments on a loan is equal to the principal that was borrowed.

Rule of 72

A rule of thumb stating that 72 divided by the annually compounded rate of return closely approximates the number of years required for money to double.

The Formulas You Need to Know

Symbols Used

CY or C/Y = Compounds per year, or compounding frequency

CY_{New} = the new compounding frequency an interest rate is converted to

CY_{Old} = the old compounding frequency an interest rate is converted from

FV = Future value, or maturity value

i = Periodic interest rate

i_{New} = the new periodic interest rate after a conversion

i_{Old} = the old periodic interest rate before a conversion

IY or I/Y = Nominal interest rate per year

$\ln$ = natural logarithm

N = number of compound periods

PV = Present value, or principal

Formulas Introduced

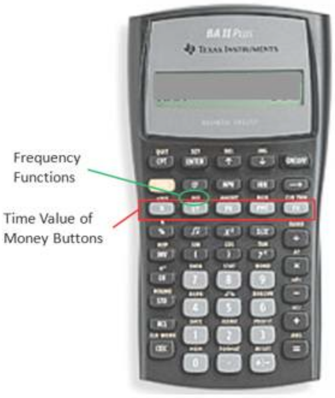
Formula 9.1 Periodic Interest Rate: $i=\dfrac{IY}{CY}$

Formula 9.2 Number of Compound Periods for Single Payments: $N=CY\times\text{ Years}$

Formula 9.3 Compound Interest for Single Payments: $FV = PV(1+t)^N$

Formula 9.4 Interest Rate Conversion: $i_{New} = (1+i_{Old})^{\frac{CY_{Old}}{CY_{New}}} - 1$

Technology



Calculator

Time Value of Money Buttons

1. The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
N	N	The number of compounding periods (from Formula 9.2)	An integer or decimal number; no negatives
I/Y	IY	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	PV	Present value or principal	An integer or decimal number
PMT	PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations	An integer or decimal number
FV	FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

2. The frequency function is logically placed above the I/Y button and is labelled P/Y. This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing 2nd P/Y to find the following entry fields, through which you can scroll using your i and h arrow buttons.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
P/Y	PY	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y	A positive, nonzero number only
C/Y	CY	Compounds per year (compounding frequency)	A positive, nonzero number only

- To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press Enter.
- When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually.
- To exit the P/Y window, press 2nd Quit.

Keying in a Question

- You must load the calculator with six of the seven variables.
- To solve for the missing variable, press CPT followed by the variable.

Cash Flow Sign Convention

- The cash flow sign convention is used for the PV, PMT, and FV buttons.
- If money leaves you, you must enter it as a negative.
- If money comes at you, you must enter it as a positive.

Clearing the Memory

- Once you enter data into any of the time value buttons, it is permanently stored until one of the following happens:
    - You override it by entering another piece of data and pressing the button.
    - You clear the memory of the time value buttons by pressing 2nd CLR TVM (a step that is recommended before you proceed with a separate question).
    - You press the reset button on the back of the calculator.

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9.CS: Case Study - Exploring Different Sources Of Compound Financing

The Situation

Quentin is the human resources manager for Lightning Wholesale. Recently, he was approached by Katherine, who is one of the employees in the sporting goods department. She enquired about the possibility of getting an advance on her next paycheck. Quentin informed her that Lightning Wholesale's payroll policy is not to provide advances on paychecks. Katherine indicated that she was having some financial problems, did not have a very good credit rating, and would be forced on her way home to stop in at a payday loan company to take out an advance.

Concerned for his employee, Quentin visited Katherine later in the day and sat down with her to show her the true costs of using a payday loan company. He also wanted to show her a better alternative to get some short-term financing.

The Data

Canadian payday loan companies generally operate under one of three structures:

1. The Traditional Model. These companies incur all operating costs, provide their own capital for any loan, and collect interest and charges or fees for their services. These companies assume all of the risk.
2. The Broker Model. These companies incur all operating costs but do not provide the capital for the loan. A third-party partner provides the capital and the payday loan company charges a brokerage fee for its services. The third-party partner collects all interest and assumes all risk.
3. The Insurance Model. These companies incur all operating costs and recover these costs through fees and insurance premiums on the loan. An insurance company (usually owned by the payday loan company) provides all capital and assumes all risk.

The table below summarizes a sample of charges that could be imposed under each model.

	Traditional Model	Broker Model	Insurance Model
Effective Rate of Interest	59%	21%	48%
Per Transaction Fee	$9.99	$10.00	N/A
Check Cashing Fee	7.99% of principal and interest combined	$8.00	N/A
Insurance Fee	N/A	N/A	19% of principal and interest combined
Brokerage Fee	N/A	29.5% of principal and interest combined	N/A

- Observe that under Section 347 of the Criminal Code, any charges related to the borrowing of money are considered interest. This includes any types of fees and charges, although in name they may not be called interest.
- As an alternative to using a payday loan company, Katherine could use a finance company that targets people with poor credit ratings or those in quick need of money. These companies typically charge 28% compounded monthly on loans.
- A second alternative is to take a cash advance on her credit card. Most credit card companies charge around 18% compounded daily.

Important Information

- Like most people who borrow money from payday loan companies, Katherine needs to borrow a small sum of money for a short period of time. Her requirements are to borrow $300 for a period of seven days, or one week.
- Assume exactly 52 weeks in a year.

Your Tasks

1. Show Katherine the true effective rate of interest she is being charged if she borrows the money from any of these payday loan companies. For each model:
    a. Convert the effective rate into a nominal weekly compounded rate.

   b. Calculate the future value of the loan after one week using the weekly periodic rate.
   c. Calculate any cheque cashing, brokerage, or insurance fees on the future value.
   d. Take the interest on the loan and add all fees charged, including any flat fees. This is the total interest amount on the loan.
   e. Convert the interest amount into a percentage of principal. This is the periodic interest rate per week.
   f. Take the periodic interest rate per week and convert it into a nominal weekly compounded rate.
   g. Convert the nominal weekly compounded rate into an effective rate.
2. Now make Katherine aware of what the alternative sources of financing will cost.
   a. Convert both the finance company's interest rate and the credit card's interest rate into effective rates and weekly compounded nominal rates.
   b. For each option, calculate the future value of her loan and determine the amount of interest charged.
3. Examine the effective rates for all of the options. Rank them from the best alternative to the worst alternative.
4. Look at the amount of interest paid under the best alternative compared to the worst alternative. Express the worst alternative as a percentage of the best alternative.
5. Summarize your findings for Katherine.

Contributors and Attributions

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9.E: Compound Interest- Working With Single Payments (Exercises)

9.1: Compound Interest Fundamentals

For each of the following questions, round all percentages to four decimals where needed.

Mechanics

For questions 1–9, solve for the unknown variables (identified with a ?) based on the information provided.

Nominal Interest Rate	Compounding Frequency	Periodic Interest Rate
1. 7.2%	Monthly	?
2. 5.85%	Semi-annually	?
3. ?	Quarterly	1.95%
4. ?	Daily	0.08%
5. 9.2%	?	2.3%
6. 5.5%	?	$0.458\overline{3}\%$
7. ?	Annually	14.375%
8. 3.9%	Quarterly	?
9. 20.075%	?	0.055%

Applications

10. Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.
11. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.
12. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.
13. Lori hears her banker state, “We will nominally charge you 10.68% on your loan, which works out to 0.89% of your principal every time we charge you interest.” What is her compounding frequency?
14. You just received your monthly MasterCard statement and note at the bottom of the form that interest is charged at 19.5% compounded daily. What interest rate is charged to your credit card each day?
15. Larry just purchased a new Ford Mustang from his local Ford dealer. His contract states that he will be charged interest at $0.658\overline{3}\%$ per month. What is his nominal interest rate?

Challenge, Critical Thinking, & Other Applications

16. Betty just signed a contract for her business that will allow her to make interest-only payments for the next 12 months. If her interest rate is 12% compounded monthly and her outstanding principal is $5,000, how much will she pay in interest every month?
17. Geoff is shopping around for a car loan. At three different websites he saw posted rates of 6.14% compounded semiannually, 3.06% compounded per six months, and 1.49% compounded every quarter. Which is the lowest nominal rate?
18. After a period of three months, Alese saw one interest deposit of $176.40 for a principal of $9,800. What nominal rate of interest is she earning?
19. Take a nominal interest rate of 10.4% and convert to its matching periodic interest rate if interest is compounded semiannually, quarterly, monthly, or daily.
20. Take a periodic interest rate of 1.1% and convert to its matching nominal interest rate if the compounding period is per month, per quarter, or per six months.

9.2: Determining the Future Value

Mechanics

For questions 1–3, solve for the future value at the end of the term based on the information provided.

Principal	Interest Rate	Term
1. $7,500	8% compounded quarterly	3 years
2. $53,000	6% compounded monthly	4 years, 3 months
3. $19,000	5.75% compounded semi-annually	6 years, 6 months

For questions 4–6, solve for the future value at the end of the sequence of interest rate terms based on the information provided.

Principal	Interest Rates and Term
4. $3,750	4.75% compounded annually for 3 years; then 5.5% compounded semi-annually for 2 years
5. $11,375	7.5% compounded monthly for 2 years, 9 months; then 8.25% compounded quarterly for 3 years, 3 months
6. $24,500	4.1% compounded annually for 4 years; then 5.15% compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months

For questions 7–9, solve for the future value at the time period specified based on the information provided.

Principal	Payments or Deposits	Interest Rates and Term	Find Future Balance At
7. $2,300 loan	$750 payment at 9 months; $1,000 payment at 2 years	9% compounded monthly for 9 months; then 8.75% compounded quarterly for 2.25 years	3 years
8. $4,000 investment	$1,500 deposit at 6 months; $1,500 deposit at 1½ years	10% compounded semi-annually for 6 months; then 12% compounded quarterly for 1½ years	2 years
9. $3,000 loan	$300 payment at 4 months; $1,200 payment at 1 year, 9 months	8% compounded monthly for 1 year, 9 months; then 9% compounded quarterly for 1 year, 9 months	3½ years

Applications

10. You are planning a 16-day African safari to Rwanda to catch a rare glimpse of the 700 remaining mountain gorillas in the world. The estimated cost of this once-in-a-lifetime safari is $15,000 including the tour, permits, lodging, and airfare. Upon your graduation from college, your parents have promised you a $10,000 graduation gift. You intend to save this money for five years in a long-term investment earning 8.3% compounded semi-annually. If the cost of the trip will be about the same, will you have enough money five years from now to pay for your trip?
11. Your investment of $9,000 that you started six years ago earned 7.3% compounded quarterly for the first 3¼ years, followed by 8.2% compounded monthly after that. How much interest has your investment earned so far?
12. What is the maturity value of your $7,800 investment after three years if the interest rate was 5% compounded semiannually for the first two years, 6% compounded quarterly for the last year, and 2½ years after the initial investment you made a deposit of $1,200? How much interest is earned?

13. Nirdosh borrowed $9,300 4¼ years ago at 6.35% compounded semi-annually. The interest rate changed to 6.5% compounded quarterly 1¾ years ago. What amount of money today is required to pay off this loan?
14. Jason invested $10,000 into his RRSP when he turned 20 years old. Maria invested $10,000 into her RRSP when she turned 35 years old. If both earned 9% compounded semi-annually, what percentage more money (rounded to one decimal) will Jason have than Maria when they both turn 65 years old?
15. Cristy borrowed $4,800 from a family friend 2½ years ago at 7% compounded annually for the first year and 8% compounded semi-annually thereafter. She made a payment 1½ years into the loan for $2,500. How much should Cristy pay today to clear her loan?

Challenge, Critical Thinking, & Other Applications

16. A product manager wants to understand the impact of inflation on her gross profit margin. Inflation is expected to remain constant at 3.5% per year for the next five years. The price of the product is expected to remain unchanged at $99.99 over the five years and the current cost of the product today is $59.
    a. On a per-unit basis, in what dollar amount is the markup reduced after five years?
    b. What percent change does this represent in the gross profit margin?
17. The Teachers' Association just negotiated a four-year contract for its members, who will receive a 3.5% wage increase immediately followed by annual increases of 3.75%, 4.25%, and 4.1% on the anniversaries of the agreement. In the final year of the contract, how much more would the human resources manager for the school division need to budget for salaries if the average teacher currently earns $72,000 per year and the division employs 34 teachers?
18. You invested $5,000 10 years ago and made two further deposits of $5,000 each after four years and eight years. Your investment earned 9.2% compounded quarterly for the first two years, 8.75% compounded monthly for the next six years, and 9.8% compounded semi-annually for the remaining years. As of today, how much interest has your investment earned?
19. Suppose you placed $10,000 into each of the following investments. Rank the maturity values after five years from highest to lowest.
    a. 8% compounded annually for two years followed by 6% compounded semi-annually
    b. 8% compounded semi-annually for two years followed by 6% compounded annually
    c. 8% compounded monthly for two years followed by 6% compounded quarterly
    d. 8% compounded semi-annually for two years followed by 6% compounded monthly
20. You made the following three investments:
    - $8,000 into a five-year fixed rate investment earning 5.65% compounded quarterly.
    - $6,500 into a five-year variable rate investment earning 4.83% compounded semi-annually for the first 2½ years and 6.5% compounded monthly for the remainder.
    - $4,000 into a five-year variable rate investment earning 4.75% compounded monthly for the first two years and 5.9% compounded quarterly thereafter, with a $4,000 deposit made 21 months into the investment.

What is the total maturity value of all three investments after the five years, and how much interest is earned?

9.3: Determining the Present Value

Mechanics

For questions 1–3, solve for the present value at the beginning of the term based on the information provided.

Maturity Value	Interest Rate	Term
1. $14,000	9% compounded semi-annually	14 years
2. $202,754.18	7.85% compounded quarterly	11 years
3. $97,000	6% compounded monthly	9 years, 3 months

For questions 4–6, solve for the present value at the beginning of the sequence of interest rate terms based on the information provided.

Maturity Value	Interest Rates (in order from the start of the transaction) and Term
4. $45,839.05	4.5% compounded semi-annually for 4½ years; then 3.25% compounded annually for 4 years
5. $7,223.83	8.05% compounded semi-annually for 2 years, 6 months; then 7.95% compounded quarterly for 1 year, 3 months; then 7.8% compounded monthly for 2 years, 9 months
6. $28,995.95	18.75% compounded monthly for 4 years, 4 months; then 19.1% compounded daily for 3 years; then 18% compounded monthly for 1 year 7 months

For questions 7–9, solve for the present value (today) based on the information provided.

Maturity Amount and Date	Payments or Deposits (all times are from today)	Interest Rates (starting from today) and Term
7. $7,500 due 4 years from today	$3,000 payment at 2 years	8% annually for 1 year; then 6% semi-annually for 3 years
8. $108,300 due 7 years from today	$50,000 payment at 3½ years; $25,000 payment at 6 years	4.8% monthly for 3 years, 6 months; then 5.2% semi-annually for 3 years, 6 months
9. $25,000 saved 5 years from today	$5,000 deposited at 2 years, 3 months; $5,000 deposited at 4 years	9% quarterly for 1 year; Then 9¼% quarterly for 4 years

Applications

10. Dovetail Industries needs to save $1,000,000 for new production machinery that it expects will be needed six years from today. If money can earn 8.35% compounded monthly, how much money should Dovetail invest today?

11. A debt of $37,000 is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?
12. Rene wants to invest a lump sum of money today to make a $35,000 down payment on a new home in five years. If he can place his money in an investment that will earn 4.53% compounded quarterly in the first two years followed by 4.76% compounded monthly for the remaining years, how much money does he need to invest today?
13. Amadala owes Nik $3,000 and $4,000, due nine months and two years from today, respectively. If she wants to pay off both debts today, what amount should she pay if money can earn 6% quarterly in the first year and 5.75% monthly in the second year?
14. Cheyenne just received her auto insurance bill. If she pays it in full today, she can deduct $15 off her total $950 premium. Alternatively, she can make two semi-annual payments of $475. If her money can earn 3% compounded monthly, which alternative should she pursue? How much money in today's dollars will she save in making that choice?
15. Geoff is placing his money into a three-year investment that promises to pay him 8%, 8.25%, and 8.5% in consecutive years. All interest rates are compounded quarterly. If he plans to make a deposit to this investment in the amount of $15,000 18 months from now and his goal is to have $41,000, what amount does he need to invest today?
16. In August 2004, Google Inc. made its initial stock offering. The value of the shares grew to $531.99 by July 2011. What was the original value of a share in August 2004 if the stock has grown at a rate of 26.8104% compounded monthly?

Challenge, Critical Thinking, & Other Applications

17. In 2000 and 2009, Canada's population was estimated at 31,496,800 and 33,487,208, respectively. In 2009, an estimated 74.9% of Canadians were known Internet users. If the historic annual rate of growth in Internet usage in Canada averaged 7.8568% per year, what percentage of the population in the year 2000 were Internet users?
18. If a three-year and seven-month investment earned $8,879.17 of interest at 3.95% compounded monthly, what amount was originally placed into the investment?
19. A lottery ticket advertises a $1 million prize. However, the fine print indicates that the winning amount will be paid out on the following schedule: $250,000 today, $250,000 one year from now, and $100,000 per year thereafter. If money can earn 9% compounded annually, what is the value of the prize today?
20. Your company is selling some real estate and has received three potential offers:
    - $520,000 today plus $500,000 one year from now.
    - $200,000 today, $250,000 six months from now, and $600,000 15 months from now.
    - $70,000 today, $200,000 in six months, then four quarterly payments of $200,000 starting six months later.

    Prevailing interest rates are expected to be 6.75% compounded semi-annually in the next year, followed by 6.85% compounded quarterly afterwards. Rank the three offers from best to worst based on their values today.

9.4: Equivalent Payments

Mechanics

For questions 1–6, solve for the equivalent values (X) at the time period specified based on the information provided.

Original Agreement	Proposed Equivalent Agreement	Prevailing Interest Rate
1. $5,000 due today; $5,000 due in 3 years	$X due in 27 months	6% compounded monthly
2. $4,385 due 1 year ago; $6,000 due in 4 years	$X due in 2 years	8.5% compounded quarterly
3. $16,000 due today; $8,000 due in 10 months; $19,000 due in 33 months	$X due in 16 months	6.65% compounded monthly
4. $2,500 due in 3, 6, 9, and 12 months	$X due in 7 months	8.88% compounded monthly
5. $38,000 due 1½ years ago; $17,000 due ½ year ago; $45,000 due in 1 year	$X today	9.5% compounded semi-annually
6. $3,750 due 2½ years ago; $2,400 due in 3¼ years; $1,950 due in 5 years	$X due in 2 years	10.75% compounded quarterly

For questions 7–9, solve for the missing payment(s) on the loan based on the information provided.

Loan Amount	Scheduled Payments	Loan Interest Rate
7. $35,000	$15,000 due in 6 months; $X due in 2 years	5.95% compounded semi-annually
8. $51,000	$15,000 due in 3 months; $10,000 due in 1 year; $X due in 1½ years	6.32% compounded quarterly
9. $44,000	$X due in 6 months; $2X due in 15 months	5.55% compounded monthly

Applications

10. A winning lottery ticket offers the following two options:
    a. A single payment of $1,000,000 today or
    b. $250,000 today followed by annual payments of $300,000 for the next three years.

If money can earn 9% compounded annually, which option should the winner select? How much better is that option in current dollars?

11. Darwin is a young entrepreneur trying to keep his business afloat. He has missed two payments to a creditor. The first was for $3,485 seven months ago and the second was for $5,320 last month. Darwin has had discussions with his creditor, who is willing to accept $4,000 one month from now and a second payment in full six months from now. If the agreed upon interest rate is 7.35% compounded monthly, what is the amount of the second payment?
12. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: $1,600 18 months ago, $2,300 nine months ago, and $5,100 three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of $1,000 today, $4,000 nine months from now, and a final payment two years from now. The client normally charges 16.5% compounded quarterly on all outstanding debts. What is the amount of the third payment?
13. Working in the accounting department, Ariel has noticed that a customer is having trouble paying its bills. The customer has missed a payment of $2,980 two years ago, $5,150 14 months ago, and $4,140 four months ago. The customer proposes making two payments instead. The first payment would be in six months and the second payment, one-quarter the size of the first payment, would be in 18 months. If the agreed-upon interest rate is 6.89% compounded monthly, what are the amounts of each payment?
14. Seth works in the finance department of a large corporation. His company has the following debts to the same creditor: $98,000 due in 2 years; $203,000 due in 4¼ years, and $157,000 due in 6½ years. Seth wants to align these payments with the maturity dates of his company's investments. He proposes to the creditor three payments due 1¾ years, 4½ years, and 5½ years from today. The second payment is to be twice the size of the first payment and the third payment is to be three-quarters the size of the second payment. If the creditor agrees to an interest rate of 7.25% compounded quarterly, calculate the amounts of each payment.
15. A $30,000 loan at 4.9% compounded semi-annually is to be repaid with four equal semi-annual payments. The first payment is one year after the loan. Calculate the amount of each payment.

Challenge, Critical Thinking, & Other Applications

16. The Ontario Labour Relations Board is reviewing a human resource complaint. At the time of filing, a construction worker indicated that her employer failed to pay her monthly wages of $3,400, $3,200, $3,600, and $3,475 starting four months prior. It has taken the Ontario Labour Relations Board nine months since the time of filing to gather the needed information and make a judgment in favour of the complainant. If the standard interest rate used in its judgment is 9% compounded monthly, what amount is awarded to the construction worker?
17. Larry is a financial adviser helping a client save up $100,000 in five years' time. The client has the financial means to pursue either of the following two alternatives:
    a. Make three equal deposits due today, in 2 years, and 4 years.
    b. Make four equal deposits due today, in 1 year, in 3 years, and 4½ years.

    If Larry can invest the funds at 9% compounded semi-annually, which option is in the best interests of the client? In current dollars, how much better for the client is that recommended option?
18. Four years ago Aminata borrowed $5,000 from Randal with interest at 8% compounded quarterly to be repaid one year from today. Two years ago Aminata borrowed another $2,500 from Randal at 6% compounded monthly to be repaid two years from today. Aminata would like to restructure the payments so that she can pay 15 months from today and 2½ years from today. The first payment is to be twice the size of the second payment. Randal accepts an interest rate of 6.27% compounded monthly on the proposed agreement. Calculate the amounts of each payment.
19. Yi-Fang is the store owner of a franchise. In flipping through her records, she notices the following debts to the same supplier: $2,389.56 due eight months ago, $3,478.34 due six months ago, $1,694.32 due four months ago, $6,497.98 due two months ago, $4,611.03 due today, $5,784.39 due in two months, and $5,177.44 due in four months. She would like to clear all of these debts with a single payment next month. If the supplier charges 18.1% compounded monthly on overdue balances and provides a credit of 9% compounded monthly on early payments, calculate the amount of the payment.
20. The City of Winnipeg has received two different offers to purchase a parcel of real estate in the southeast quadrant of the city. The Qualico Group has bid $12 million today, along with annual payments of $5 million for the first two years and $6 million in the subsequent two years. The Genstar Development Company has bid $7.5 million today, along with $8 million after one

year and three subsequent annual payments of $7 million. If the prevailing interest rates are 8.75% compounded semi-annually, which offer should the City of Winnipeg accept? In current dollars, how much better is the highest bid?

9.5: Determining the Interest Rate

Mechanics

For questions 1–5, solve for the nominal interest rate compounded based on the information provided.

Present Value (PV)	Future Value (FV)	Term	IY Compounding
1. $101,000.00	$191,981.42	10 years	monthly
2. $78,500.00	$144,935.53	7½ years	quarterly
3. $59,860.48	$78,500.00	5.5 years	semi-annually
4. $31,837.58	$1,000,000	40 years	annually
5. $5,000.00	$20,777.73	5 years	daily

For questions 6–8, solve for the equal fixed nominal interest rate based on the information provided. Unless otherwise specified, assume each variable interest rate is for a one-year period.

Variable Interest Rates	Fixed IY Compounding
6. 6% compounded semi-annually for 3 years; then 6.5% compounded quarterly for 2 years	annually
7. 3%, 3.5%, 4.25%, 5%, 6% all compounded quarterly	quarterly
8. 2.4%, 3.6%, 4.2% all compounded monthly; then 6.25% and 8% compounded semi-annually	monthly

Applications

9. Your company paid an invoice five months late. If the original invoice was for $6,450 and the amount paid was $6,948.48, what monthly compounded interest rate is your supplier charging on late payments?
10. In a civil lawsuit, a plaintiff was awarded damages of $15,000 plus $4,621.61 in interest for a period of 3¼ years. What quarterly compounded rate of interest was used in the settlement?
11. Muriel just received $4,620.01 including $840.01 of interest as payment in full for a sum of money that was loaned 2 years and 11 months ago. What monthly compounded rate of interest was charged on the loan?
12. Three years ago, Beverly made an investment that paid 4.95%, 5.55%, and 6.15% in each subsequent year. All interest rates are compounded monthly. What annually compounded fixed rate of return did she earn?
13. Indiana just received a maturity value of $30,320.12 from a semi-annually compounded investment that paid 4%, 4.1%, 4.35%, 4.75%, and 5.5% in consecutive years. What amount of money did Indiana invest? What fixed quarterly compounded nominal interest rate is equivalent to the variable rate his investment earned?
14. The five-year RRSP rate escalator program at TD Canada Trust offers semi-annually compounded interest rates of 1%, 2%, 3%, 4%, and 5%. A similar program at the Bank of Montreal posted rates of 1.1%, 1.55%, 2.75%, 3.25%, and 6.75%. Calculate the fixed semi-annual rates of return for each program and recommend where to invest in your RRSP. How much better is your recommended equivalent fixed rate?

Challenge, Critical Thinking, & Other Applications

15. At what monthly compounded interest rate does it take five years for an investment to double?
16. In 2003, a home in Winnipeg was purchased for $214,000. In 2011, the same home was appraised at $450,000. What annually compounded rate of growth does this reflect?
17. On October 1, 1975, the minimum wage in Manitoba was $2.60 per hour. It rose to $10 per hour by October 1, 2011. What is the annually compounded growth rate for minimum wage in Manitoba during this period?

18. Jean-Luc's first month's gross salary in June 1994 was $800. By June 2012 his monthly gross salary was $1,969.23 higher. What monthly compounded rate did his salary increase by over the period?
19. Listed below are the historical populations (census metropolitan areas) of seven Canadian cities. Calculate and rank the annual growth rates from highest to lowest.

City	1996	2010
Vancouver	1,602,590	2,391,252
Calgary	754,033	1,242,624
Regina	191,692	215,138
Winnipeg	660,450	753,555
Toronto	3,898,833	5,741,419
Montreal	3,208,970	3,859,318
Halifax	332,518	403,188

20. Listed below are several options for five-year investments posted on the same day. All rates are compounded quarterly. For each, calculate the equivalent fixed annually compounded rate of return and rank these rates from highest to lowest.

Company	Year 1	Year 2	Year 3	Year 4	Year 5
Affinity Credit Union	1%	1.45%	2.75%	3.5%	6%
MGI Financial	1.85%	2.6%	3.3%	3.41%	3.75%
GIC Max	2.1%	2.75%	3.4%	3.75%	3.8%
CIBC	0.5%	1.5%	2%	3.5%	6.5%
Laurentian	1.1%	1.75%	2.5%	3.4%	7%

9.6: Equivalent and Effective Interest Rates

Mechanics

Convert the following nominal interest rates to their effective rates.

Nominal Interest Rate
1. 4.75% compounded quarterly
2. 7.2% compounded monthly
3. 3.95% compounded semi-annually

Convert the following effective rates to their nominal rates.

Effective Interest Rate	Nominal Interest Rate
4. 10%	Semi-annually
5. 12%	Monthly
6. 8%	Quarterly

Convert the following nominal interest rates to their equivalent rates.

Nominal Interest Rate	Equivalent Rate
7. 6% compounded monthly	Semi-annually
8. 4.5% compounded quarterly	Monthly
9. 8% compounded semi-annually	Quarterly
10. 7.05% compounded monthly	Semi-annually

Applications

11. The HBC credit card has a nominal interest rate of 26.44669% compounded monthly. What effective rate is being charged?
12. Your company's union just negotiated a salary increase of 0.5% every three months. What effective increase in your salary was negotiated?
13. What is the effective rate on a credit card that charges interest of 0.049315% per day?
14. RBC offers two different investment options to its clients. The first option is compounded monthly while the latter option is compounded quarterly. If RBC wants both options to have an effective rate of 3.9%, what nominal rates should it set for each option?
15. Your competitor reported to its shareholders that growth in profits in the previous year averaged 0.1% per week (assume 52 weeks in a year). Your financial department shows your company’s profits have grown 0.45% per month. Compare your monthly profit growth to your competitor’s monthly profit growth. How much better is the higher growth?
16. Louisa is shopping around for a loan. TD Canada Trust has offered her 8.3% compounded monthly, Conexus Credit Union has offered 8.34% compounded quarterly, and ING Direct has offered 8.45% compounded semi-annually. Rank the three offers and show calculations to support your answer.

Challenge, Critical Thinking, & Other Applications

17. Your three-year monthly compounded investment just matured and you received $9,712.72, of which $2,212.72 was interest. What effective rate of interest did you earn?
18. GenX Holdings received an invoice from its supplier that indicates the penalty on overdue invoices will be charged at a rate of 3.4% per month. What effective rate of interest is being charged on overdue invoices?
19. The TD Emerald Visa card wants to increase its effective rate by 1%. If its current interest rate is 19.067014% compounded daily, what new daily compounded rate should it advertise?
20. Five investors reported the following results. Rank the effective rates of return realized by each investor from highest to lowest and show outputs to support your work. Investor

9.7: Determining the Number of Compounds

Mechanics

Solve for the number of compounds involved in each transaction based on the information provided. Express your answer in a more common format.

Present Value	Future Value	Nominal Interest Rate
1. $68,000.00	$89,032.97	4.91% compounded monthly
2. $41,786.68	$120,000.00	8.36% compounded quarterly
3. $10,000.00	$314,094.20	9% compounded annually
4. $111,243.48	$1,000,000.00	8.8% compounded semi-annually
5. $25,000.00	$125,000.00	5.85% compounded monthly
6. $8,000.00	$10,000.00	18% compounded daily
7. $110,000.00	$250,000.00	6.39% compounded weekly
8. $500,000.00	$2,225,500.00	7.1% compounded quarterly

Applications

9. You just took over another financial adviser's account. The client invested $15,500 at 6.92% compounded monthly and now has $24,980.58. How long has this client had the money invested?
10. A debt of $7,500 is owed. Suppose prevailing interest rates are 4.9% compounded quarterly. How far in advance was the debt paid if the creditor accepted a payment of $6,721.58?
11. Wayne was late in making a $3,500 payment to Dora. If Dora accepted a payment of $3,801.75 and charged 5.59% compounded semi-annually, how late was the payment?
12. How long will it take $5,750 to become $10,000 at 6.25% compounded weekly?
13. A friend of yours just won the 6/7 category on the Lotto Max (matching six out of seven numbers), and her share of the prize was $275,000. She wants to pay cash for a new home that sells for $360,000. If she can invest the money at 7.45% compounded semi-annually, how long will she have to wait to purchase the home assuming its sale price remains the same?
14. Lakewood Properties anticipates that the City of Edmonton in the future will release some land for a development that costs $30 million. If Lakewood can invest $17.5 million today at 9.5% compounded monthly, how long will it take before it will have enough money to purchase the land?
15. As marketing manager, you want to pursue a new product development for which you require $1 million for research. However, budgetary constraints mean you can only receive $850,000. If you take your budget and invest it at 8.7% compounded monthly, how long will it be before you can pursue the necessary research for the project?

Challenge, Critical Thinking, & Other Applications

16. How long will it take money to double if it earns 8.2% compounded quarterly? First use the Rule of 72 to estimate the time, then compute the actual time. What is the difference between the two numbers (assume 182 days in a half year)?
17. Your organization has a debt of $30,000 due in 13 months and $40,000 due in 27 months. If a single payment of $67,993.20 was made instead using an interest rate of 5.95% compounded monthly, when was the payment made?
18. A $9,500 loan requires a payment of $5,000 after 1½ years and a final payment of $6,000. If the interest rate on the loan is 6.25% compounded monthly, when should the final payment be made?
19. Your finance department has a policy of maturing its investments from longest held first to shortest when it needs the money for other purposes. Rank the following investments in the order that they should be matured as needed.

Investment	Amount Invested	Current Valuation	Nominal Interest Rate
A	$50,000	$75,549.62	7.94% compounded quarterly
B	$75,000	$104,830.46	6.83% compounded monthly
C	$35,000	$51,231.69	7.25% compounded semi-annually
D	$110,000	$161,643.96	8.1% compounded quarterly

20. A loan requires five payments of $1,000 today, $1,500 due in 9 months, $3,000 due in 15 months, $2,500 due in 21 months, and $4,000 due in 33 months. Using an interest rate of 4.4% compounded monthly, a single payment of $11,950 was made to clear all debts. When was the single payment made

Review Exercises

Mechanics

1. If you invest $10,000 at 7.74% compounded quarterly for 10 years, what is the maturity value?
2. Ford Motor Company is considering an early retirement buyout package for some employees. The package involves paying out today's fair value of the employee's final year of salary. Shelby is due to retire in one year. Her salary is at the company maximum of $72,000. If prevailing interest rates are 6.75% compounded monthly, what buyout amount should Ford offer to Shelby today?

3. Polo Park Bowling Lanes owes the same supplier $3,000 today and $2,500 one year from now. The owner proposed to pay both bills with a single payment four months from now. If interest rates are 8.1% compounded monthly, what amount should the supplier be willing to accept?
4. What is the effective rate of interest on your credit card if you are being charged 24.5% compounded daily?
5. Your 2½-year investment of $5,750 just matured for $6,364.09. What weekly compounded rate of interest did you earn?
6. Vienna just paid $9,567.31 for an investment earning 5.26% compounded semi-annually that will mature for $25,000. What is the term of the investment?

Applications

7. Merryweather's union just negotiated a new four-year contract with her employer. The terms of the contract provide for an immediate wage increase of 3.3%, followed by annual increases of 3.5%, 4.25%, and 2.75%. If she currently earns $61,500, what will her salary be in the final year of the contract?
8. Bronco's four-year investment just matured at $26,178.21. If the investment earned semi-annually compounded interest rates of 4.5% and 4.75% in the first two years, followed by monthly compounded interest rates of 5% and 5.1% in the last two years, how much money did Bronco initially invest?
9. Jay's Pharmacy owes the same creditor two debts of $6,000 due one year ago and $7,500 due in two years. Jay has proposed making two alternative payments of $10,000 due in three months and a final payment in 2½ years. If the creditor is agreeable to this proposal and wants an interest rate of 9% compounded quarterly, what is the amount of the final payment?
10. Over a 10-year period, the Growth Fund of America had annual returns of 0.02%, 29.8%, 14.84%, 26.86%, 31.78%, 45.7%, 7.49%, −12.2%, −22.02%, and 32.9%. What fixed annually compounded rate of return did the fund realize over the 10 years?
11. A sum of $84,100 was invested for 2½ years and matured at $101,268 using quarterly compounding. What quarterly compounded interest rate was realized? What is this effectively?
12. A $10,000 loan at 8.15% compounded quarterly is to be repaid by two payments. The first payment is due in 9 months and the second payment, 1⅕ times the size of the first payment, is due in 33 months. Determine the amount of each payment.
13. Exactly how long will it take for your money to quadruple at 6.54% compounded monthly?
14. Jack is considering alternative three-year investment proposals from different banks for his $20,000 principal. RBC says that if he invests his money with them, he will have $23,841.78 using quarterly compounding. CIBC indicates that he would have $23,607.15 using monthly compounding. What nominal and effective rates are being offered by each bank?

Challenge, Critical Thinking, & Other Applications

15. For the past four years, Darren has been saving up for a college fund for his oldest daughter. He deposited $5,000 initially, followed by annual deposits of $5,000, $4,000, $3,600, and $5,000. The money was initially invested at 5.75% compounded semi-annually for the first two years before increasing to 6% compounded quarterly for the balance of the investment. How much money is in the college fund today?
16. Six months ago Old Dutch Foods purchased some new machinery for a new product line that they just developed. The supplier agreed to three payments on the machinery of $40,000 due today, $85,000 due in six months, and $75,000 due in 15 months. The new product line has not been as successful as initially planned, so Old Dutch Foods has proposed an alternative agreement involving three payments, each due at 3 months, 9 months, and 21 months. The second payment is to be double the first payment, and the last payment is to be double the second payment. If the supplier is agreeable to this and wants an interest rate of 8.55% compounded monthly, determine the payments required in the proposed agreement.
17. Louisa owns a furniture store and decided to help a friend out by allowing him to purchase $5,000 of furniture using her credit at 6.25% compounded semi-annually. The furniture loan is to be repaid in four years. However, after 2½ years Louisa can no longer have the loan outstanding and needs the money. Avco Financial has agreed to purchase the maturity amount of this loan from Louisa using a discount rate of 17.1% compounded monthly. If Louisa proceeds with selling the loan contract to Avco Financial, what sum of money can she expect to receive?
18. Franklin owes the following amounts to the same person: $16,000 due today, $11,500 due in 1¼ years, $17,000 due in 2¾ years, and $15,000 due in 4¼ years. He wants to make a single payment of $56,500 instead. Using an interest rate of 8% compounded quarterly, when should this payment be made?
19. For each of the following investments, calculate the nominal rate of interest and convert to its effective rate of interest. Rank the investments from best to worst based on their effective rates.

Investment	Present Value	Future Value	Compounded	Term
A	$31,000.00	$39,190.09	Weekly	4 years, 8 weeks

Investment	Present Value	Future Value	Compounded	Term
B	$45,000.00	$59,738.80	Annually	5 years
C	$37,310.00	$43,534.31	Quarterly	2 years, 9 months
D	$16,175.00	$19,701.35	Semi-annually	3 years, 6 months
E	$3,688.00	$4,088.69	Monthly	1 year, 10 months

20. Three payments of $4,000 each are overdue to the same creditor by 1.5 years, 1 year, and 0.5 years. Three future payments of $4,000 each are due in 0.5 years, 1 year, and 1.5 years. Using an interest rate of 5.35% compounded quarterly, the debtor wants to make a single payment of $25,750. When should this payment be made?

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CHAPTER OVERVIEW

10: Compound Interest- Applications Involving Single Payments

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10.0: Introduction

Everywhere you look, financial advertisements appeal for a share of your hard-earned money. It is hard to go a day without seeing at least one of the following:

- Television ads about GICs that earn you the maximum money
- Bank billboards and signage indicating maximum interest rates
- Magazine ads for high-interest bonds
- Internet banner ads and unsolicited emails promising safe investments and high interest rates

Headlines on news websites blare that costs are rising from inflation and how your purchasing power is dropping. Revealing articles tell how underfunding of the Canada Pension Plan (CPP) and Old Age Security (OAS) will leave you without sufficient retirement income in your golden years. The CPP in 2012 offered a maximum benefit of $986.67 per month to a limited few, with the average CPP collector earning only $529.09 per month. OAS pays a maximum $544.98 per month, with the average pensioner receiving $514.74. If you retired today your monthly income for life (to be indexed for inflation) is $1,043.83 with a $1,531.65 maximum. Very few people receive the maximum. To make matters worse, OAS and CPP income is taxable by the government! If you don't start investing your money into your tax-sheltered RRSP or TFSA (Tax-Free Savings Account) soon and provide your own source of retirement funds, you will be cash poor in retirement. It is no wonder that many mid-size to large companies help their loyal workers save for retirement by offering employee benefit plans from company-sponsored GIC plans to high-interest savings.

In retail, endless sources of credit and no-payment plans are available on a wide range of merchandise. These highly successful plans allow consumers to buy merchandise they probably couldn't otherwise get. The Brick offers no payments for at least 15 months, while Leon's hosts the "Don't pay a cent" event. Canadian consumers increasingly purchase merchandise through such plans. Meanwhile, behind the scenes, retailers are buying and selling these plans to finance companies to liquidate their outstanding accounts receivables.

This whole chapter uses compound interest involving single payments to deal with real-life applications. For all of them your knowledge of future values, present values, interest rates, and terms is vital. This chapter introduces a range of investment options such as long-term GICs, Canada Savings Bonds, and strip bonds. You will see how businesses buy and sell promissory notes. You will also learn how to extend time value of money concepts to applications involving rates of inflation, purchasing power, and rates of change.

Your knowledge of compound interest concepts is about to become very practical. So buckle up your seatbelt as this chapter takes you on an exciting ride!

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10.1: Application - Long-Term GICs

Recall that Guaranteed Investment Certificates (GICs) are investments offering a guaranteed rate of interest over a predetermined time period. Whereas short-term GICs in Section 8.3 involved terms less than one year, most long-term GICs range from one to five years. Though terms longer than this are available, they are not very common.

Also recall that Section 8.3 discussed the factors that determine interest rates for short-term GICs. The same factors apply to long-term GICs: To receive the highest interest rate on a GIC, you should still invest a large principal in a non-redeemable GIC for the longest term possible.

The key difference between short- and long-term GICs lies in the compounding of interest. Long-term GICs do not wait until the end of the term for interest on them to appear and be paid out. Rather, in line with the definition of compound interest, a long-term GIC periodically converts the accrued interest into principal throughout the transaction. Although GICs come in many varieties (remember, financial institutions try to market these products attractively to investors), three structures are commonly available:

1. **Interest Payout GICs**. An **interest payout GIC** uses interest rates that by all appearances you might assume to be compounded periodically since they are listed side-by-side with compound interest rates. In practice, though (and by reading the fine print), you will find that the periodically calculated interest is never added to the principal of the GIC, and in essence the concepts of simple interest are used. Instead, the interest is paid out to the investor (perhaps into a chequing account) and does not actually compound unless the investor takes the interest payment received and invests it in another compounding investment. Interest payout GIC interest rates can take either a fixed or variable format. For example, in an online browsing of long-term GICs you may find a posted rate on a three-year GIC at 2% semi-annually. The fine print and footnotes may show that the interest is paid out on a simple interest basis at the end of each six months.
2. **Compound Interest GICs**. A **compound interest GIC** uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding. Interest rates can be either fixed or variable.
3. **Escalator Interest GICs**. An **escalator interest GIC** uses compound interest rates that usually remain constant during each of a series of time intervals, always rising stepwise throughout the term of the investment with any accrued interest being converted to principal.

When you invest in a GIC, you are in essence lending the bank your money in return for earning an interest rate. Financial institutions commonly use the money raised from GICs to fund mortgages. As a result, the posted interest rates on GICs hover at 1% to 2% lower than the interest rates on mortgages. Thus, the bank earns the interest from its mortgagors, paying only a portion of it to its GIC investors. The rest of this section discusses each of the three types of GICs separately.

Interest Payout GIC

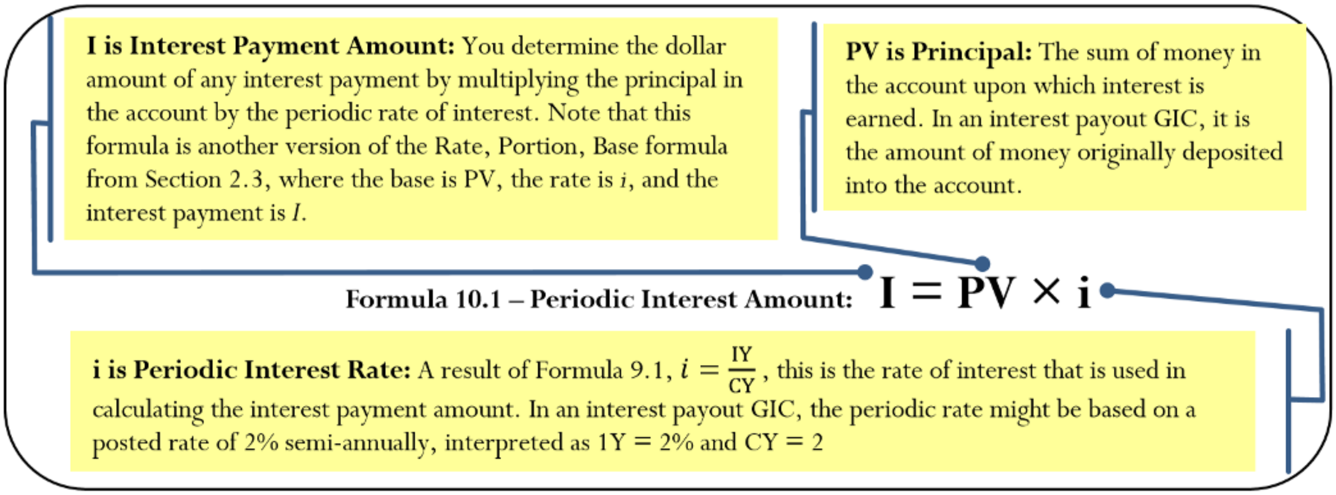
The interest payout GIC is mathematically the least interesting of the three types of GICs since the interest, while periodically calculated based on compound interest rates, does not get converted to principal. Instead, the interest payment is paid out to an account specified by the investor. Therefore, usually the only variable of concern on an interest payout GIC is the amount of the interest payment.

As a source of confusion, in the marketplace many financial institutions refer to interest payout GICs as simple interest GICs, because the interest never converts to principal. Examining bank websites such as CIBC (www.cibc.com/ca/gic/index.html), TD Canada Trust (www.tdcanadatrust.com/GICs/index.jsp), or HSBC (www.hsbc.ca/1/2/en/personal/investing-retiring/gics) reveals a variety of long-term GICs referred to as using simple interest. Why do these GICs then appear in this chapter and not previously in Chapter 8? The answer is threefold:

1. **Periodic Interest Payments**. In interest payout GICs, interest is periodically paid out throughout the term of the GIC. This differs from the GICs discussed in Chapter 8 in that simple interest required interest to be calculated and added to the principal only at the end of the transaction's time period.
2. **Rates Used in Calculating the Interest Amount**. In interest payout GICs, the posted rates appear intermixed with the posted compound interest rates. For example, a rate of 2% semi-annually may be posted (the word compounded is usually omitted to reduce confusion with compound interest GICs), which means that 1% interest is paid out every six months.
3. **Length of Time**. The terms involved with interest payout GICs are longer than one year, meeting the definition of "long-term" GICs.

The Formula

Calculating the interest payment requires a simplification of Formula 9.3 involving compound interest for single payments. As with compound interest GICs, you use the periodic interest rate calculated through Formula 9.1. However, in an interest payout GIC you never add the interest to the principal, so you do not need the 1+ term in the formula. There is also no future value to calculate, just an interest amount. This changes Formula 9.3 from $FV = PV \times (1+i)N$ to $I = PV \times iN$. Simplifying further, you calculate the interest one compound period at a time, where $N = 1$. This eliminates the need for the N exponent and establishes Formula 10.1.



I is Interest Payment Amount: You determine the dollar amount of any interest payment by multiplying the principal in the account by the periodic rate of interest. Note that this formula is another version of the Rate, Portion, Base formula from Section 2.3, where the base is PV, the rate is i, and the interest payment is I.

PV is Principal: The sum of money in the account upon which interest is earned. In an interest payout GIC, it is the amount of money originally deposited into the account.

Formula 10.1 – Periodic Interest Amount: $\mathbf{I = PV \times i}$

i is Periodic Interest Rate: A result of Formula 9.1, $i = \frac{\text{IY}}{\text{CY}}$, this is the rate of interest that is used in calculating the interest payment amount. In an interest payout GIC, the periodic rate might be based on a posted rate of 2% semi-annually, interpreted as 1Y = 2% and CY = 2

Formula 10.1

How It Works

Follow these steps to calculate the interest payment for an interest payout GIC:

Step 1: Identify the amount of principal invested. This is the PV. Also determine the nominal interest rate (IY) and interest payout frequency (CY).

Step 2: Using Formula 9.1, calculate the periodic interest rate (i).

Step 3: Apply Formula 10.1 to calculate the interest amount.

Assume $5,000 is invested for a term of three years at 5% quarterly in an interest payout GIC. Calculate the amount of the quarterly interest payment.

Step 1: The principal invested is $5,000, or PV = $5,000. The nominal interest rate is IY = 5%. The frequency is CY = 4 for quarterly.

Step 2: Applying Formula 9.1 results in i = 5%/4 = 1.25%.

Step 3: Applying Formula 10.1, I = $5,000 × 0.0125 = $62.50.

The investor receives an interest payment of $62.50 every quarter throughout the term of the investment. A term of three years then means that there are $N = 4 \times 3 = 12$ payments totaling $62.50 × 12 = $750 of interest. At the end of the three years, the GIC matures and the investor is paid out the principal of $5,000.

Important Notes

If the interest rate is variable, you must apply Formula 10.1 to each of the variable interest rates in turn to calculate the interest payment amount in the corresponding time segment. For example, if a $1,000 GIC earns 2% quarterly for the first year and 2.4% monthly for the second year, then in the first year $I = \$1,000 \times \frac{2\%}{4} = \$5$, and in the second year $I = \$1,000 \times \frac{2.4\%}{12} = \$2$.

? Exercise 10.1.1: Give It Some Thought

In an interest payout GIC, is the maturity value of the investment higher than, lower than, or the same as the principal invested in the GIC?

Answer

The maturity value and the principal are the same since the interest is never converted to principal

✓ Example 10.1.1: An Interest Payout GIC

Jackson placed $10,000 into a four-year interest payout GIC earning 5.5% semi-annual interest. Calculate the amount of each interest payment and the total interest earned throughout the term.

Solution

Calculate the periodic interest payment amount (I). Then calculate the total interest paid for the term based on N.

What You Already Know

Step 1:

PV = $10,000, IY = 5.5%, CY = 2

How You Will Get There

Step 2:

Apply Formula 9.2.

Step 3:

Apply Formula 10.1.

Step 4:

To calculate the total interest, determine N using Formula 9.2 and multiply the payment by N.

Perform

Step 2:

$$i=\frac{5.5\%}{2}=2.75\%$$

Step 3:

$$I=\$10,000\times 0.0275=\$275$$

Step 4:

N = 2 × 4 = 8; total interest = $275 × 8 = $2,200

Every six months, Jackson receives an interest payment of $275. Over the course of four years, these interest payments total $2,200.

Compound Interest GIC

Throughout the term of a compound interest GIC, interest is periodically converted to principal. A starting amount, called the principal, remains in the account for the entire term and compounds interest. Therefore, you treat a compound interest GIC exactly like a future value compound interest calculation on a single payment amount.

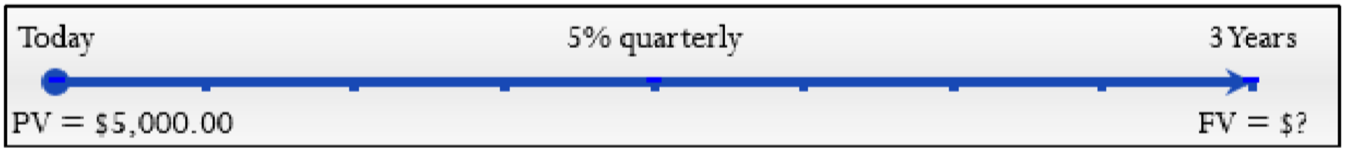
How It Works

Compound interest GICs do not require any new formulas or techniques. Most commonly, the variables of concern are either the maturity value of the investment or the compound interest rate.

1. **Maturity Value**. If the compound interest rate is fixed, then you find the maturity value by applying Formula 9.3 once, where $FV = PV(1+i)N$. These 4 steps were introduced in section 9.2. If the compound interest rate is variable, then to find the maturity value you must apply Formula 9.3 once for each segment of the timeline. These 7 steps were also introduced in section 9.2. Note that in step 5, no principal adjustment needs to be made since only the interest rate variable changes.
2. **Compound Interest Rate**. In the event that the unknown variable is the interest rate, recall the 6 steps were introduced in section 9.5.

Assume an investment of $5,000 is made into a three-year compound interest GIC earning 5% compounded quarterly. Solve for the maturity value.

Step 1: The timeline below illustrates this investment. This is a fixed rate compound interest GIC with a term of three years and PV = $5,000. The nominal interest rate is IY = 5%. The compounding frequency is CY = 4.



Step 2: The periodic interest rate is i = 5%/4 = 1.25%.

Step 3: The number of compound periods is N = 4 × 3 = 12.

Step 4: Applying Formula 9.3, $FV = \$5,000(1+0.0125)^{12} = \$5,803.77$. Hence, at maturity the GIC contains $5,803.77, consisting of $5,000 of principal and $803.77 of compound interest.

? Exercise 10.1.2: Give It Some Thought

Supposing that each investment is held until maturity, which earns more interest, an interest payout GIC or a compound interest GIC?

Answer

The compound interest GIC earns more interest since the interest is converted to principal and therefore earns even more interest. The interest payout GIC does not compound.

✓ Example 10.1.2: How Much Do You Have at Maturity?

Andrej invested $23,500 into a three-year variable compound interest GIC. The quarterly compounded interest rate was 3.8% for the first 15 months, 3.7% for the next 12 months, and 3.65% after that. What is the maturity value of Andrej’s GIC?

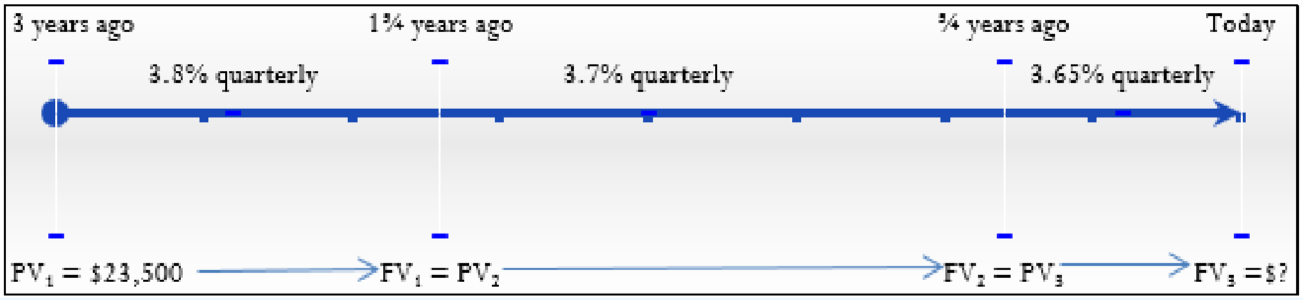
Solution

Calculate the maturity value (FV) of Andrej’s variable rate compound interest GIC.

What You Already Know

Step 1:

The principal, terms, and interest rates are known, as shown in the timeline.


3 years ago
1¾ years ago
¾ years ago
Today
3.8% quarterly
3.7% quarterly
3.65% quarterly
PV₁ = $23,500
FV₁ = PV₂
FV₂ = PV₃
FV₃ =$?

PV = $23,500

First time segment: IY = 3.8%, CY = quarterly = 4 Term = 1¼ years

Second time segment: IY = 3.7%, CY = quarterly = 4 Term = 1 year

Third time segment: IY = 3.65%, CY = quarterly = 4 Term = ¾ year

How You Will Get There

Step 2:

For each time segment, calculate the periodic interest rate by applying Formula 9.2.

Step 3:

For each time segment, calculate the number of compound periods by applying Formula 9.2.

Step 4:

Calculate the future value (FV_1) of the first time segment using Formula 9.3.

Step 5:

Let $FV_1 = PV_2$.

Step 6:

Calculate the future value (FV_2) of the second time segment using Formula 9.3. Repeat Step 5: Let $FV_2 = PV_3$. Repeat Step 6: Calculate the future value (FV_3) of the third time segment using Formula 9.3.

Step 7:

FV_3 is the final future value amount.

Perform

Step 2:

First Time Segment:

$$i = \frac{3.8\%}{4} = 0.95\%$$

Second Time Segment:

$$i = \frac{3.7\%}{4} = 0.925\%$$

Third Time Segment:

$$i = \frac{3.65\%}{4} = 0.9125\%$$

Step 3:

First Time Segment:

$$N = 4 \times 1\tfrac{1}{4} = 5$$

Second Time Segment:

$$N = 4 \times 1 = 4$$

Third Time Segment:

$$N = 4 \times \frac{3}{4} = 3$$

Step 4:

$$FV_1 = \$23,500 \times (1 + 0.0095)^5 = \$24,637.66119$$

Step 5 - Step 6:

$$FV_2 = \$24,637.66119 \times (1 + 0.00925)^4 = \$25,561.98119$$

Repeat Step 5 - Step 6:

$$FV_3 = \$25,561.98119 \times (1 + 0.009125)^3 = \$26,268.15$$

Calculator Instructions

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	5	3.8	-23500	0	Answer: -24,637.66119	4	4
2	4	3.7	24,637.66119	✓	Answer: -25,561.98119	✓	✓
3	3	3.65	25,561.98119	✓	Answer: -26,268.14515	✓	✓

At the end of the three-year compound interest GIC, Andrej has $26,268.15, consisting of the $23,500 principal plus $2,768.15 in interest.

Example 10.1.3: Equivalent Interest Rate on the GIC

Using Example 10.1.2, what equivalent fixed quarterly compounded interest rate did Andrej earn on his GIC?

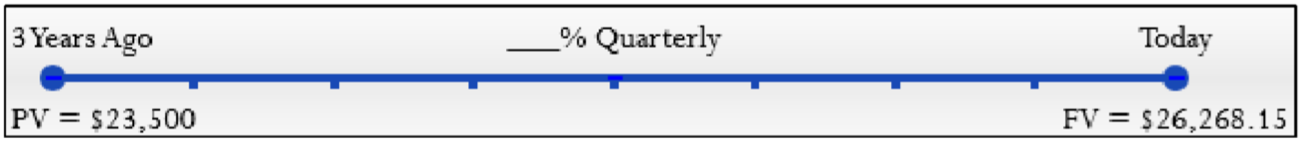
Solution

Calculate the fixed quarterly compounded interest rate (IY) that is equivalent to the three variable interest rates.

What You Already Know

Step 1:

From Example 10.1.2, the principal, maturity value, compounding frequency, and term are known, as illustrated in the timeline.
CY = 4, Term = 3 years



How You Will Get There

Step 2:

Calculate N using Formula 9.2.

Step 3:

Substitute into Formula 9.3 and rearrange for i.

Step 4:

Substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2:

$$N=4\times3=12$$

Step 3:

$$\begin{aligned}\$26,268.15&=\$23,500(1+i)^{12}\\1.117793&=(1+i)^{12}\\1.117793^{\frac{1}{12}}&=1+i\\1.009322&=1+i\\0.009322&=i\end{aligned}$$

Step 4:

$$0.009322=\frac{IY}{4}$$

$$IY=0.037292=3.7292\%\text{ compounded quarterly}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
12	Answer: 3.729168	-23500	0	26268.15	4	4

A fixed rate of 3.7292% compounded quarterly is equivalent to the three variable interest rates that Andrej realized.

Escalator Interest GIC

An escalator interest GIC is a compound interest rate GIC with four distinguishing characteristics:

1. The interest rate is variable throughout the term.
2. The nominal interest rate always increases with each change so that higher returns on longer terms will encourage the investor to keep the sum of money invested in this GIC.
3. The interest rates are known in advance and fixed for the duration of each time segment of the investment.
4. Each time segment is most commonly one year in length.

Various financial institutions call escalator interest GICs by many names to differentiate their products from others on the market. Some of the names include Rate Riser, Stepper, Multi-Rater, Stepmaker, and RateAdvantage. Regardless of the actual name used, if the GIC fits the above characteristics it is an escalator interest GIC.

How It Works

The escalator interest GIC is a special form of the compound interest GIC, so the exact same formulas and procedures used for compound interest GICs remain applicable. The most common applications with escalator GICs involve finding one of the following:

1. The maturity value of the GIC.
2. The equivalent fixed rate of interest on the GIC so that the investor can either compare it to that of other options or just better understand the interest being earned. Recall from Chapter 9 the 6 steps you need to calculate equivalent fixed rates.

Example $10.1.4$: Understanding Your Escalator Rate GIC

Antoine is thinking of investing $8,000 into a five-year CIBC Escalating Rate GIC with annually compounded rates of 0.5%, 1.5%, 2%, 3.5%, and 6.5% in each subsequent year. Determine the maturity value of Antoine's investment along with the equivalent fixed annually compounded rate.

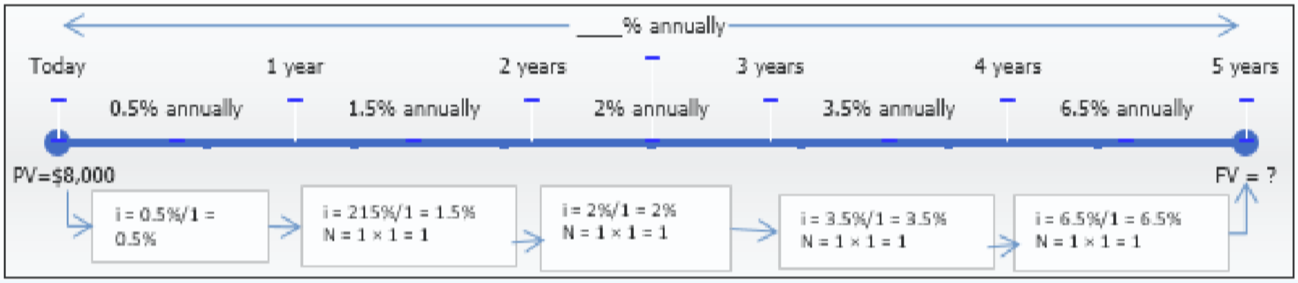
Solution

First, calculate the maturity value of Antoine's investment (FV) at the end of the five-year term. Then calculate the equivalent fixed nominal interest rate (IY).

What You Already Know

Step 1:

The present value, term, and escalating nominal interest rates are known, as shown in the timeline.


____% annually
Today
1 year
2 years
3 years
4 years
5 years
0.5% annually
1.5% annually
2% annually
3.5% annually
6.5% annually
PV=$8,000
FV = ?
i = 0.5%/1 = 0.5%
i = 215%/1 = 1.5%
N = 1 × 1 = 1
i = 2%/1 = 2%
N = 1 × 1 = 1
i = 3.5%/1 = 3.5%
N = 1 × 1 = 1
i = 6.5%/1 = 6.5%
N = 1 × 1 = 1

How You Will Get There

Step 2:

For each time segment, calculate the i and N in the timeline using Formula 9.1 and Formula 9.2. Note that since all rates are compounded annually (CY = 1), Formula 9.1 results in $i = IY$ for all time segments. As well, in Formula 9.2 the N always equals 1 since both CY = 1 and Years = 1 for every time segment.

Step 3: Solve for FV using Formula 9.3. Since only the interest rate changes, expand the formula:

$$FV = PV \times (1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_5)^{N_5}$$

Step 4: To reflect the entire five-year term compounded annually, calculate a new value of N using Formula 9.2.

Step 5: Substitute into Formula 9.3 and rearrange for i.

Step 6: With CY = 1, then $IY = i$.

Perform

Step 2:

The above figure shows the successive calculated values of i and N.

Step 3:

$$FV_1 = \$8,000(1+0.005)^1(1+0.015)^1(1+0.02)^1(1+0.035)^1(1+0.065)^1 = \$9,175.13$$

Step 4:

$$N = 1 \times 5 = 5$$

Step 5:

$$\begin{aligned}\$9,175.13 &= \$8,000(1+i)^5 \\ 1.146891 &= (1+i)^5 \\ 1.146891^{\frac{1}{5}} &= 1+i \\ 1.027790 &= 1+i \\ 0.027790 &= i\end{aligned}$$

Step 6:

$$IY = i = 0.027790 \text{ or } 2.779\% \text{ compounded annually}$$

Calculator Instructions

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	0.5	-8000	0	Answer: 8,040	2	2
2	√	1.5	-8040	√	Answer: 8,160.60	√	√
3	√	2	-8,160.60	√	Answer:8,323.812	√	√
4	√	3.5	-8,323.812	√	Answer: 8,615.14542	√	√
5	√	6.5	-8,615.14542	√	Answer: 9,175.129872	√	√
All	5	Answer: 2.779014	-8000	√	9175.13	√	√

If Antoine invests in the CIBC Escalating Rate GIC, the maturity value after five years is $9,175.13. He realizes 2.779% compounded annually on his investment.

Contributors and Attributions

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10.2: Application - Long-Term Promissory Notes

How do all of those "don't pay until . . ." plans work? The retail industry overflows with financing plans specifically designed to attract customers to purchase merchandise on credit. Most of these offers include terms such as "no money down" and "no payments for x months." You can find these plans at most furniture retailers such as The Brick and Leon's, electronic retailers such as Best Buy, along with many other establishments including jewelry stores and sleep centers.

But have you ever considered how this works on the business side? If every customer purchased merchandise under these no-money-required plans, how would the retailer stay in business? For example, perhaps The Brick’s customers purchase furniture in January 2014 that they do not have to pay for until January 2016. During these two years, The Brick does not get paid for the products sold; however, it has paid its suppliers for the merchandise. How can a retailer stay in business while it waits for all those postponed payments?

Consumers generally do not read the fine print on their contracts with these retailers. Few consumers realize that the retailers often sell these contracts (sometimes immediately) to finance companies they have partnered with. While the consumer sees no noticeable difference, behind the scenes the retailer receives cash today in exchange for the right to collect payment in the future when the contract becomes due. That way the finance company becomes responsible for collecting on the loan to the consumer.

This section introduces the mathematics behind the sale of promissory notes between companies. Recall from Section 8.4 that a *promissory note*, more commonly called a note, is a written debt instrument that details a promise made by a buyer to pay a specified amount to a seller at a predetermined and specified time. If the debt allows for interest to accumulate, then it is called an *interest-bearing promissory note*. If there is no allowance for interest, then it is called a *non-interest-bearing promissory note*. Interest-bearing notes are covered in this section and non-interest notes are discussed in the next section. When promissory notes extend more than one year, they involve compound interest instead of simple interest.

Interest-Bearing Promissory Notes

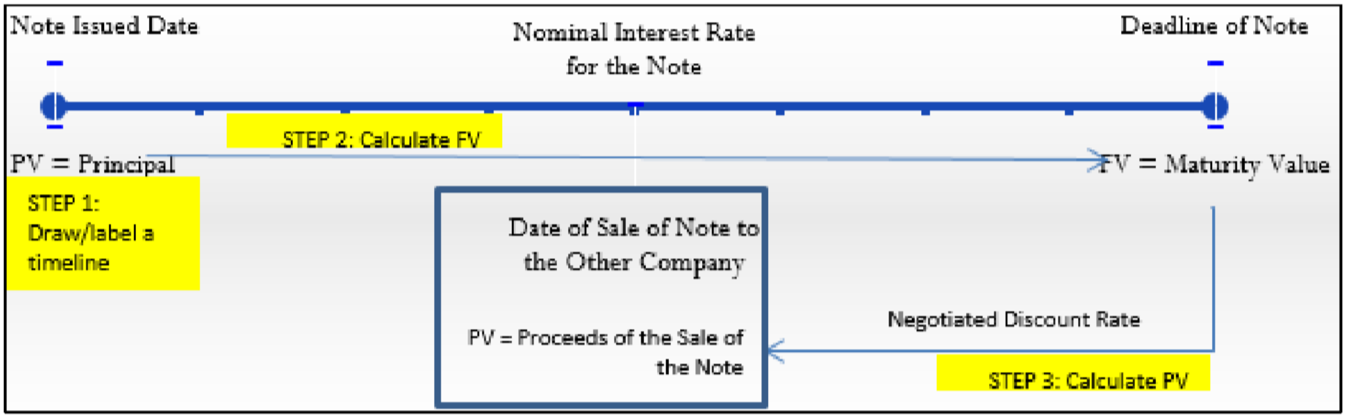
When you participate in "don't pay until . . ." promotions, you create a promissory note in which you promise to pay for your goods within the time interval stated. These promotions commonly carry 0% interest if paid before the stated deadline, so the notes are non-interest-bearing promissory notes. However, failure to pay the note before the deadline transforms the note into an interest-bearing note for which interest is retroactive to the date of sale, usually at a very high rate of interest such as 21%.

The mathematics of interest-bearing promissory notes deal primarily with the sale of long-term promissory notes between organizations. When the note is sold, the company buying it (usually a finance company) purchases the maturity value of the note and not the principal of the note. To the finance company, the transaction is an investment from which it intends to earn a profit through the difference between maturity value and purchase price. Thus, the finance company discounts the maturity value of the note using a discount rate that permits it to invest a smaller sum of money today to receive a larger sum of money in the future. The company selling the note is willing to take the smaller sum of money to cash in its accounts receivables and eliminate the risk of default on the debt.

How It Works

Recall that in simple interest the sale of short-term promissory notes involved three steps. You use the same three step sequence for long-term compound interest promissory notes. On long-term promissory notes, a three-day grace period is not required, so the due date of the note is the same as the legal due date of the note.

Step 1: Draw a timeline, similar to the one on the next page, detailing the original promissory note and the sale of the note.


Note Issued Date
Nominal Interest Rate for the Note
Deadline of Note
STEP 2: Calculate FV
PV = Principal
FV = Maturity Value
STEP 1: Draw/label a timeline
Date of Sale of Note to the Other Company
PV = Proceeds of the Sale of the Note
Negotiated Discount Rate
STEP 3: Calculate PV

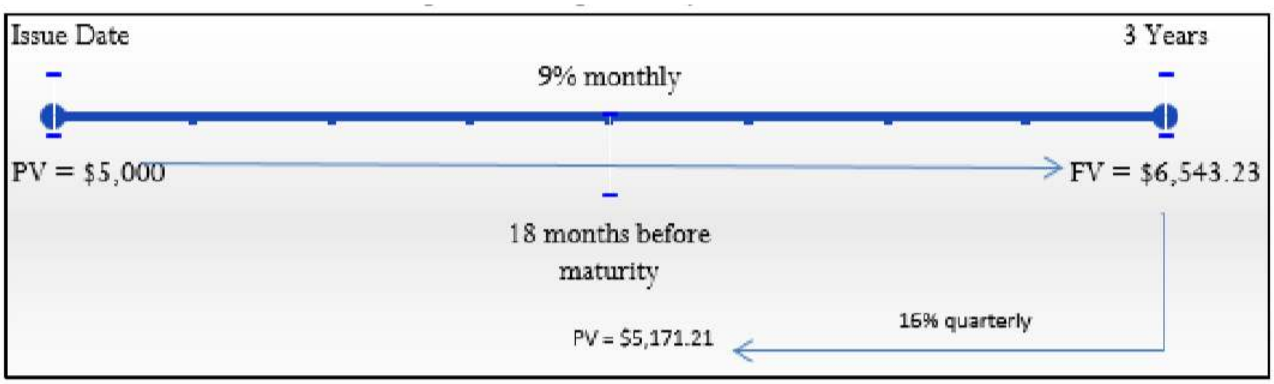
Step 2: Take the initial principal on the date of issue and determine the note's future value at the stated deadline using the stated rate of interest attached to the note. As most long-term promissory notes have a fixed rate of interest, this involves a future value calculation using Formula 9.3.

a. Calculate the periodic interest rate using Formula 9.1, $i = \frac{1Y}{CY}$.
b. Calculate the number of compounding periods between the issue date and due date using Formula 9.2, $N = CY \times \text{Years}$.
c. Solve for the future value using Formula 9.3, $FV = PV(1+i)^N$.

Step 3: Using the date of sale, discount the maturity value of the note using a new negotiated discount rate of interest to determine the proceeds of the sale. Most commonly the negotiated discount rate is a fixed rate and involves a present value calculation using Formula 9.3.

a. Calculate the new periodic interest rate using Formula 9.1, $i = \frac{1Y}{CY}$.
b. Calculate the number of compounding periods between the date of sale of the note and the due date using Formula 9.2, $N = CY \times \text{Years}$. Remember to use the CY for the discount rate, not the CY for the original interest rate.
c. Solve for the present value using Formula 9.3, $FV = PV(1+i)^N$, rearranging for PV.

Assume that a three-year $5,000 promissory note with 9% compounded monthly interest is sold to a finance company 18 months before the due date at a discount rate of 16% compounded quarterly.


Issue Date
3 Years
9% monthly
PV = $5,000
FV = $6,543.23
18 months before maturity
16% quarterly
PV = $5,171.21

Step 1: The timeline to the right illustrates the situation.

Step 2a: The periodic interest rate on the note is i = 9%/12 = 0.75%.

Step 2b: The term is three years with monthly compounding, resulting in N = 12 × 3 = 36.

Step 2c: The maturity value of the note is FV = $5,000(1 + 0.0075)36 = $6,543.23.

Step 3a: Now sell the note. The periodic discount rate is i = 16%/4 = 4%.

Step 3b: The time before the due date is 1½ years at quarterly compounding. The number of compounding periods is N = 4 × 1½ = 6.

Step 3c: The proceeds of the sale of the note is $\$6,543.23 = PV(1+0.04)^6$, where PV= $5,171.21. The finance company purchases the note (invests in the note) for $5,171.21. Eighteen months later, when the note is paid, it receives $6,543.23.

Important Notes

The assumption behind the three-step procedure for selling a long-term promissory note is that the process starts with the issuance of the note and ends with the proceeds of the sale. However, mathematically you may deal with any part of the transaction as an unknown. For example, perhaps the details of the original note are known, the finance company's offer on the date of sale is known, but the quarterly discounted rate used by the finance company needs to be calculated.

The best strategy in any of these scenarios is always to execute step 1 and create a timeline. Identify the known variables to visualize the process, then recognize any variable(s) remaining unknown. Keeping in mind how the selling of a promissory note works, you can adapt the three-step promissory note procedure using any of the techniques discussed in Chapter 9. Some examples of these adaptations include the following:

1. The discounted rate is unknown. Execute steps 1 and 2 normally. In step 3, solve for i (then IY) instead of PV.
2. The original principal of the note is unknown. Execute step 1 normally. Work with step 3, but solve for FV instead of PV. Then work with step 2 and solve for PV instead of FV.
3. The length of time by which the date of sale precedes the maturity date is unknown. Execute steps 1 and 2 normally. In step 3, solve for N instead of PV. As you can see, the three steps always stay intact. However, you may need to reverse steps 2 and 3 or calculate a different unknown variable.

Things To Watch Out For

In working with compound interest long-term promissory notes, the most common mistakes relate to the maturity value and the two interest rates.

1. **Maturity Value**. Remember that the company purchasing the note is purchasing the maturity value of the note, not its principal on the issue date. Any promissory note situation always involves the maturity value of the promissory note on its due date.
2. **Two Interest Rates**. The sale involves two interest rates: an interest rate tied to the note itself and an interest rate (the discount rate) used by the purchasing company to acquire the note. Do not confuse these two rates.

✓ Example 10.2.1: Proceeds on an Interest-Bearing Note

Jake's Fine Jewelers sold a diamond engagement ring to a customer for $4,479.95 and established a promissory note under one of its promotions on January 1, 2014. The note requires 6% compounded semi-annually interest and is due on January 1, 2017. On October 1, 2015, Jake's Fine Jewellers needed the money and sold the note to a finance company at a discount rate of 11% compounded quarterly. What are the proceeds of the sale?

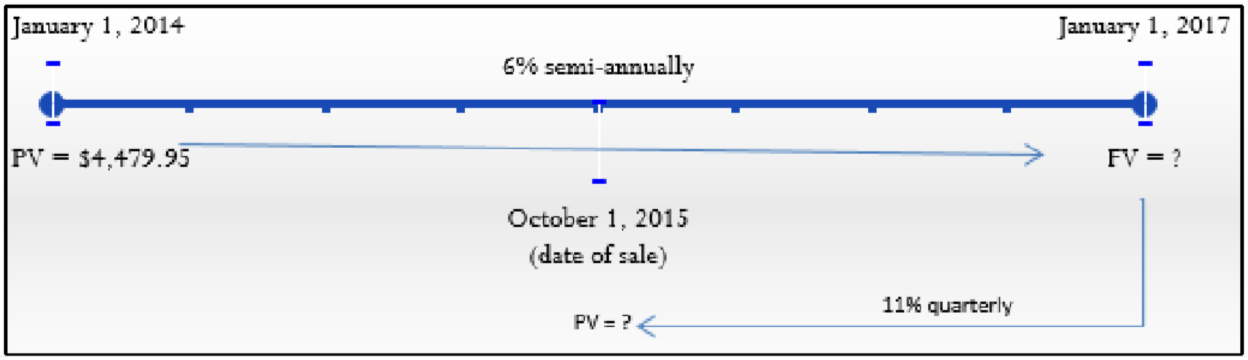
Solution

Find the proceeds of the sale for Jake's Fine Jewellers on October 1, 2013. This is the present value of the note (PV) based on the maturity value and the discount rate.

What You Already Know

Step 1:

The issue date, maturity date, principal, interest rate on the note, date of sale, and the discount rate are known, as illustrated in the timeline.


January 1, 2014
January 1, 2017
6% semi-annually
PV = $4,479.95
FV = ?
October 1, 2015
(date of sale)
11% quarterly
PV = ?

How You Will Get There

Step 2a:

Working with the promissory note itself, calculate the periodic interest rate by applying Formula 9.1.

Step 2b:

Calculate the number of compound periods using Formula 9.2.

Step 2c:

Calculate the maturity value of the note using Formula 9.3.

Step 3a:

Working with the sale of the note, calculate the discount periodic interest rate by applying Formula 9.1.

Step 3b:

Calculate the number of compound periods elapsing between the sale and maturity. Use Formula 9.2.

Perform

Step 2a:

$$IY = 6\%, CY = 2, i = 6\%/2 = 3\%$$

Step 2b:

Years = January 1, 2017 − January 1, 2014 = 3 Years, N = 2 × 3 = 6

Step 2c:

$$\begin{aligned} PV &= \$4,479.95 \\ FV &= \$4,479.95(1+0.03)^6 = \$5,349.29 \end{aligned}$$

Step 3a:

$$IY = 11\%, CY = 4, i = 11\%/4 = 2.75\%$$

Step 3b:

Years = January 1, 2017 − October 1, 2015 = 1¼ Years, N = 4 × 1¼ = 5

Step 3c:

$$\begin{aligned} \$5,349.29 &= PV(1+0.0275)^5 \\ PV &= \frac{\$5,349.29}{1.0275^5} \\ &= \$4,670.75 \end{aligned}$$

Calculator Instructions

Part	N	I/Y	PV	PMT	FV	P/Y	C/Y

Part	N	I/Y	PV	PMT	FV	P/Y	C/Y
Maturity	6	6	-4479.95	0	Answer: 5,349.294586	2	2
Sale	5	11	Answer: -4,670.75395 4	√	5349.29	4	4

Jake's fine Jewelers made the sale for $4,479.95 on January 1, 2014. On October 1, 2015, it receives $4,670.75 in proceeds of the sale to the finance company. The finance company holds the note until maturity and receives $5,349.29 from the customer.

✓ Example 10.2.2: Finding an Unknown Discount Rate

A $6,825 two-year promissory note bearing interest of 12% compounded monthly is sold six months before maturity to a finance company for proceeds of $7,950.40. What semi-annually compounded discount rate was used by the finance company?

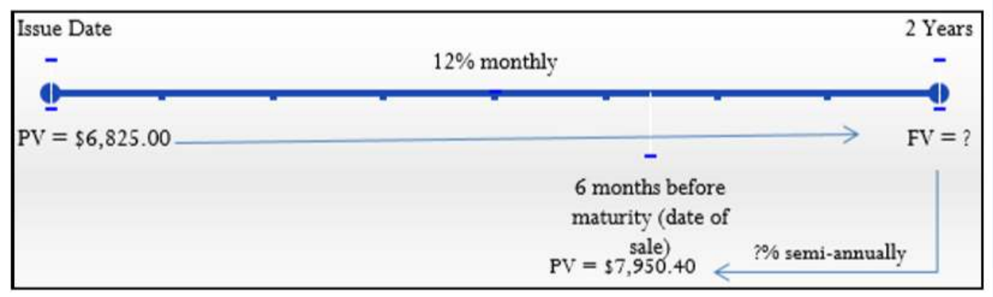
Solution

Calculate the semi-annually compounded negotiated discount rate (IY) used by the finance company when it purchased the note.

What You Already Know

Step 1:

The term, principal, promissory note interest rate, date of sale, and proceeds amount are known, as shown in the timeline.



How You Will Get There

Step 2a:

Working with the promissory note itself, calculate the periodic interest rate by applying Formula 9.1.

Step 2b:

Calculate the number of compound periods using Formula 9.2.

Step 2c:

Calculate the maturity value of the note using Formula 9.3.

Step 3a:

Note the CY on the discount rate.

Step 3b:

Calculate the number of compound periods elapsing between the sale and before maturity. Use Formula 9.2.

Step 3c:

Calculate the periodic discount rate used by the finance company by applying Formula 9.3 and rearranging for i. Then substitute into Formula 9.1 and rearrange for IY.

Perform

Step 2a:

$$IY = 12\%, CY = 12, i = 12\%/12 = 1\%$$

Step 2b:

Years = 2, N = 12 × 2 = 24

Step 2c:

$$PV = \$6,825.00, FV = \$6,825.00(1+0.01)^{24} = \$8,665.94$$

Step 3a:

CY = 2

Step 3b:

Years = 6 months = ½ Year, N = 2 × ½ = 1

Step 3c:

Solve for i:

$$\begin{aligned} \$8,665.94 &= \$7,950.40(1+i)^{1} \\ 1.090000 &= 1+i \\ i &= 0.090000 \end{aligned}$$

Solve for $IY : 0.090000 = IY \div 2$

IY = 0.180001 or 18.0001% compounded semi-annually (most likely 18%; the difference is due to a rounding error)

Calculator Instructions

Part	N	I/Y	PV	PMT	FV	P/Y	C/Y
Maturity	24	12	-6825	0	Answer: 8,665.938976	12	12
Discount Rate	1	Answer: 18.0001	-7950.40	√	8665.94	2	2

The sale of the promissory note is based on a maturity value of \$8,665.94. The finance company used a discount rate of 18% compounded semi-annually to arrive at proceeds of \$7,950.40.

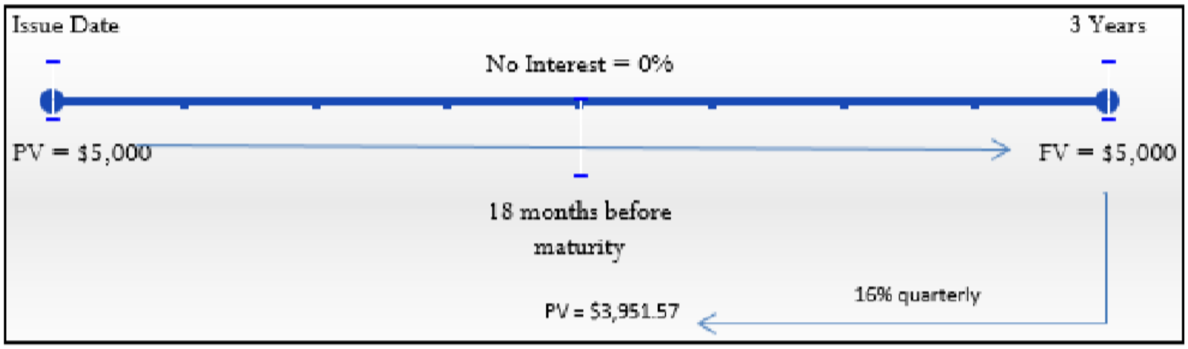
Non-interest-Bearing Promissory Notes

A non-interest-bearing promissory note involves either truly having 0% interest or else already including a flat fee or rate within the note’s face value. Therefore, the principal amount and maturity amount of the promissory note are the same.

How It Works

A non-interest-bearing note simplifies the calculations involved with promissory notes. Instead of performing a future value calculation on the principal in step 2, your new step 2 involves equating the present value and maturity value to the same amount ($PV = FV$). You then proceed with step 3.

Assume a three-year \$5,000 non-interest-bearing promissory note is sold to a finance company 18 months before the due date at a discount rate of 16% compounded quarterly.


Issue Date
3 Years
No Interest = 0%
PV = $5,000
FV = $5,000
18 months before maturity
16% quarterly
PV = $3,951.57

Step 1: The timeline is illustrated here.

Step 2: The maturity value of the note three years from now is the same as the principal, or FV = $5,000.

Step 3a: Now sell the note. The periodic discount rate is i = 16%/4 = 4%.

Step 3b: The time before the due date is 1½ years at quarterly compounding. The number of compounding periods is N = 4 × 1½ = 6.

Step 3c: The proceeds of the sale of the note is $\$5,000 = PV(1+0.04)^6$, or PV = $3,951.57. The finance company purchases the note (invests in the note) for $3,951.57. Eighteen months later, when the note is paid, it receives $5,000.

? Exercise 10.2.1: Give It Some Thought

If a non-interest-bearing note is sold to another company at a discount rate, are the proceeds of the sale more than, equal to, or less than the note?

Answer

Less than the note. The maturity value is discounted to the date of sale.

✓ Example 10.2.3: Selling a Noninterest-Bearing Long-Term Promissory Note

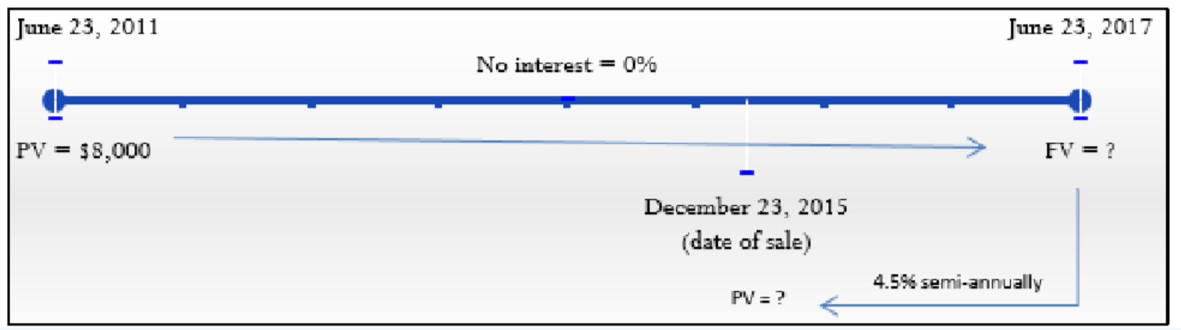
A five-year, noninterest-bearing promissory note for $8,000 was issued on June 23, 2011. The plan is to sell the note at a discounted rate of 4.5% compounded semi-annually on December 23, 2015. Calculate the expected proceeds on the note.

Solution

What You Already Know

Step 1:

The principal, non-interest on the note, issue date, due date, discount rate, and date of sale are known, as illustrated in the timeline.


June 23, 2011
June 23, 2017
No interest = 0%
PV = $8,000
FV = ?
December 23, 2015 (date of sale)
4.5% semi-annually
PV = ?

How You Will Get There

Step 2:

Equate the FV of the note with the principal of the note.

Step 3a:

Working with the sale of the note, calculate the discount periodic interest rate by applying Formula 9.1.

Step 3b:

Calculate the number of compound periods elapsing between the sale and maturity. Use Formula 9.2.

Step 3c:

Calculate the proceeds of the sale by using Formula 9.3, rearranging for PV.

Perform

Step 2:

$FV = PV$ = $8,000

Step 3a:

$$IY = 4.5\%, CY = 2, i = 4.5\%/2 = 2.25\%$$

Step 3b:

Years = June 23, 2017 − December 23, 2015 = 1½ Years, N = 2 × 1½ = 3

Step 3c:

$$\begin{aligned} \$8,000 &= PV(1+0.0225)^3 \\ PV &= \frac{\$8,000}{1.0225^3} \\ &= \$7,483.42 \end{aligned}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
6	4.5	Answer: -7,483.418564	0	8000	2	2

The expected proceeds are $7,483.42 on December 23, 2015, with a maturity value of $8,000.00 on June 23, 2017.

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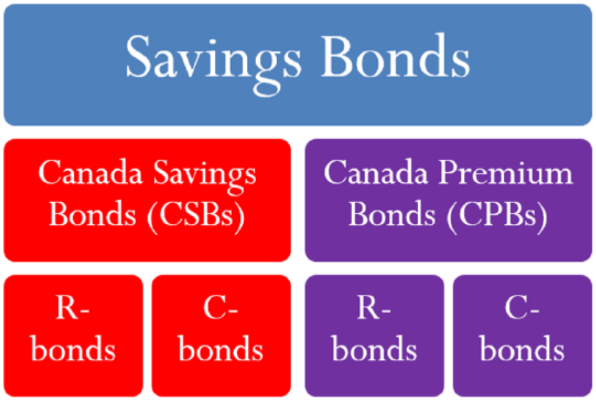
10.3: Application - Savings Bonds

How does Canada finance its long-term debt? One way is through savings bonds, or SBs for short, which are longterm financial instruments with 10-year maturities issued only by the federal Canadian government. Approximately $8.8 billion of savings bonds were outstanding in August 2012, representing approximately 1.4% of the gross national debt. Compare this with the 27% of Canada's national debt financed through treasury bills, the short-term instrument you studied in Section 8.6. For up-to-date information on CSBs, visit http://csb.gc.ca/home.

Key Characteristics of Savings Bonds

Similar to T-bills, SBs are purchased by financial institutions who, in turn, sell the SBs to investors like you and me. Savings bonds have six key characteristics:



1. **Denomination**. SBs are issued only in denominations of $100, $300, $500, $1,000, $5,000, and $10,000.
2. **Term**. SBs are issued with 10-year maturity dates and can be purchased only on their issue date.
3. **Issuance**. From 1998 to 2010, SBs have been issued on the first of every month from November through April inclusive (six times per year). Starting in 2011, SBs are issued only in November and December.
4. **Calculation of Interest**. Interest on SBs is always calculated using one of two methods:
    1. Regular interest SBs, called R-bonds, pay the interest annually to the owner of the bond and do not convert the interest to principal. This is identical to an interest payout GIC. The lowest denomination available for these bonds is $300.1
    2. Compound interest SBs, called C-bonds, annually convert the interest earned from the bond to principal. This is identical to a compound interest GIC. These bonds are available in any denomination.
5. **Types of Bonds**. SBs are issued in two versions distinguished by when they can be redeemed. The figure to the right illustrates the combination of versions and interest calculations available.
    1. Canada Savings Bonds (CSBs) are redeemable at any time. If they are redeemed during an anniversary month (that is, a multiple of 12 months from the issuance date), they receive interest for the full year, according to the R-bond or C-bond rules as appropriate. If they are redeemed during a non-anniversary month, they receive the annual interest plus interest for the partial year using simple interest calculations on the basis of the number of full months. For example, if a CSB with an anniversary date of November 1 is redeemed on January 23, simple interest for the partial year is paid only for the two full months from November 1 to January 1. CSBs pay a lower interest rate because of the privilege of redemption. When they are issued, the interest rate is generally known for the first year only, though sometimes multiyear rates are posted. Usually, each subsequent year will have its new interest rate announced in a timely fashion.
    2. Canada Premium Bonds (CPBs) have all the same characteristics of CSBs except that they are redeemable only during the anniversary month based on the month of issue. The actual day of the month does not matter. These bonds pay a higher interest rate premium, historically 0.6% to 2.65% higher, for not being redeemable at any

time. When CPBs are issued, the interest rate is known for the first three years, and thereafter announced for periods of one to three years at a time.

6. **Identification**. All SBs when issued are assigned a series letter and number.
    i. CSBs start with the letter "S" and as of November 2012 were on issue number "130." This is referred to as "Series S130."
    ii. CPBs start with the letter "P" and as of December 2012 were on issue "81." This is referred to as "Series P81."

Interest Rates on Savings Bonds

The posted interest rates on savings bonds are based on current market conditions either at the time of issue or upon each anniversary. The rates are related to the Bank of Canada interest rate. Generally, for CSBs the interest rate set is slightly above the Bank of Canada rate by approximately 0.15% to 0.25%, though this margin is not fixed. Similarly for CPBs, the rate is commonly 0.5% to 1% higher than the Bank of Canada rate, though again the margin is not fixed. The historic interest rates since 2004 for CSBs are listed in the table below. Since 2004, all series of CSBs (starting with S86 up to S130) have received the same annual interest rate based on their month of issue. Series prior to 2004 (S85 and earlier) may have individually different interest rates; you should not use this table for those bonds.

Interest Rates on Savings Bonds

The posted interest rates on savings bonds are based on current market conditions either at the time of issue or upon each anniversary. The rates are related to the Bank of Canada interest rate. Generally, for CSBs the interest rate set is slightly above the Bank of Canada rate by approximately 0.15% to 0.25%, though this margin is not fixed. Similarly for CPBs, the rate is commonly 0.5% to 1% higher than the Bank of Canada rate, though again the margin is not fixed.

The historic interest rates since 2004 for CSBs are listed in the table below. Since 2004, all series of CSBs (starting with S86 up to S130) have received the same annual interest rate based on their month of issue. Series prior to 2004 (S85 and earlier) may have individually different interest rates; you should not use this table for those bonds. Issue

Table 10.3.1

Issue/Anniversary Date*	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
January 1	1.65%	1.65%	2.62%	2.90%	3.10%	1.65%	0.40%	0.65%	0.65%	0.5%
February 1	1.55%	1.55%	2.68%	2.80%	2.85%	1.25%	0.40%	0.65%	0.65%	0.5%
March 1	1.30%	1.55%	2.75%	3.10%	2.50%	1.00%	0.40%	0.65%	0.65%	0.5%
April 1	1.25%	1.65%	2.75%	3.10%	2.45%	0.75%	0.40%	0.65%	0.65%	0.5%
November 1	1.50%	2.50%	3.00%	3.25%	2.00%	0.40%	0.65%	0.5%	0.5%	N/A**
December 1	1.50%	2.56%	3.00%	3.25%	1.85%	0.40%	0.65%	0.5%	0.5%	N/A**
Series Issued (in monthly order)	S86 to S91	S92 to S97	S98 to S103	S104 to S109	S110 to S115	S116 to S121	S122 to S127	S128 to S129†	S130‡	No series issued**

All interest rates are annual.

***** This table is only valid for all CSBs with issue dates of January 1, 2004, and later, starting with S86 up to S130.

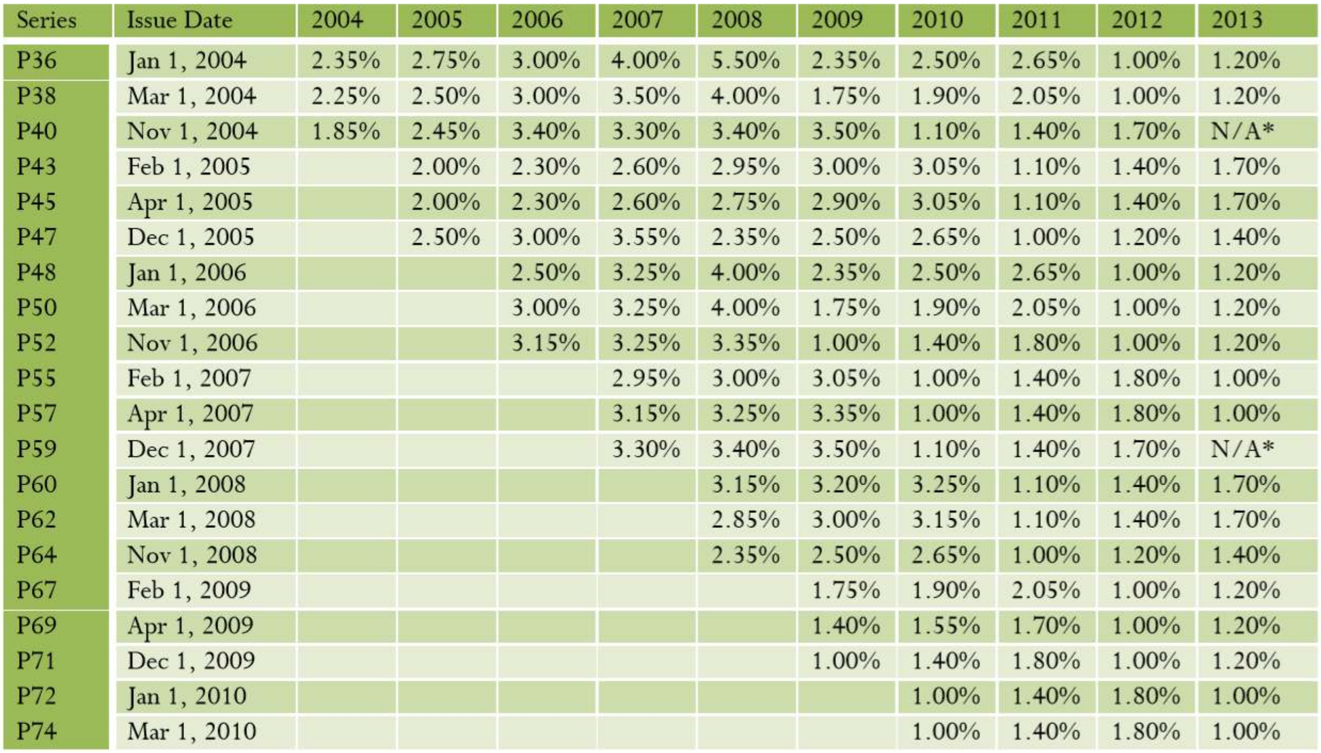
† In 2011, the Canadian Government ceased issuing CSBs in January through April. The series numbers represent November and December series.

‡ No CSBs were issued in December 2012. Thus, the series number is for November 2012.

** At time of writing, these rates were not available yet.

From 1998 to 2010, six series were issued annually, with the posted interest rate lasting for only one year. For example, Series S105 was issued on February 1, 2007, at an interest rate of 2.80%, with annual changes on February 1 to 2.85% in 2008, 1.25% in 2009, 0.40% in 2010, and 0.65% in both 2011 and 2012. As another example, Series S113 was issued on April 1, 2008, at an interest rate of 2.45%, with annual changes on April 1 to 0.75% in 2009, 0.40% in 2010, and 0.65% in both 2011 and 2012.

The historic interest rates for a sample of CPBs since 2004 are listed in the next table. Unlike their CSB counterparts, each CPB has its own individual interest rate. To see more information on historic CPB rates or rates on older CSBs, visit http://csb.gc.ca/about/rates.



Series	Issue Date	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
P36	Jan 1, 2004	2.35%	2.75%	3.00%	4.00%	5.50%	2.35%	2.50%	2.65%	1.00%	1.20%
P38	Mar 1, 2004	2.25%	2.50%	3.00%	3.50%	4.00%	1.75%	1.90%	2.05%	1.00%	1.20%
P40	Nov 1, 2004	1.85%	2.45%	3.40%	3.30%	3.40%	3.50%	1.10%	1.40%	1.70%	N/A*
P43	Feb 1, 2005		2.00%	2.30%	2.60%	2.95%	3.00%	3.05%	1.10%	1.40%	1.70%
P45	Apr 1, 2005		2.00%	2.30%	2.60%	2.75%	2.90%	3.05%	1.10%	1.40%	1.70%
P47	Dec 1, 2005		2.50%	3.00%	3.55%	2.35%	2.50%	2.65%	1.00%	1.20%	1.40%
P48	Jan 1, 2006			2.50%	3.25%	4.00%	2.35%	2.50%	2.65%	1.00%	1.20%
P50	Mar 1, 2006			3.00%	3.25%	4.00%	1.75%	1.90%	2.05%	1.00%	1.20%
P52	Nov 1, 2006			3.15%	3.25%	3.35%	1.00%	1.40%	1.80%	1.00%	1.20%
P55	Feb 1, 2007				2.95%	3.00%	3.05%	1.00%	1.40%	1.80%	1.00%
P57	Apr 1, 2007				3.15%	3.25%	3.35%	1.00%	1.40%	1.80%	1.00%
P59	Dec 1, 2007				3.30%	3.40%	3.50%	1.10%	1.40%	1.70%	N/A*
P60	Jan 1, 2008					3.15%	3.20%	3.25%	1.10%	1.40%	1.70%
P62	Mar 1, 2008					2.85%	3.00%	3.15%	1.10%	1.40%	1.70%
P64	Nov 1, 2008					2.35%	2.50%	2.65%	1.00%	1.20%	1.40%
P67	Feb 1, 2009						1.75%	1.90%	2.05%	1.00%	1.20%
P69	Apr 1, 2009						1.40%	1.55%	1.70%	1.00%	1.20%
P71	Dec 1, 2009						1.00%	1.40%	1.80%	1.00%	1.20%
P72	Jan 1, 2010							1.00%	1.40%	1.80%	1.00%
P74	Mar 1, 2010							1.00%	1.40%	1.80%	1.00%

All interest rates are annual.

* At time of writing, these rates were not available yet.

Calculating Interest Amounts and Maturity Values on Savings Bonds

The mathematics of savings bonds focuses on the two things most important to investors:

1. What amount of interest is earned?
2. What is the maturity value?

The Formula

Savings bonds require a combination of seven previously introduced formulas:

1. Formula 8.1 Simple Interest: $I = PVrt$
2. Formula 8.2 Simple Interest for Single Payments: $FV = PV(1 + rt)$
3. Formula 8.3 Interest Amount for Single Payments: $I = FV - PV$
4. Formula 9.1 Periodic Interest Rate: $i = \dfrac{IY}{CY}$
5. Formula 9.2 Number of Compound Periods for Single Payments: $N = CY \times \text{Years}$

6. Formula 9.3 Compound Interest for Single Payments: $FV = PV(1+i)^N$
7. Formula 10.1 Periodic Interest Amount: $I = PV \times i$

Savings bonds are always compounded annually, or $CY = 1$. Therefore, it is possible to simplify a few variables and calculations in these formulas:

- In Formula 9.1, the periodic interest rate (i) is always the same as the nominal interest rate (IY), or $i = IY$.
- When you calculate the future value using Formula 9.2 and Formula 9.3, each time segment is one year in duration, or Years = 1. Therefore, Formula 9.2 always produces N = 1. Additionally, you can then simplify Formula 9.3 to read $FV = PV(1+i)$.

How It Works

Follow these steps to work with regular interest savings bonds:

Step 1: Determine the principal (PV) invested in the bond on the issue date.

Step 2: Locate the historic interest rates for the applicable CSB or CPB. Once for each complete year that the bond is held, apply Formula 10.1 to calculate the annual interest amount paid out to the investor that year as of the most recent anniversary month. Round all amounts to two decimals.

Step 3: Sum the annual interest amounts that you calculated in step 2 to arrive at the total interest (I). Note that the maturity value (FV) is the same as the principal (PV). If the bond is being redeemed in an anniversary month, this is the last step. For CSBs being redeemed in a non-anniversary month, proceed with step 4.

Step 4: Calculate the number of complete months (t) that the CSB is held after the most recent anniversary date. Using the interest rate (r) for the partial year, apply Formula 8.1 to calculate the interest payment amount for the partial year. Add this amount to the previously totaled interest amount in step 3 to arrive at a final total interest earned (I). The maturity value (FV) remains the same as the principal (PV).

Follow these steps to work with compound interest savings bonds:

Step 1: Determine the principal (PV) invested in the bond on the issue date.

Step 2: Locate the historic interest rates for the applicable CSB or CPB. Once for each complete year that the bond is held, apply Formula 9.3 to calculate the maturity amount as of the most recent anniversary month. Recall in each repetition that the FV of a prior calculation becomes the new PV in the subsequent calculation.

Step 3: The maturity value (FV) is the final result of step 2. If the interest amount (I) earned to date is required, apply Formula 8.3. If the bond is being redeemed in an anniversary month, this is the last step. For CSBs being redeemed in a non-anniversary month, proceed with step 4.

Step 4: Take the unrounded maturity value from step 3 and make it the new present value (PV). Calculate the number of complete months (t) that the CSB is held after the most recent anniversary date. Using the interest rate (r) for the partial year, apply Formula 8.2 to arrive at the final maturity value (FV) for the CSB. To calculate the total interest on the bond from purchase to redemption, take this final maturity value (FV) and subtract the principal (PV) that you identified in step 1.

Things To Watch Out For

When working with the simple interest formulas (8.1 and 8.2), recall that the interest rate (r) and the time (t) need to be in the same units. With annual interest rates on savings bonds, you must convert the monthly time to an annual format by taking the number of months and dividing by 12 ($t = \text{months}/12$).

Paths To Success

Two methods can be used to speed up the calculations required for compound interest savings bonds.

1. **Step 2 Formula Adaptation**: Recall that when calculating the future value of a single payment for which only the interest rate fluctuates, you can calculate the maturity amount by formula in a single multiplication:

$$FV = PV \times (1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_n)^{N_n}$$

   where n represents the total complete years that the savings bond is held. With annual compounding where $N = 1$ and $IY = i$, you can simplify this further:

$$FV = PV \times (1+IY_1) \times (1+IY_2) \times \ldots \times (1+IY_n)$$

2. **Step 4 Simple Interest BAII Plus Calculator Adaptation**: If you have used the calculator for compound interest calculations, you can continue to use it to compute the FV for the final partial year at simple interest. Just input $r \times t$ into the I/Y button while leaving N, P/Y, and C/Y all equaling one.

? Give It Some Thought

1. What is the maturity value of a $10,000 Series S91 CSB R-bond redeemed on December 1, 2009?
2. Look at Table $10.3.1$. As an example, note that S86 received a new interest rate of 0.40% on January 1, 2010. Also note that S122 was issued on January 1, 2010, at the same rate of 0.40%. Why do older CSBs such as S86 receive the same interest rate as newer CSBs such as S122?
3. Which of the following investment amounts are not possible for R-bonds?
    a. $300
    b. $400
    c. $500
    d. $600
    e. $700
    f. $800

Answer

1. $10,000. An R-bond does not convert interest to principal, resulting in $PV = FV$.
2. Since CSBs can be redeemed at any time, if a newer CSB is given a more favourable interest rate than an older CSB, all investors would cash in their old CSBs and purchase the new CSB, effectively cancelling the old CSB.
3. $400 and $700. The smallest denominations of R-bonds are $300 and $500.

✓ Example $10.3.1$: CSB R-Bond Redemptions

Elisabeth invested in five $10,000 denomination Series S106 R-bond CSBs. How much interest will she earn in total if she redeems the bond on the following dates:

1. March 1, 2011
2. June 23, 2011

Solution

Calculate the total interest (I) that Elisabeth earned while in possession of the bond for each date.

What You Already Know

Step 1:

She has five bonds at $10,000 each, or PV = $50,000.

How You Will Get There

Step 2:

From the savings bond table, S106 was issued on March 1, 2007. The annual interest rates (i) from that point are 3.1% in 2007, 2.5% in 2008, 1.00% in 2009, 0.40% in 2010, and 0.65% in 2011. Remembering that $i = IY$, for each of 2007, 2008, 2009, and 2010 apply Formula 10.1. This calculates all of the annual interest amounts paid each year up to March 1, 2011.

Step 3:

Total all of the interest from the previous step. This answers part (a).

Step 4:

For the second question, calculate t, which is the number of complete months after March 1, 2010. Then taking the new interest rate (r), apply Formula 8.1. Add this to the interest from Step 3 to arrive at the total interest for part (b).

Perform

Step 2:

PMT(2007) = \$50,000 × 0.031 = \$1,550

PMT(2008) = \$50,000 × 0.025 = \$1,250

PMT(2009) = \$50,000 × 0.01 = \$500

PMT(2010) = \$50,000 × 0.004 = \$200

Step 3:

I = \$1,550 + \$1,250 + \$500 + \$200 = \$3,500 (if redeemed on March 1, 2011)

Step 4:

June 23 − March 1 = 3 complete months = t; r = 0.65%

I = \$50,000(0.0065)(3 12) = \$81.25

Total I = \$3,500 + \$81.25 = \$3,581.25 (if redeemed on June 23, 2010)

If Elisabeth redeems her five \$10,000 S106 R-bond CSBs on March 1, 2011, she will have earned \$3,500 in total interest. By holding onto it until June 23, 2011, she acquires an additional \$81.25 of interest for a total of \$3,581.25.

✓ CPB C-Bond Redemption

Dusty invested into two \$10,000, three \$1,000, and one \$500 denomination Series P60 C-bond CPBs. She redeemed the bond in 2012.

a. What was the maturity value?
b. How much interest did she earn?

Solution

Calculate the maturity value of the bond (FV). Then determine the total interest (I).

What You Already Know

Step 1:

Her total investment was: PV = 2 × \$10,000 + 3 × \$1,000 + 1 × \$500 = \$23,500.

How You Will Get There

Step 2:

From the premium bond table, P60 was issued on January 1, 2008. The annual interest rates (i) from that point were 3.15% in 2008, 3.2% in 2009, 3.25% in 2010, and 1.10% in 2011. Since this is a CPB bond, the redemption in 2012 must be in January 2012 (the only month it can be redeemed). Remembering that $N = 1$ and $i = IY$, adapt Formula 9.3 to calculate the maturity value of the bond:

$$FV = PV \times (1 + i_{2008}) \times (1 + i_{2009}) \times (1 + i_{2010}) \times (1 + i_{2011})$$

Step 3:

To calculate the interest, apply Formula 8.3.

Perform

Step 2:

FV = \$23,500 × (1 + 0.0315) × (1 + 0.032) × (1 + 1.0325) × (1 + 1.011) = \$26,113.07

Step 3:

I = \$26,113.07 − \$23,500.00 = \$2,613.07

Calculator Instructions

Year	N	I/Y	PV	PMT	FV	P/Y	C/Y

Year	N	I/Y	PV	PMT	FV	P/Y	C/Y
2008	1	3.15	-23500	0	Answer: 24,240.25	1	1
2009	$\surd$	3.2	$\pm FV$	$\surd$	Answer: 25,015.938	$\surd$	$\surd$
2010	$\surd$	3.25	$\pm FV$	$\surd$	Answer: 25,828.95599	$\surd$	$\surd$
2011	$\surd$	1.1	$\pm FV$	$\surd$	Answer: 26,113.0745	$\surd$	$\surd$

On her investment of \$23,500, Dusty will have a maturity value of \$26,113.07 in January of 2012. She earned \$2,613.07 of interest.

✓ Example 10.3.3: CSB C-Bond Redemptions

What is the maturity value and amount of interest earned on a \$10,000 denomination Series S95 C-bond CSB if it is redeemed on the following dates:

a. April 1, 2011
b. December 28, 2011

Solution

Calculate the maturity value (FV) and the total interest (I) on each of the noted dates.

What You Already Know

Step 1:

PV = \$10,000.

How You Will Get There

Step 2:

From the savings bond table, S95 was issued on April 1, 2005. The annual interest rates (i) from that point on were 1.65% in 2005, 2.75% in 2006, 3.1% in 2007, 2.45% in 2008, 0.75% in 2009, 0.40% in 2010, and 0.65% in 2011. Remembering that N = 1 and $i = IY$, adapt Formula 9.3 to calculate the maturity value of the bond:

$$FV = PV \times (1 + i_{2005}) \times (1 + i_{2006}) \times (1 + i_{2007}) \times (1 + i_{2008}) \times (1 + i_{2009}) \times (1 + i_{2010})$$

Step 3:

To calculate the interest as of April 1, 2011, apply Formula 8.3.

Step 4:

For the second question, calculate t, which is the number of complete months after April 1, 2010. The FV from step 2 becomes the new PV. Then take the new interest rate (r) and apply Formula 8.2. Reapply Formula 8.3 to calculate the interest at this point.

Perform

Step 2:

FV = \$10,000 × (1 + 0.0.0165) × (1 + 0.0275) × (1 + 1.031) × (1 + 1.0245) × (1 + 1.0075) × (1 + 1.004)= \$11,159.34255

Step 3:

I = \$11,159.34 − \$10,000.00 = \$1,159.34 (on April 1, 2011)

Step 4:

December 28, 2011 − April 1, 2011 = 8 complete months = t; r = 0.65%

$$FV = \$11,159.34255(1+0.0065)\left(\frac{8}{12}\right) = \$11,207.70$$

I = \$11,207.70 − \$10,000.00 = \$1,207.70 (on December 28, 2011)

Calculator Instructions

Year	N	I/Y	PV	PMT	FV	P/Y	C/Y
2005	1	1.65	-10000	0	Answer: 10,165	1	1
2006	√	2.75	$\pm FV$	√	Answer: 10,444.5375	√	√
2007	√	3.1	$\pm FV$	√	Answer: 10,768.31816	√	√
2008	√	2.45	$\pm FV$	√	Answer: 11,032.14196	√	√
2009	√	0.75	$\pm FV$	√	Answer: 11,114.88302	√	√
2010	√	0.4	$\pm FV$	√	Answer: 11,159.34255	√	√
2011 Partial Year	√	$0.65 \times 8 \div 12 =$	$\pm FV$	√	Answer: 11,207.69971	√	√

If the bond is redeemed on April 1, 2011, it matures at \$11,159.34, earning \$1,159.34 in interest. If the bond is redeemed on December 28, 2011, it matures at \$11,207.70, earning \$1,207.70 in interest.

References

1. Like interest payout GICs, these R-bonds essentially apply the concepts of simple interest. However, because of the long-term nature of savings bonds, this concept is discussed here instead of with simple interest in Chapters 7 and 8.

Contributors and Attributions

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10.4: Application - Strip Bonds

A **strip bond** is a marketable bond that has been stripped of all interest payments and is one of the many financial tools through which you may earn nontaxable income inside your Registered Retirement Savings Plan (RRSP). A marketable bond is a long-term debt of a corporation or government and represents a means by which large sums of money are "borrowed" from investors. In a marketable bond, interest is regularly paid out to investors. For example, since 1989 Manitoba Hydro has issued over $5.3 billion in bonds. A more detailed discussion on marketable bonds is found in Chapter 14.

Characteristics of Strip Bonds

Mathematically, a strip bond essentially is a long-term version of the treasury bill. Whereas T-bills are found with terms of less than one year, strip bonds have longer terms because of the large sums of money that governments or large corporations require. These large sums take a long time to pay back (think of how long it takes most Canadians to pay off a mortgage, for example). In fact, Canada holds the record of the longest strip bond, with a 99-year term!

Much like T-bills, strip bonds have the following characteristics:

1. Strip bonds do not earn interest. They are sold at a discount and redeemed at full value. This follows the line of "buy low, sell high." The compound percentage by which the value of the strip bond grows from sale to redemption is called the yield or rate of return. From a mathematical perspective, you calculate the yield in the exact same manner as an interest rate. Yields are normally specified in nominal form, that is, as a numerical rate followed by words stating the compounding frequency, which is semi-annual unless otherwise specified. Yields remain fixed for the entire term.
2. The face value of a strip bond is the maturity value payable at the end of the term. It includes both the principal and yield together.
3. Strip bonds are traded between investors at any point in secondary financial markets according to prevailing strip bond rates at the time of sale. Historical strip bond yields can be found at www.bankofcanada.ca/en/rates/yield_curve.html (note that the Bank of Canada refers to strip bonds as zero-coupon bonds). Current yields on strip bonds are found on any online financial source and are usually published daily in the financial sections of newspapers such as the Globe and Mail.
4. No minimum investment is required for strip bonds, which are available in any increment of $1.

Purchase Price of a Strip Bond

Strip bonds are financial investment tools that have a known payout (face value) upon maturity. The most common mathematical application involving strip bonds is to calculate the price that must be paid today, or the purchase price, to acquire the strip bond.

The Formula

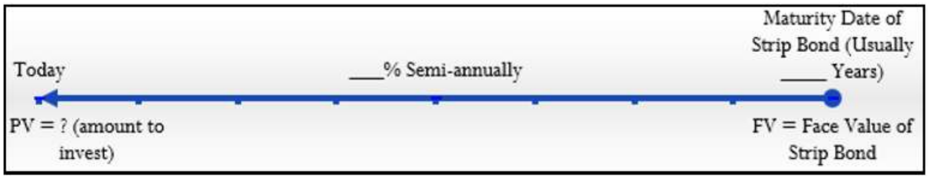
When you work with strip bonds, you may need up to four formulas that have been previously introduced:

1. Formula 8.3 Interest Amount for Single Payments: $I=FV-PV$
2. Formula 9.1 Periodic Interest Rate: $i=\dfrac{IY}{CY}$
3. Formula 9.2 Number of Compound Periods for Single Payments: $N=CY\times \text{Years}$
4. Formula 9.3 Compound Interest for Single Payments: $FV=PV(1+i)^N$

How It Works

The future value of a strip bond is always known since it is the face value. You calculate the present value or purchase price by applying the following steps:

Step 1: Determine the face value (FV) of the strip bond, the years between the date of the sale and the maturity date (Years), the yield (IY) on the date of the sale, and the compounding frequency (CY). If needed, draw a timeline similar to the one below, which illustrates a typical strip bond timeline.


Today
___% Semi-annually
Maturity Date of Strip Bond (Usually ____ Years)
PV = ? (amount to invest)
FV = Face Value of Strip Bond

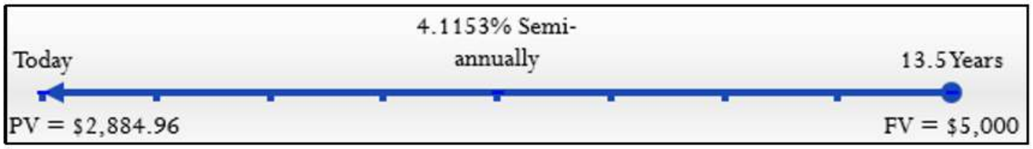
Step 2: Calculate the periodic interest rate (i) and the number of compounding periods (N) using Formula 9.1 and Formula 9.2, respectively.

Step 3: Calculate the present value of the strip bond using Formula 9.3, rearranging for PV.

Step 4: If you are interested in the actual dollar amount that the investor gains by holding onto the strip bond until maturity, apply Formula 8.3.

Assume that when prevailing market rates are 4.1153% you want to invest in a $5,000 face value strip bond with 13½ years until maturity. How much money is required to purchase the strip bond today?

Step 1: The known variables are FV = $5,000, Years = 13½, IY = 4.1153%, and CY = 2. The timeline illustrates the strip bond.



Step 2: The periodic interest rate is i = 4.1153%/2 = 2.05765%. The number of compounding periods remaining on the strip bond is N = 2 × 13½ = 27.

Step 3: The present value or purchase price of the strip bond is calculated as $\$5,000 = PV(1+0.0205765)^{27}$ or PV = $2,884.96. Thus, you can purchase the strip bond for $2,884.96

Step 4: If you hold onto the strip bond for the remaining 13½ years, you will receive $5,000 upon maturity. This represents a gain of I = $5,000.00 − $2,884.96 = $2,115.04 over the term of the investment.

Things To Watch Out For

Don't forget that the compounding on strip bonds is assumed to be semi-annual unless stated otherwise. Thus, CY = 2 in most scenarios.

Exercise 10.4.1: Give It Some Thought

a. If the yield remains constant, what is the relationship between the purchase price of a strip bond and the time to maturity?
b. What happens to the purchase price of a strip bond if the yield increases while the time to maturity remains constant?

Answer

a. The longer the strip bond needs to be discounted, the lower the purchase price becomes; the shorter the time to maturity, the greater the purchase price.
b. The purchase price becomes lower since the future value is discounted using a higher yield

Example 10.4.1: Purchase Price of a Strip Bond

Johansen is considering purchasing a $50,000 face value Government of Canada strip bond with 23½ years until maturity. The current market yield for these bonds is posted at 4.2031%. What is the price of the strip bond today, and how much money is gained if the bond is held until maturity?

Solution

Calculate the purchase price (PV) of the strip bond today along with the total interest (I).

What You Already Know

Step 1:

The face value, years to maturity, and nominal interest rate are known: FV = $50,000, Years = 23.5, IY = 4.2031%, CY = default = semi-annual = 2

How You Will Get There

Step 2:

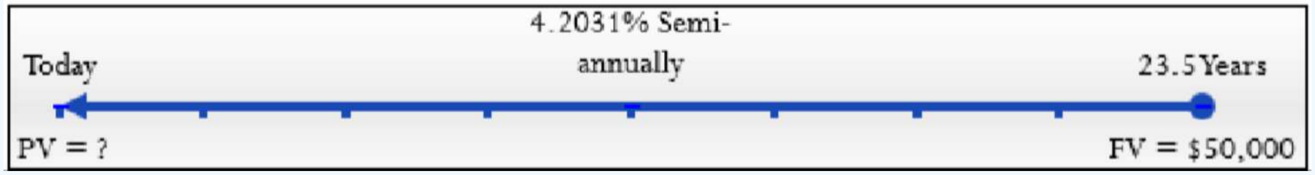
Apply Formula 9.1 and Formula 9.2.

Step 3:

Apply Formula 9.3 and rearrange for PV.

Step 4:

To calculate the total gain, apply Formula 8.3.



Perform

Step 2:

$i = \frac{4.2031\%}{2} = 2.10155\%$, $N = 2 \times 23.5 = 47$

Step 3:

$PV = \$50,000(1 + 0.0210155)^{-47} = \$18,812.66$

Step 4:

$I = \$50,000 - \$18,812.66 = \$31,187.34$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
47	4.2031	Answer: -18,812.66199	0	50000	2	2

The strip bond has a purchase price of $18,812.66 today. If you hold onto the strip bond until maturity, you will receive a payment of $50,000 23½ years from today. This represents a $31,187.34 gain.

Nominal Yields on Strip Bonds

In two instances it is common to calculate the nominal yield on strip bonds:

1. At times, various newspapers or online sites may list a purchase price per $100 of face value but not actually publish the yield on the strip bond. A smart investment decision always requires you to know the interest rate being earned.
2. Strip bonds are actively traded on a market. Many investors sell strip bonds to other investors prior to maturity. In these cases, the investor selling the bond needs to calculate the actual yield realized on the investment.

To calculate nominal yields you need the same four formulas as for the strip bond's purchase price.

How It Works

Follow these steps to calculate the nominal yield of a strip bond:

Step 1: How much was paid for the strip bond? Determine the present value or purchase price of the strip bond (PV). This may already be known, or you may have to calculate this amount using the previously introduced steps for calculating the present value.

Step 2: How much did the strip bond sell for? This is the future value (FV) of the strip bond. If it is the maturity of the strip bond, then the future value is the face value of the bond. If the investor is selling the strip bond prior to maturity, then this number is based on a present value calculation using the yield at the time of sale and time remaining until maturity.

Step 3: Determine the years (Years) between the purchase and the sale of the strip bond. Using $CY = 2$ (unless otherwise stated), apply Formula 9.2 and calculate the number of compounding periods between the dates (N).

Step 4: Calculate the periodic interest rate (i) using Formula 9.3, rearranging for i.

Step 5: Convert the periodic interest rate to its nominal interest rate by applying Formula 9.1 and rearranging for IY.

Step 6: If you are interested in the actual dollar amount that the investor gained while holding the strip bond, apply Formula 8.3.

Assume that when market yields were 4.254% an investor purchased a $25,000 strip bond 20 years before maturity. The purchase price was $10,772.52. The investor then sold the bond five years later, when current yields dropped to 3.195%. The selling price was $15,539.94. The investor wants to know the gain and the actual yield realized on her investment.

Step 1: The PV is known at $10,772.52. This is the purchase price paid for the strip bond.

Step 2: The FV is known at $15,539.94. This is the selling price received for the strip bond.

Step 3: Between the purchase and sale, the Years = 5. With CY = 2, the strip bond was held for N = 2 × 5 = 10 compounding periods.

Step 4: Applying Formula 9.3, $\$15,539.94 = \$10,772.52(1+i)^{10}$ or$i$ = 0.037321.

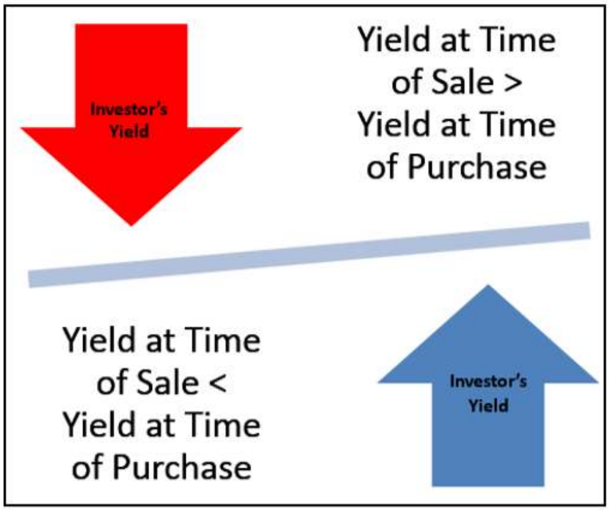
Step 5: Converting to the nominal rate, $0.037321 = IY \div 2$ or IY = 7.4642%.

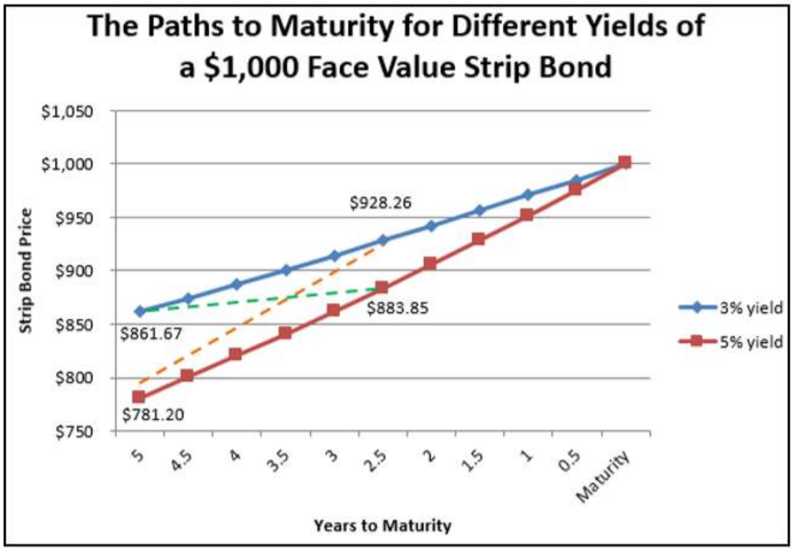
Step 6: The actual gain is I = $15,539.94 − $10,772.52 = $4,767.42.

The investor gained $4,767.42 on the investment, representing an actual yield of 7.4642% compounded semiannually.

Important Notes

The relationship between the yield at which you purchased the strip bond versus the current yield in the market at the time of sale is illustrated in the figure below.


Investor's Yield
Yield at Time of Sale > Yield at Time of Purchase
Yield at Time of Sale < Yield at Time of Purchase
Investor's Yield



Each relationship illustrated in the figure is discussed below:

Current Yield > Purchase Yield. If the current yield is higher at the time of sale than the yield at time of purchase, then the strip bond price at the time of sale is pushed downward from the original path to maturity. Consider two situations to illustrate this point, demonstrated in the figure on the next page. If $1,000 is present valued over five years at a 3% semi-annual yield, the price is $861.67. At a 5% semi-annual yield, the same strip bond would have a lower price of $781.20. In both of these cases, though, as the time of sale advances toward the maturity date over the next five years, the prices will gradually increase toward $1,000. However, the price of the 5% yield strip bond will always be lower than that of the 3% strip bond. Thus, if you purchase a 3% strip bond and sell when yields are 5%, the original sale price is lowered from the blue line to the red line. Thus, if the sale occurs with 2.5 years remaining until maturity, the price drops from $928.26 to $883.85. Since the future value drops, the actual investor's yield will become lower than the original purchase yield (as reflected by the dashed green arrow).

Current Yield < Purchase Yield. If the current yield is lower at the time of sale than the yield at the time of purchase, then correspondingly the strip bond price at the time of sale is pushed upward from the original path to maturity. Using the same example from above, recall that the price on the 3% yield strip bond is always higher than the 5% yield strip bond. Thus, if you had purchased the strip bond when yields were 5%, selling the strip bond when yields are 3% means that the future value jumps upward from the red line to the blue line. Thus, if the sale occurs with 2.5 years remaining until maturity, the price increases from $883.85 to $928.26. Therefore, the actual investor's yield will become higher than the original purchase yield (as reflected by the dashed orange line). This concept is exemplified by the investor above. She originally purchased when discount rates were 4.254% and sold when rates had dropped to 3.195%. This increased the sale price of her strip bond and translated into the investor actually realizing 7.4642% instead of the original yield of 4.254%!

? Exercise $10.4.2$: Give It Some Thought

In each of the following situations, determine whether the investor achieves a yield higher than, lower than, or equal to the yield at the time of purchase.

a. Yield at time of purchase = 5.438%; Yield at time of sale = 6.489%
b. Yield at time of purchase = 2.888%; Yield at time of sale = 2.549%
c. Yield at time of purchase = 4.137%; Yield at time of sale = 4.137%

Answer

a. Lower than; current yield > purchase yield.
b. Higher than; current yield < purchase yield.
c. Equal to; both yields are the same.

Example 10.4.2: The Investor's Actual Yield

Cameron invested in a \$97,450 face value Government of British Columbia strip bond with 25 years until maturity. Market yields at that time were 9.1162%. Ten and half years later Cameron needed the money, so he sold the bond when current yields were 10.0758%. Calculate the actual yield and gain that Cameron realized on his investment.

Solution

Calculate the nominal interest rate (IY) that Cameron realized while in possession of the strip bond. Also calculate the total gain (I).

What You Already Know

The following information is known about the bond at the time of purchase:

FV = \$97,450, Years = 25, IY = 9.1162%, CY = 2

The following information is known about the bond at the time of sale:

Years strip bond held = 10.5, FV = \$97,450, CY = 2, Years = 25 − 10.5 = 14.5, IY = 10.0758%

How You Will Get There

Step 1:

To figure out the purchase price, apply Formula 9.1, Formula 9.2, and Formula 9.3 based on the purchase date.

Step 2:

To figure out the selling price, apply Formula 9.1, Formula 9.2, and Formula 9.3 based on the selling date. Note that once the PV is calculated, it becomes the FV for subsequent calculations.

Step 3:

Taking the years the strip bond was held, apply Formula 9.2 to determine the number of compounding periods.

Step 4:

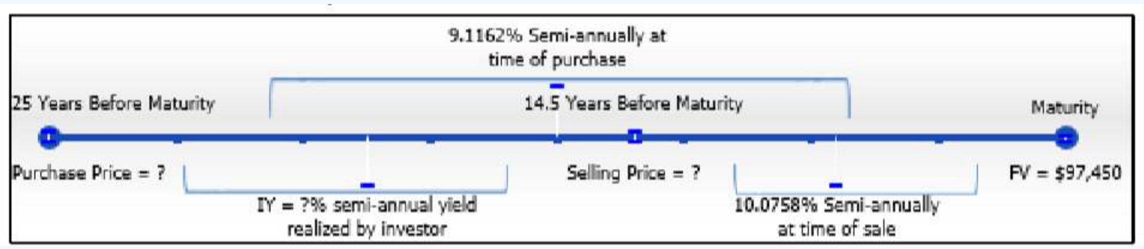
Calculate the periodic interest rate by applying Formula 9.3, rearranging for i.

Step 5:

Calculate the nominal interest rate by applying Formula 9.1, rearranging for IY.

Step 6:

Calculate the total gain by applying Formula 8.3.


9.1162% Semi-annually at
time of purchase
25 Years Before Maturity
14.5 Years Before Maturity
Maturity
Purchase Price = ?
Selling Price = ?
FV = $97,450
IY = ?% semi-annual yield
realized by investor
10.0758% Semi-annually
at time of sale

Perform

Step 1:

i = 9.1162%/2 = 4.5581%; N = 2 × 25 = 50

$$\begin{aligned} \$97,450 &= PV(1+0.045581)^{50} \\ PV &= \$97,450 \div 1.045581^{50} \\ &= \$10,492.95 \end{aligned}$$

Step 2:

i = 10.0758%/2 = 5.0379%; N = 2 × 14.5 = 29

$$\begin{aligned} \$97,450 &= PV(1+0.050379)^{29} \\ PV &= \$97,450 \div 1.050379^{29} \\ &= \$23,428.63 \end{aligned}$$

Step 3:

CY = 2; N = 2 × 10.5 = 21

Step 4:

PV = $10,492.95; FV = $23,428.63

$$\begin{aligned} \$23,428.63 &= \$10,492.95(1+i)^{21} \\ i &= \left(\frac{\$23,428.63}{\$10,492.95}\right)^{\frac{1}{21}} = 0.038991 \end{aligned}$$

Step 5:

0.038991 = IY ÷ 2 or IY = 0.038991 × 2 = 7.7982%

Step 6:

I = $23,428.63 − $10,492.95 = $12,935.68

Calculator Instructions

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	50	9.1162	Answer: -10,492.95246	0	97450	2	2
2	29	10.0758	Answer: -23,428.63365	√	√	√	√
4 and 5	21	Answer: 7.798240	-10492.95	√	23428.63	√	√

Cameron sold the strip bond when prevailing market rates were higher than the rate at the time of purchase. He has realized a lower yield of 7.7982% compounded semi-annually, which is a gain of $12,935.68.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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10.5: Application - Inflation, Purchasing Power, and Rates Of Change

How much does inflation cut into people's incomes and wealth? In 1955, the average Canadian worker earned an annual gross salary of about $2,963.1 By 2010, the average Canadian worker brought home annual gross earnings approximating $44,366.2 Does this mean that Canadians today are 15 times richer than our grandparents? From earlier discussions in section 4.3 on inflation, real income, and purchasing power, you already can tell we are not. The actual increase is by something less than 15 times. But how much less is it?

To answer the question, you have to express both incomes with reference to the same year. Either convert the 1955 income to its 2010 equivalent or vice versa. To help, what if you are told the rate of inflation from 1955 to 2010 averaged 3.91%3 per year? To convert, you might use either the percent change method or the real income method. Each of these poses problems, though:

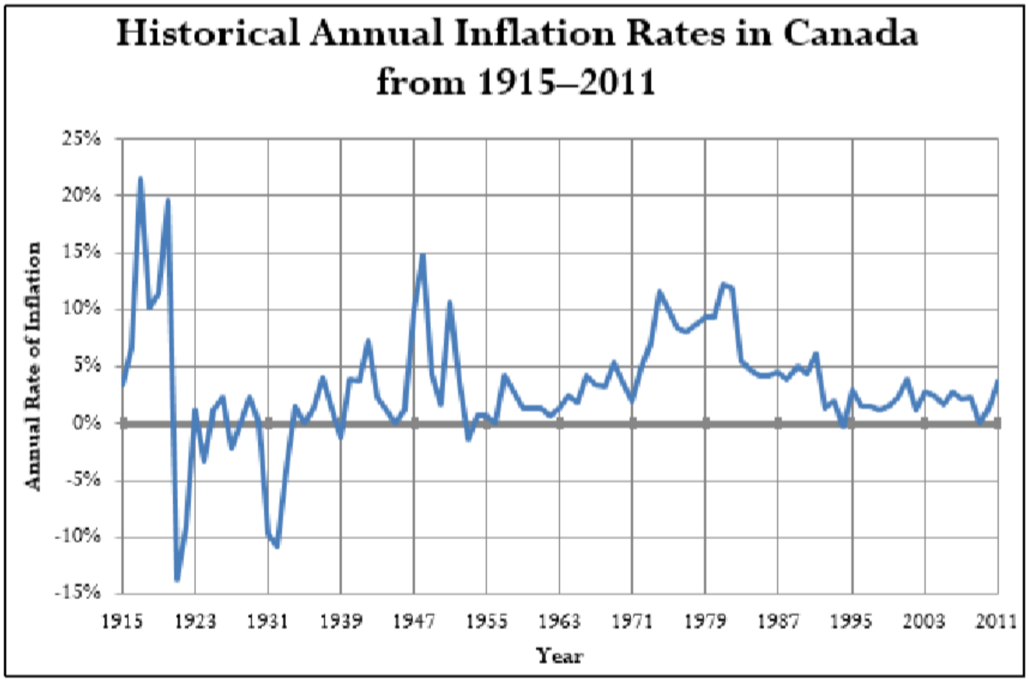
• To use the percent change formula from section 3.1, you would have to apply it 55 consecutive times! Clearly, this is impractical.

• To use the real income formula from section 4.3, you would need the consumer price index numbers for both years. Locating CPI values can be time consuming, especially for years in the distant past. Another complication is that if you want to project future values (such as the equivalent income in, say, 2020 instead of 2010), CPI values do not exist for future years.

There has to be an easier way! In this section, you will see how to adapt the compound interest concepts and formulas to suit such applications as inflation, purchasing power, and even percent change.

Inflation

Inflation is the overall upward price movement of products in an economy. It is measured by positive change in the consumer price index. Historical inflation rates in Canada are illustrated in the figure below.



Note that historically prices have always risen over the long term. In the short term, though, there have been periods during which prices moved downward. This is known as **deflation**, which is measured by negative change in the consumer price index. Such periods of deflation usually do not last very long; the longest period recorded was during the Great Depression. Most recently, deflation occurred for a few months in 2009.

Inflation is most commonly expressed as an annual rate; therefore, you treat it mathematically as an annually compounded interest rate. This is the nominal interest rate (IY) with a compounding frequency of one, or CY = 1. Note that if deflation has occurred during the time period in question, the interest rate is a negative number. If you have a series of inflation rates, you treat this as a variable interest rate. If you are finding an average inflation rate for some time period, you treat this like finding the equivalent fixed interest rate. In using the compound interest formulas, assign the present value to any beginning value in the problem at hand, while the future value is any ending value. The number of compounding periods (N) still reflects the number of compounds between the two values.

How It Works

You can use any of the formulas and techniques from Chapter 9 to work with inflation. The opening example of incomes in 1955 and 2010 will illustrate the processes.

Solving for Future Value

If the unknown variable is an ending value, apply Formula 9.3, solving for the future value. If only one inflation rate (a fixed rate) applies, this requires only one application of the formula. If the inflation rate changes over time, you apply the formula multiple times or use the quick method of calculation:

$$FV = PV \times (1+i_1)^{N_1} \times (1+i_2)^{N_2} \times \ldots \times (1+i_n)^{N_n}$$

In the example, you could move the 1955 income to 2010. In this case, the PV = \$2,963, IY = 3.91%, CY = 1, and N = 55. With a fixed interest rate, apply Formula 9.3 and calculate $FV = \$2,963(1+0.0391)^{55} = \$24,427.87$. Therefore, the 2010 equivalent income of 1955 is approximately \$24,427. Note the actual 2010 income is about 81% higher, which would imply that Canadians have indeed become wealthier.

Solving for Present Value

If the unknown variable is a beginning value, apply Formula 9.3, rearranging for present value. Depending on whether the inflation rate is fixed or variable, solve using the same technique as for future value. In the example, you could move the 2010 income to 1955. In this case, the FV = \$44,366, IY = 3.91%, CY = 1, and N = 55. With a fixed interest rate, apply Formula 9.3, rearranging for PV, and calculate PV = \$44,366 ÷ (1 + 0.0391)55 = \$5,381.41. Therefore, the 1955 equivalent income of 2010 is approximately \$5,381. Note this income is the same percentage (81%) higher than the actual 1955 income, as you found by the first method.

Solving for the Rate

If the average inflation rate is the unknown variable, then the beginning and ending values must be known. In the example, assume the average inflation rate is unknown, but the equivalent incomes of PV =\$2,963 in 1955 and FV = \$24,427.87 in 2010 are known. The CY = 1 and N = 55 years. Applying Formula 9.3 results in $\$24,427.87 = \$2,963(1+i)^{55}$. Solving for i you get 3.91%. This is the average inflation rate (since CY = 1, then $i = IY$).

Solving for the Term

If the unknown variable is the amount of time between the beginning and ending values, once again apply Formula 9.3 and rearrange for N. In the example, assume the beginning and ending values of PV = \$2,963 in 1955 and FV = \$24,427.87 are known, but the year for the future value is unknown. The IY = 3.91% and CY = 1. Applying Formula 9.3 gives you $\$24,427.87 = \$2,963(1+0.0391)^N$. Solving for N gives 55 years. The starting year of 1955 + 55 years is the ending year of 2010, in which the equivalent income of \$24,427.87 applies.

Important Notes

Your BAII Plus Calculator

The time value of money buttons are designed for financial calculations and require you to follow the cash flow sign convention at all times. Remember that this convention requires money leaving you to be entered as a negative number, and money being received by you to be entered as a positive number.

When you adapt this function to economic calculations such as inflation, no money is being invested or received— numbers are being moved across time. To obey the calculator requirement of the cash flow sign convention, ensure that the signs attached to the present (PV) and future values (FV) are opposites. The choice as to which is positive and which is negative is arbitrary and does not affect the outcome of the calculation. Ignore the cash flow sign displayed on any solutions.

Many experts today claim that the average Canadian retiree in 2012 needs approximately $40,000 in gross annual income to retire comfortably. Assume that you are 20 years old in 2012. Historically in Canada, the inflation rate has averaged 3.16%. If the average inflation rate continues in the future, what gross income will you require when you retire at age 65?

Solution

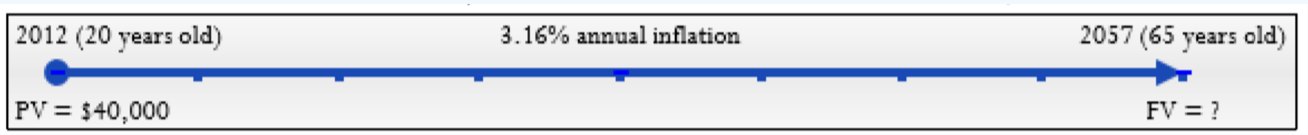
Calculate the ending value (FV) for your annual gross income when you retire at age 65.

What You Already Know

Step 1:

The present value, fixed inflation rate, and time frame are known, as illustrated in the timeline.

Years = 45



How You Will Get There

Step 2:

With a CY = 1, the $i = IY$.

Step 3:

Apply Formula 9.2 to calculate the number of compounding periods.

Step 4:

Calculate the future value by applying Formula 9.3.

Perform

Step 2:

i = 3.16%

Step 3:

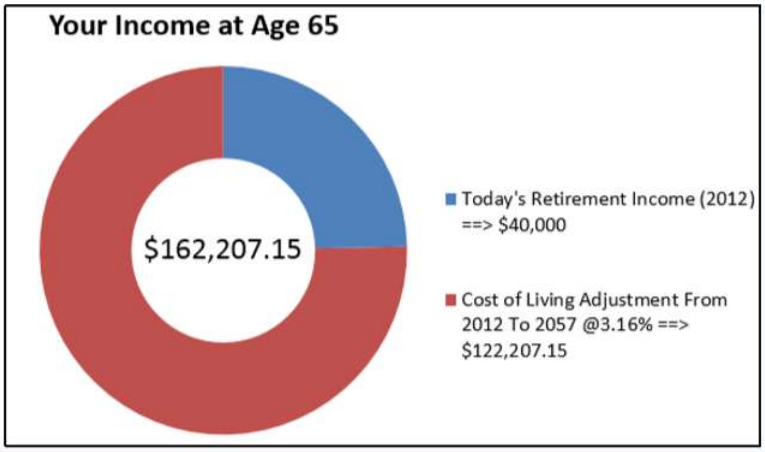
N = 1 × 45 = 45

Step 4:

$FV = \$40,000(1+0.0316)^{45} = \$162,207.15$

N	I/Y	PV	PMT	FV	P/Y	C/Y
45	3.16	-40000	0	Answer: 160,798.0247	1	1

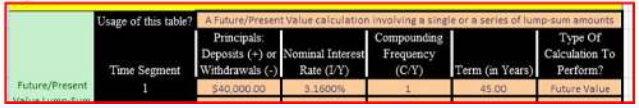
When you retire in 2057, if inflation continues to average 3.16% annually, according to the experts your annual gross income must be $162,207.15 for you to live comfortably.



Excel Instructions

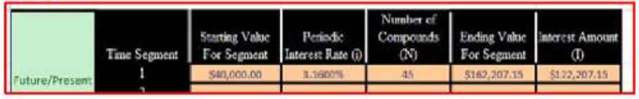
Open the Excel template entitled “Chapter 9: Single Payments and Compound Interest Template.”

Input



	Usage of this table?	A Future/Present Value calculation involving a single or a series of lump-sum amounts				
	Time Segment	Principals: Deposits (+) or Withdrawals (-)	Nominal Interest Rate (I/Y)	Compounding Frequency (C/Y)	Term (in Years)	Type Of Calculation To Perform?
Future/Present Value Lump-Sum	1	$40,000.00	3.1600%	1	45.00	Future Value

Output



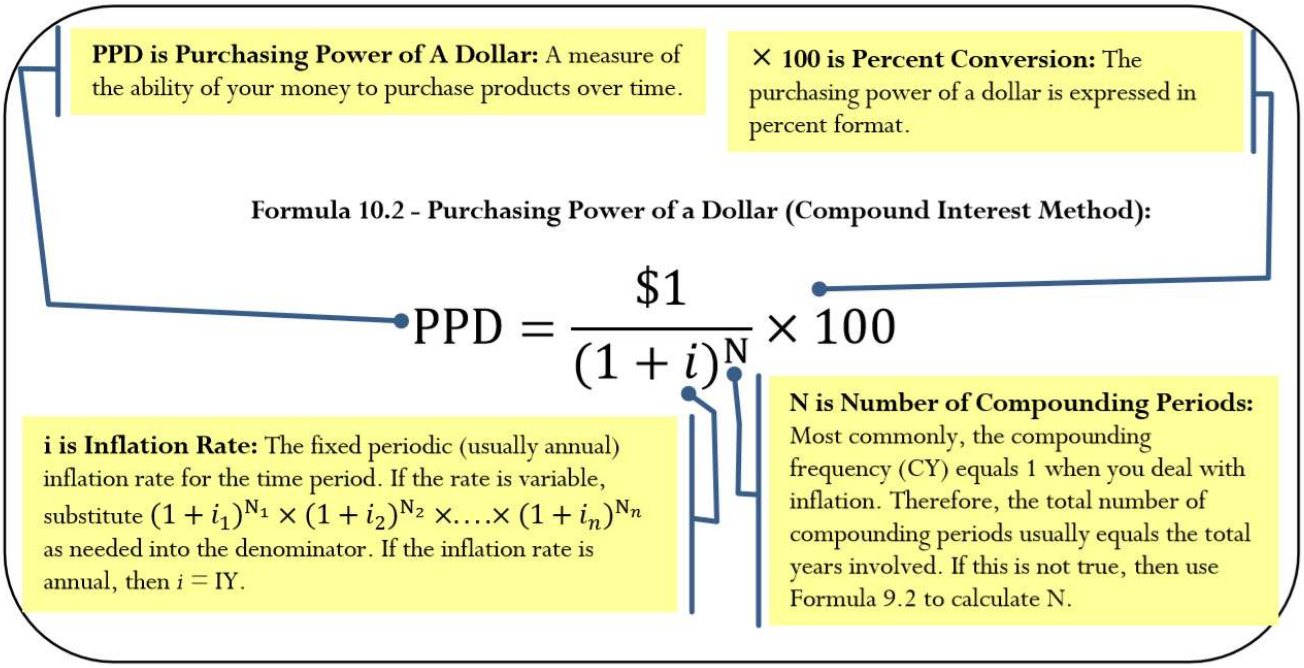
	Time Segment	Starting Value For Segment	Periodic Interest Rate (i)	Number of Compounds (N)	Ending Value For Segment	Interest Amount (I)
Future/Present	1	$40,000.00	3.1600%	45	$162,207.15	$122,207.15

Purchasing Power

Recall from section 4.3 that the purchasing power of a dollar is the amount of goods and services that can be exchanged for a dollar. Purchasing power has an inverse relationship to inflation. When inflation occurs and prices increase, your purchasing power decreases.

The Formula

In the formula introduced in section 4.3, the denominator used the CPI to represent the change in product prices. In compound interest applications, you instead use the inflation rate, as shown in Formula 10.2.



PPD is Purchasing Power of A Dollar: A measure of the ability of your money to purchase products over time.

× 100 is Percent Conversion: The purchasing power of a dollar is expressed in percent format.

Formula 10.2 - Purchasing Power of a Dollar (Compound Interest Method):

$$\mathrm{PPD} = \frac{\$1}{(1+i)^{\mathrm{N}}} \times 100$$

i is Inflation Rate: The fixed periodic (usually annual) inflation rate for the time period. If the rate is variable, substitute $(1+i_1)^{\mathrm{N}_1} \times (1+i_2)^{\mathrm{N}_2} \times \ldots \times (1+i_n)^{\mathrm{N}_n}$ as needed into the denominator. If the inflation rate is annual, then $i = \mathrm{IY}$.

N is Number of Compounding Periods: Most commonly, the compounding frequency (CY) equals 1 when you deal with inflation. Therefore, the total number of compounding periods usually equals the total years involved. If this is not true, then use Formula 9.2 to calculate N.

Formula 10.2

How It Works

Follow these steps to solve a purchasing power question using compound interest:

Step 1: Identify the inflation rate (IY), the compounding on the inflation rate (CY), and the term (Years). Normally, $i = IY$ and $N =$ Years; however, apply Formula 9.1 and Formula 9.2 if you need to calculate i or N.

Step 2: Apply Formula 10.2, solving for the purchasing power of a dollar.

Using the income example, determine how an individual's purchasing power has changed from 1955 to 2010. Recall that the average inflation during the period was 3.91% per year.

Step 1: The IY = 3.91%, CY = 1, and Years = 55. The annual rate lets i = 3.91% and N = 54.

Step 2: From Formula 10.2, the $PPD = \frac{\$1}{(1+0.0391)^{55}} \times 100 = 12.1296\%$. As a rough interpretation, this means that if someone could purchase 100 items with \$100 in 1955, that same \$100 would only purchase about 12 of the same items in 2010.

Things To Watch Out For

Determine exactly what the question is asking you to calculate with respect to the purchasing power of a dollar.

- If the question asks, "What is the purchasing power of a dollar?" then your answer is the result of Formula 10.2.
- If the questions asks, "How has your purchasing power decreased?" then your answer is 100% minus the result of Formula 10.2.

It is critical to Plan when solving these questions so that your solution addresses the question asked.

Exercise $10.5.1$: Give It Some Thought

What is the purchasing power of your dollar if the prices on products:

a. Double?
b. Triple?
c. Quadruple?

Answer

a. 1/2 = 50%
b. 1/3 = $3.3\bar{3}\%$
c. 1/4 = 25%

The historical inflation rates in Canada were 3.13% from June 2007 to June 2008, −0.26% from June 2008 to June 2009, and 0.96% from June 2009 to June 2010. Comparing June 2007 to June 2010, what is the purchasing power of a 2007 dollar in 2010?

Solution

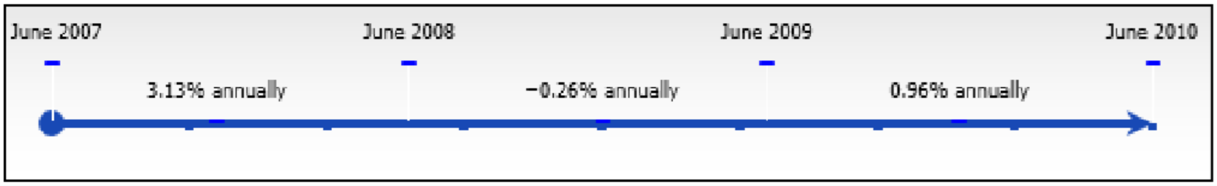
Calculate the purchasing power of a 2007 dollar (PPD) in 2010.

What You Already Know

Step 1:

The inflation rates and terms are known, as illustrated in the timeline.

CY (for each time segment) = 1, Term (for each time segment) = 1 Year



How You Will Get There

Step 1 (continued):

Identify the periodic interest rate (i) and the number of compounds (N) for each time segment. With CY = 1, then $i = IY$ and N = Years.

Step 2:

Apply Formula 10.2, substituting the variable interest rate version in the denominator.

Perform

Step 1 (continued):

i_1 = 3.13%, i_2 = −0.26%, i_3 = 0.96%; N_1, N_2, N_3 = 1

Step 2:

$$\begin{aligned} PPD &= \frac{\$1}{(1+0.0313)\times(1-0.0026)\times(1+0.0096)} \times 100 \\ &= 96.2933\% \end{aligned}$$

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	3.13	Answer: 0.969649	0	-1	1	1
2	√	-0.26	Answer: 0.972177	√	-0.969649	√	√
3	√	0.96	Answer: 0.962933	√	-0.972177	√	√

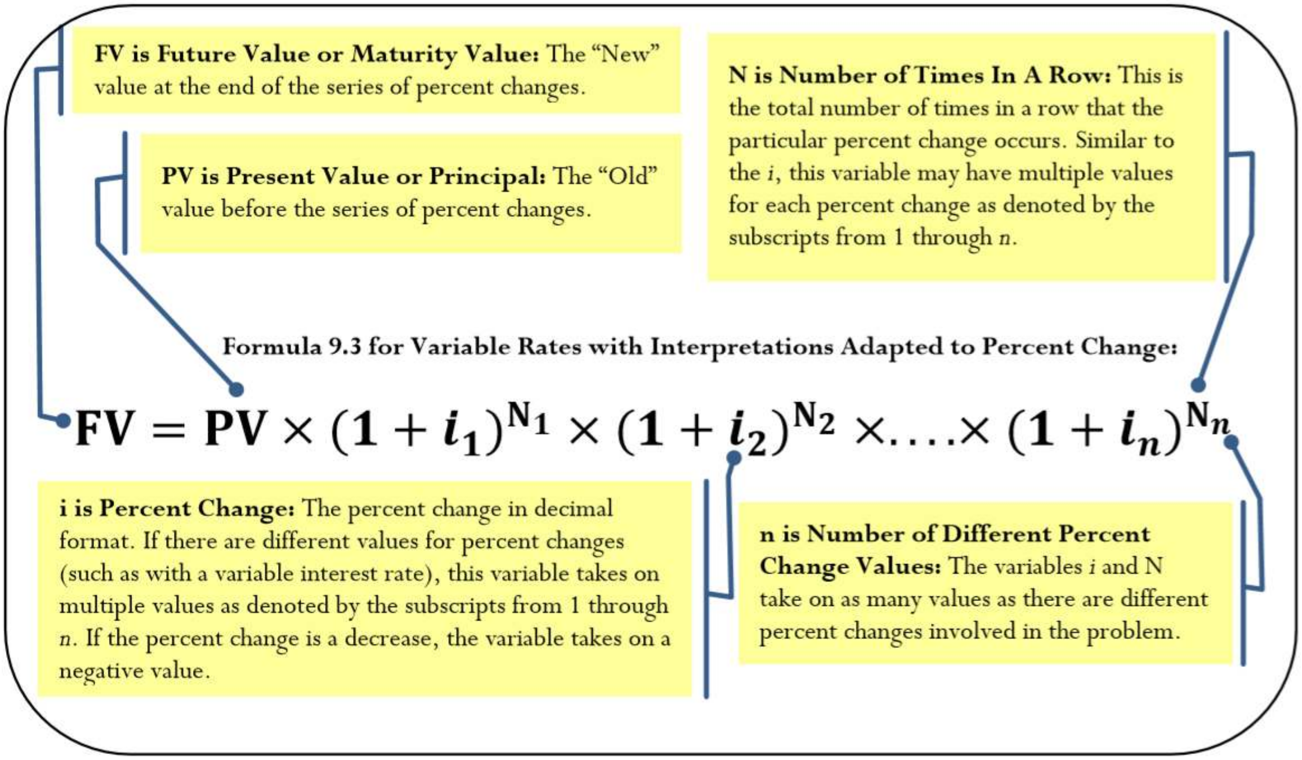
The purchasing power of a June 2007 dollar in June 2010 is 96.2933%. If $100 in June 2007 could purchase 100 items, in June 2010 $100 could only purchase about 96 items.

Rates of Change

Recall from section 3.1 that you could calculate a percent change between Old and New numbers. While this formula works well when you are interested in just a single percent change, it becomes time consuming and tedious when you work with a series of percent changes. For example, assume the TSX has a value of 12,000. The TSX then drops by 4% each month for five months, and then rises by 4% each month for five months. What is the “New” value for the TSX? It is not 12,000! If you use the formula for percent change, you need a series of 10 calculations solving for “New” each time—one calculation for each month! Not much fun. Mathematically, in a series of percent changes, each change compounds on the previous one. Therefore, you can use compound interest formulas to work with any series of percent changes.

The Formula

You can solve any series of percent changes by applying an adapted version of Formula 9.3 for variable interest rates:



FV is Future Value or Maturity Value: The “New” value at the end of the series of percent changes.

PV is Present Value or Principal: The “Old” value before the series of percent changes.

N is Number of Times In A Row: This is the total number of times in a row that the particular percent change occurs. Similar to the i, this variable may have multiple values for each percent change as denoted by the subscripts from 1 through n.

Formula 9.3 for Variable Rates with Interpretations Adapted to Percent Change:

$$FV = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \ldots \times (1 + i_n)^{N_n}$$

i is Percent Change: The percent change in decimal format. If there are different values for percent changes (such as with a variable interest rate), this variable takes on multiple values as denoted by the subscripts from 1 through n. If the percent change is a decrease, the variable takes on a negative value.

n is Number of Different Percent Change Values: The variables i and N take on as many values as there are different percent changes involved in the problem.

How It Works

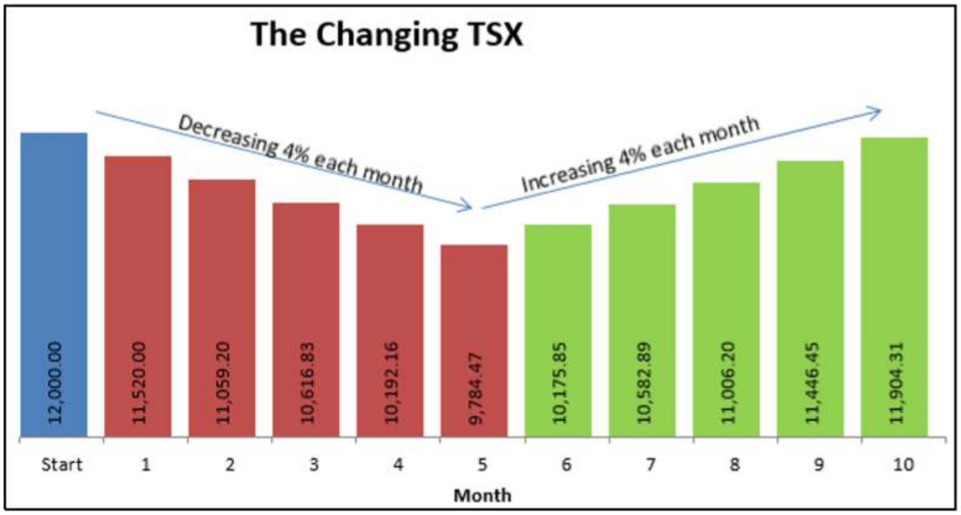
Follow these steps to adapt Formula 9.3 for percent change applications:

Step 1: Assign either the “Old” value to PV or the “New” value to FV (depending on which you know).

Step 2: Identify your series of percent changes (i_1 through i_n) and how many times in a row each percent change value occurs (N_1 through N_n). Remember that decreases are negative values.

Step 3: Apply the adapted version of Formula 9.3, solving for either FV or PV.

As an example, find out the new value for the TSX based on a starting value of 12,000 with decreases of 4% for five months followed by increases of 4% for five months.



Step 1: The “Old” value is PV=12,000.

Step 2: The first change is a decrease of 4%, or i_1 = −4%. It occurs for five months in a row, thus N_1 = 5. Next, an increase of 4%, or i_2 = 4%, also occurs five months in a row, thus N_2 = 5.

Step 3: Use Formula 9.3 to calculate $FV = 12,000 \times (1-0.04)^5 \times (1+0.04)^5 = 11,904.31$. The “New” value of the TSX after the 10 months of change is 11,904.31.

Important Notes

You can use your financial calculator for these nonfinancial calculations just as you did for inflation. You must follow the cash flow sign convention by ensuring that PV and FV have opposite signs. However, once again the signs do not have any meaning in these calculations.

Things To Watch Out For

When you work with variable inflation rates, purchasing power of a dollar, or rates of change, you can simplify the variable rates into a single fixed rate using a geometric average before applying the compound interest formula. For example, using the TSX example above, the geometric average of the 10 months of change is:

$$GAvg = \left[\sqrt[10]{(1-0.04)^5 \times (1+0.04)^5} - 1\right] \times 100 = -0.080032\%$$

Using this calculated value as the periodic interest rate and substituting into Formula 9.3 produces:

$$FV = 12,000 \times (1-0.00080032)^{10} = 11,904.31$$

Similarly, you could solve Example $10.5.2$ with this approach. Some students have found this technique easier than working with the variable rates. Either way, whether you use variable rates or the geometric average as the fixed rate, both techniques produce the same solution.

? Exercise $10.5.2$: Give It Some Thought

If a quantity decreases by $x\%$ and then increases by the same percent, why do you not arrive at the original quantity as your “New” value?

Answer

The initial decrease will make the base number smaller. If the new base number is then increased by the same percentage, that percentage is determined from a smaller base. It will not produce the same value as the decrease.

✓ Example $10.5.3$: Changing Employment in Vancouver

The City of Vancouver tracks employment in its Metro Core area. In 1971, approximately 45,000 employees in the Metro Core area were classified as "professional and commercial service" workers. From 1971 to 1981, the number of workers grew by about 4.5% per year. From 1981 to 1991, the growth rate was about 3.1% per year, and from 1991 to 2001 the growth rate was about 1.5% per year.4 Rounded to the nearest thousand, how many people were employed in the "professional and commercial services" field in the Metro Core area of Vancouver in 2001?

Solution

You are looking for the “New” value (FV) of the number of employees after the 30 years of percent changes.

What You Already Know

Step 1:

The beginning value and structure of the sequence of percent changes are known, as shown in the timeline.

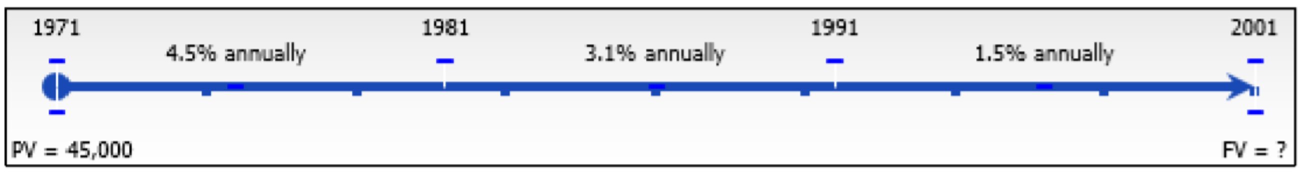
How You Will Get There

Step 2:

Capture the values of i_n and N_n for each consecutive percent change value. Note that the timeline has three time segments, so each variable has three values.

Step 3:

Apply Formula 9.3.



Perform

Step 2:

$$i_1 = 4.5\%, N_1 = 10, i_2 = 3.1\%, N_2 = 10, i_3 = 1.5\%, N_3 = 10$$

Step 3:

$$\begin{aligned} FV &= 45,000 \times (1+0.045)^{10} \times (1+0.031)^{10} \times (1+0.015)^{10} \\ FV &= 45,000 \times 1.045^{10} \times 1.031^{10} \times 1.015^{10} \\ FV &= 45,000 \times 1.552969 \times 1.357021 \times 1.160540 = 110,058 = 110,000 \end{aligned}$$

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	10	4.5	-45000	0	Answer: 69,883.62398	1	1
2	√	3.1	-69,883.62398	√	Answer: 94,833.56372	√	√
3	√	1.5	-94,833.56372	√	Answer: 110,058.2223	√	√

From 1971 to 2001, the number of employees in the "professional and commercial services" field in the Metro Core area of Vancouver grew from 45,000 to 110,000.

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2. Statistics Canada, “Earnings, Average Weekly, by Province and Territory,” adapted from CANSIM table 281-0027 and Catalogue no. 72-002-X, www40.statcan.gc.ca/l01/cst01/labr79-eng.htm, accessed August 11, 2011.
3. Statistics Canada, “Consumer Price Indexes for Canada, Monthly, 1914–2012 (V41690973 Series)”; all values based in May of each year; www.bankofcanada.ca/rates/related/inflation-calculator/?page_moved=1, accessed November 26, 2012.
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10.S: Summary

Key Concepts Summary

10.1: Application: Long-Term GICs (Keeping Your Money Safe When Investing)

- The characteristics and calculations involved with an interest payout GIC
- The characteristics and calculations involved with compound interest GICs
- The characteristics and calculations involved with escalator GICs

10.2: Application: Long-Term Promissory Notes (IOUs)

- The sale of interest-bearing promissory notes
- The sale of noninterest-bearing promissory notes

10.3: Application: Savings Bonds (You Can Personally Finance Canada’s Debt!)

- Key characteristics of savings bonds
- The interest rates for savings bonds
- Calculating interest amounts and maturity values for savings bonds

10.4: Application: Strip Bonds (Buy Low, Sell High)

- Characteristics of strip bonds
- Calculating the present value or purchase price of a strip bond
- Calculating the nominal yield on a strip bond

10.5: Application: Inflation, Purchasing Power, and Rates of Change (Your Grandparents Used to Go to a Movie for a Quarter)

- Applying the concepts of compound interest to rates of inflation
- Applying the concepts of compound interest to purchasing power
- Applying the concepts of compound interest to rates of change

The Language of Business Mathematics

Canada Premium Bond (CPB)

A savings bond that is redeemable only during its anniversary month.

Canada Savings Bond (CSB)

A savings bond that is redeemable at any time.

compound interest GIC

A GIC that uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding.

compound interest savings bonds

Called C-bonds, these bonds annually convert the interest on the savings bond to principal.

deflation

The overall downward price movement of products in an economy, which is measured by negative change in the consumer price index.

escalator interest GIC

A GIC that uses compound interest rates that usually remain constant during each of a series of time intervals, always rising stepwise throughout the term of the investment with any accrued interest being converted to principal.

inflation

The overall upward price movement of products in an economy, which is measured by positive change in the consumer price index.

interest payout GIC

A GIC where the interest is periodically paid out to the investor, but it is never added to the principal of the GIC. Because the interest does not actually compound, in essence the concepts of simple interest are used.

regular interest savings bonds

Called R-bonds, these bonds annually pay the interest to the owner of the bond and do not convert the interest to principal.

savings bonds (SBs)

Long-term financial instruments with 10-year maturities issued only by the Canadian federal government to finance the long-term national debt.

strip bond

A marketable bond that has been stripped of all interest payments.

The Formulas You Need to Know

Symbols Used

I = Interest payment amount

i = Periodic interest rate

n = number of pieces of data or variables

N = Number of compounding periods

PPD = Purchasing power of a dollar

PV = Principal or present value

Formulas Introduced

Formula 10.1 Periodic Interest Amount: $I = PV \times i$

Formula 10.2 Purchasing Power of a Dollar (Compound Interest Method): $PPD = \frac{\$1}{(1+i)^N} \times 100$

Technology

Calculator No new calculator functions were introduced in this chapter.

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10.CS: Case Study - How Much Interest Is Really Earned?

The Situation

Sharon works for Lightning Wholesale, which strongly encourages its employees to save up for retirement. After learning on June 1, 2006, that the company matches dollar for dollar any employee investments into their RRSPs, she has been pursuing her RRSP actively.

Sharon soon figured out one important and often misunderstood fact about RRSPs. Though many people think of an RRSP as a financial tool, like a savings account, into which money is invested, she realized that an RRSP is actually a taxsheltered environment. Within the RRSP you can invest your money into various financial tools to save your retirement income, and any interest or capital gains that are earned on these investments are not taxable. This is in contrast to investments held outside an RRSP environment; for the latter, all earnings are subject to annual income taxes. The financial tools available to an RRSP could include GICs, mutual funds, bonds, securities, trusts, gold bullion, and stocks, just to name a few. Sharon’s portfolio happens to include CSBs, GICs, and strip bonds.

She is interested in learning how much money she has saved up by June 1, 2010, and how much of that money is actually real, inflation-adjusted growth in her savings. In other words, she wants to account for inflation and get an understanding of how much real interest beyond cost of living adjustments she has earned in her RRSP.

The Data

- Lightning Wholesale matches any employee investment in CSBs, CPBs, and GICs dollar for dollar. For example, if Sharon's out-of-pocket investment consists of $2,000 into a savings bond, she in fact invests $4,000 into the savings bond—$2,000 from her own pocket and $2,000 from Lightning Wholesale.
- Lightning Wholesale matches any employee investment in strip bonds by providing enough money to purchase a second identical strip bond.
- Sharon's out-of-pocket investment history is as follows:
  - Each March 1 and December 1 since June 1 of 2006 she has had $1,000 to invest in C-bond CSBs.
  - She has invested $1,000 into each of Series P55, P59, P67, and P71 C-bond CPBs.
  - She placed $5,000 into a five-year escalator GIC on September 1, 2006, paying quarterly compounded rates each year of 2%, 2.4%, 3%, 4.5%, and 7%.
  - She purchased three strip bonds: - On December 1, 2006, she purchased a 20-year $20,000 face value strip bond with a market yield of 7.053%. - On June 1, 2007, she purchased a 25-year $25,000 face value strip bond with a market yield of 6.5425%. - On December 1, 2008, she purchased a 25-year $15,000 face value strip bond with a market yield of 5.9067%.
- On June 1, 2010, the market yield of strip bonds with 16½-year maturities was 6.5425%, 22-year maturities was 4.9855%, and 23½-year maturities was 4.8821%.
- Annual June to June inflation rates starting with June 2006 to June 2007 have been 2.19%, 3.13%, −0.26%, and 0.96%.

Important Information

- Although CPBs can only be cashed during the anniversary month, calculate the accrued interest to June 1, 2010, for all CPBs using simple interest for any partial years.
- Assume that inflation rates are constant throughout any given year.
- Information about CSBs and CPBs is found in the savings and premium bond tables earlier in the chapter.

Your Tasks

1. If Sharon wanted and was able to cash in all of her investments, calculate the maturity value of all of her investments on June 1, 2010. Calculate the total interest earned across all investments.
2. Using the inflation rates, calculate the equivalent value of all her money placed into investments on June 1, 2010.

3. Calculate the difference between her actual maturity value and her inflation-adjusted principal. This is the real amount of interest that she has gained over the years.

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10.E: Compound Interest- Applications Involving Single Payments (Exercises)

10.1: Application - Long-Term GICs

Mechanics

Calculate the amount of each interest payment as well as the total interest paid over the entire term for each of the following interest payout GICs.

Principal (PV)	Nominal Interest Rate (IY)	Term
1. $43,000	6% quarterly	4 years
2. $38,100	6.6% monthly	2 years

Calculate the missing pieces of the puzzle (as indicated by ?) for the following compound interest GICs. Cells with N/A are not required.

Principal	Nominal Interest Rate and Term	Compounding Frequency	Maturity Value	Interest Amount Earned	Equivalent Fixed Interest Rate
3. $17,400	4.3% for 4 years	semi-annually	?	?	N/A
4. $25,000	2.9% for first year; 3.9% for second year	quarterly	?	?	?%quarterly
5. $51,000	3.9% for 5 years	monthly	?	?	N/A
6. $72,375	1.85% for 6 months; then 2.1% for 18 months; then 2.35% for 2 years	Semi-annually	?	?	?%annually

Calculate the missing pieces of the puzzle (as indicated by ?) for the following escalator rate GICs. Each interest rate listed is for a one-year term.

Principal	Nominal Interest Rate and Term	Compounding Frequency	Maturity Value	Interest Amount Earned	Equivalent Fixed Interest Rate
7. $2,680	0.75%; 1.25%; 1.85%	Annually	?	?	?% annually
8. $134,000	1.15%; 1.95%; 2.5%; 4%; 7%	Semi-annually	?	?	?% semi-annually
9. $84,630	0.15%; 1.1%; 3.9%; 6.8%	Monthly	?	?	?% annually
10. $15,985	0.25%; 0.75%; 1.9%; 2.75%; 7.1%	Quarterly	?	?	?% annually

Applications

11. Sanchez placed $11,930 into a five-year interest payout GIC at 4.2% compounded monthly. Calculate the interest payout amount every month.
12. RBC is offering you a four-year fixed rate compound interest GIC at 3.8% compounded quarterly. TD Canada Trust is offering you a variable rate compound interest GIC at 3.4% for the first two years, followed by 4.25% for the remaining two years. If

you have $10,000 to invest, which GIC should you select? How much more interest will you earn from the selected GIC than from the alternative?

13. Your five-year variable rate GIC is due to mature. Your monthly compounded interest rate was 1.5% for 22 months, 1.75% for 9 months, 1.6% for 13 months, and 1.95% for the remainder. If you originally invested $7,165, what is the maturity value? What equivalent monthly compounded fixed rate of interest did the GIC earn?
14. For three different five-year compound interest GICs, you can choose between fixed rates of 4.8% compounded monthly, 4.83% compounded quarterly, or 4.845% compounded semi-annually. On a $25,000 investment, how much more interest will the best option earn compared to the worst option?
15. The RBC RateAdvantage escalator GIC offers annually compounded interest rates on a five-year nonredeemable GIC of 0.75%, 2.25%, 2.5%, 3%, and 5%. Calculate the maturity value of a $40,000 investment. Determine the annually compounded equivalent fixed rate earned on the investment.
16. TD Canada Trust is offering its five-year Stepper GIC at annually escalating rates of 1.15%, 2%, 2.75%, 3.5%, and 4.5%. All rates are compounded semi-annually. Alternatively, it is offering a five-year fixed rate GIC at 2.7% compounded monthly. What total interest amount does an $18,000 investment earn under each option?
17. Your escalator rate GIC has annually compounded interest rates of 1.15%, 1.95%, 3.2%, 4.7%, and 6.6% in subsequent years. On a $15,140 investment, calculate the following:
    a. The amount of interest earned in each year.
    b. The annually compounded equivalent fixed rate.

Challenge, Critical Thinking, & Other Applications

18. You have invested $6,365 into a three-year interest payout GIC earning 3.75% compounded annually. If you invest each of your interest payout amounts into a Builder GIC (which allows you to contribute regularly to the GIC) earning 2.4% compounded monthly, how much interest in total do you earn over the three-year term?
19. Three years ago you placed $50,000 into an RBC five-year fixed rate, nonredeemable, compound interest GIC earning 3.9% compounded quarterly. Interest rates have risen since that initial investment, and banks today are advertising rates of 4.5% compounded semi-annually on two-year compound interest GICs. Your RBC GIC allows you to redeem your investment early, but you would incur a 0.75% penalty per year on the maturity value of the account. Should you keep your current RBC GIC, or should you take the early withdrawal penalty and invest in a new two-year GIC? Show calculations to support your recommendation.
20. Calculate the interest earned on each of the following five-year GICs. Rank the GICs from best to worst based on the amount of interest earned on a $15,000 investment.
    a. An interest payout GIC earning 4.5% compounded quarterly.
    b. A fixed rate compound interest GIC earning 4.2% compounded monthly.
    c. A variable rate quarterly compound interest GIC earning consecutively 3.9% for 1.5 years, 4.25% for 1.75 years, 4.15% for 0.75 years, and 4.7% for 1 year.
    d. An escalator rate GIC earning semi-annually compounded rates of 1.25%, 2%, 3.5%, 5.1%, and 7.75% in successive years.

10.2: Application - Long-Term Promissory Notes

Mechanics

Calculate the unknown variable (indicated with a ?) for the following noninterest-bearing promissory notes.

Principal	Issue Date	Due Date	Sale Date	Discount Rate	Sale Proceeds
1. $10,000	August 14, 2010	November 14, 2015	February 14, 2012	5.95% compounded quarterly	$?
2. $19,000	May 29, 2010	May 29, 2015	September 29, 2012	8.7% compounded monthly	$?
3. $?	September 30, 2009	September 30, 2014	March 30, 2012	6.8% compounded semi-annually	$21,574.34
4. $31,300	June 3, 2009	June 3, 2015	November 3, 2012	?% compounded monthly	$27,268.08

Calculate the unknown variable(s) (indicated with a ?) for the following interest-bearing promissory notes.

Issue Amount	Term of Note	Interest Rate on Note	Date of Sale (before maturity)	Discount Rate	Sale Proceeds
5. $51,000	6 years	7.75% compounded quarterly	2½ years	12% compounded semi-annually	$?
6. $18,200	4½ years	9.45% compounded monthly	1 year, 6 months	15% compounded quarterly	$?
7. $5,350	3¼ years	6.95% compounded monthly	1½ years	?% compounded semi-annually	$5,587.02
8. $2,900	4 years	8.8% compounded semi-annually	9 months	?% compounded monthly	$3,570.13
9. $?	5 years	7.5% compounded quarterly	2 years, 6 months	13.25% compounded semi-annually	$11,705.14
10. $26,945.75	6 years	10.3% compounded annually	? years, ? months	19.3% compounded monthly	$32,046.52

Applications

11. Determine the proceeds of the sale on a seven-year noninterest-bearing promissory note for $1,600, discounted 45 months before its due date at a discount rate of 8.2% compounded quarterly.
12. If a noninterest-bearing four-year promissory note is discounted 1½ years before maturity at a discount rate of 6.6% compounded semi-annually to have proceeds of $14,333.63, what is the principal amount of the note?
13. Determine the proceeds of the sale on a six-year interest-bearing promissory note for $5,750 at 6.9% compounded monthly, discounted two years and three months before its due date at a discount rate of 9.9% compounded quarterly.
14. A three-year interest-bearing promissory note for $8,900 at 3.8% compounded annually is sold one year and two months before its due date at a discount rate of 7.1% compounded monthly. What is the amount of the discount on the sale?
15. Two years and 10 months before its due date, an eight-year interest-bearing promissory note for $3,875 at 2.9% compounded semi-annually is discounted to have sale proceeds of $4,182.10. What monthly compounded discount rate was used?
16. On May 1, 2012, a six-year interest-bearing note for $8,800 at 4.99% compounded monthly dated August 1, 2009, is discounted at 8% compounded semi-annually. Determine the proceeds on the note.
17. A seven-year interest-bearing note for $19,950 at 8.1% compounded quarterly is issued on January 19, 2006. Four years and 11 months later, the note is discounted at 14.55% compounded monthly. Determine the proceeds on the note and how much interest the original owner of the note realized.

Challenge, Critical Thinking, & Other Applications

18. A $36,555 interest-bearing note at 5% compounded monthly is issued on October 15, 2011, for a term of 87 months. Fifty-seven months later, the note is sold to yield a discount amount of $11,733.41. What quarterly compounded discount rate is being used?
19. On December 12, 2012, an eight-year promissory note at 6.2% compounded semi-annually with three years and three months remaining until its due date is sold. The discount rate is 10.9% compounded monthly, resulting in a discount of $49,349.87. Calculate the original principal of the note.
20. A 10-year, $100,000 note at 7.5% compounded quarterly is to be sold. The prevailing discount rate for promissory notes of this type today, six years before maturity, is 9% compounded annually. Prevailing discount rates are forecast to rise by 2% every year. The seller of the note will sell the note today, one year from today, or two years

from today, depending on which sale date produces the highest nominal proceeds. Rank the various alternatives and recommend a sale date.

10.3: Application - Savings Bonds

For all questions in this section, refer to the two tables for savings and premium bonds for series numbers and interest rates on various bonds.

Mechanics

For each of the following R-bonds, calculate the total interest earned as of the date of redemption.

Series	Principal	Redemption Date
1. P64	$33,000	November 1, 2011
2. S111	$17,800	February 1, 2012
3. P72	$52,300	January 30, 2013
4. S104	$18,600	January 2, 2012

For each of the following C-bonds, calculate the maturity value and the total interest earned as of the date of redemption.

Series	Principal	Redemption Date
5. P47	$18,900	December 15, 2011
6. S87	$76,400	September 18, 2011
7. S103	$29,900	November 12, 2011
8. P55	$2,500	February 15, 2013
9. P50	$6,200	March 21, 2012
10. S110	$45,700	October 15, 2011

Applications

11. A $24,900 Series S91 R-bond CSB is redeemed on December 1, 2011. Calculate the total interest earned.
12. A $65,500 Series P57 C-bond CPB is redeemed on April 1, 2013. Calculate the maturity value and the total interest earned.
13. A $103,100 Series S117 C-bond CSB is redeemed on September 3, 2011. Calculate the maturity value and the total interest earned.
14. Calculate the total interest earned on a $57,600 Series P59 R-bond CPB if it is redeemed in 2013.
15. Calculate the total interest earned on a $38,200 Series S98 R-bond CSB if it is redeemed on August 15, 2011.
16. What amount would the owner of a $22,200 Series S93 C-bond CSB receive if the bond is redeemed on January 31, 2012? How much of that amount is interest?
17. How much more would the owner of a Series P43 C-bond CPB have than the owner of a Series P45 C-bond CPB if both invested $10,000 and both were redeemed in 2014?
18. How much more total interest would the owner of a Series P36 R-bond CPB have earned than the owner of a Series P38 R-bond CPB if both invested $100,000 and both were redeemed in 2012?

Challenge, Critical Thinking, & Other Applications

19. Investing is always a bit of a guessing game as to when to invest. Looking at 2005 Series S92 to S97 C-bond CSBs, rank the 2005 series from best to worst if an equal amount of money had been invested into each and all were matured on their anniversary dates in 2011. Show calculations to support your rankings.
20. Most people do not have one single investment at any one time. Rather, they have multiple investments started at different points as money becomes available to them to invest. Let's say an investor is able to invest $3,000 each November 1 and April 1 starting with November 1, 2004, and he purchases C-bond CSBs with the money each time. Calculate the total maturity value of his investments on November 1, 2011 (not including the $3,000 he would invest on that day).

10.4: Application - Strip Bonds

Mechanics

For each of the following, calculate the purchase price of the strip bond and the gain realized over the term.

Face Value	Years Remaining Until Maturity	Yield
1. $157,500	15	4.1965%
2. $250,000	9.5	3.6926%
3. $12,050	30	4.0432%
4. $81,735	17.5	4.2561%

For each of the following, calculate the semi-annual yield on the strip bond and the gain realized over the term.

Purchase Price	Years Remaining Until Maturity	Face Value
5. $9,087.13	26	$100,000
6. $17,537.39	12	$50,000
7. $24,947.97	21.5	$75,000

For each of the following, determine the actual yield realized by the investor and the gain realized over the time the strip bond was held.

Purchase	Sale	Face Value
8. 22 years before maturity; Market yield 6.3815%	14.5 years before maturity; Market yield 7.8643%	$80,000
9. 14 years before maturity; Market yield 10.1831%	5 years before maturity; Market yield 9.1777%	$5,000
10. 28 years before maturity; Market yield 6.8205%	11.5 years before maturity; Market yield 7.2173%	$500,000

Applications

11. A $15,000 face value Government of Manitoba strip bond has 19.5 years left until maturity. If the current market rate is posted at 6.7322%, what is the purchase price for the bond?
12. A $300,000 face value Government of Canada strip bond will mature on June 1, 2041. If the yield on such strip bonds is 4.6849%, what was the purchase price on December 1, 2012?
13. An $87,000 face value strip bond from the Government of Alberta matures on February 14, 2033. If an investor paid $23,644.78 on February 14, 2011, what was the semi-annually compounded yield on the strip bond?
14. A $150,000 face value bond was issued on August 8, 2008, with a 30-year maturity. If an investor paid $40,865.24 on February 8, 2013, what was the market rate of return on the strip bond at the time of the purchase?
15. An investor purchased a $7,500 face value strip bond for $2,686.01 on May 29, 2006. The strip bond had been issued on May 29, 2002, with a 25 -year maturity. The investor sold the strip bond on November 29, 2012, for $3,925.28.
    a. What was the market yield when the investor purchased the strip bond?
    b. What was the market yield when the investor sold the strip bond?
    c. What actual yield did the investor realize on the strip bond?
16. A $70,000 face value strip bond was issued on October 6, 1998, with 30 years remaining until maturity. An investor purchased the bond on April 6, 2005, when market yields were 4.9529%. The investor sold the bond on October 6,

2012, when market yields were 5.2185%.

a. What was the purchase price of the strip bond?
b. What was the sale price of the strip bond?
c. What was the actual yield the investor realized on the strip bond?
d. What was the total gain realized by the investor?

Challenge, Critical Thinking, & Other Applications

17. On August 1, 2005, a $100,000 Government of Canada strip bond was issued with a 25-year maturity. An investor paid $30,597.14 for the strip bond. On February 1, 2012, the investor sold the bond for $44,604.78. What was the difference in yields in the market between the issue date and the selling date?
18. Sebastien is the finance manager for a large corporation that needs to raise $100 million for a project today. Rounded to the nearest million, what face value of strip bonds would need to be issued if they are to mature in five years and the current market yield on five-year strip bonds is posted at 5.7391%?
19. Antoine just inherited $50,000 from his grandfather's estate. If he is considering investing in 20-year maturity strip bonds posted at 5.8663%, how many $1,000 denomination strip bonds could he purchase with his inheritance?
20. Consider the following three situations. For each, use a $100,000 strip bond issued with eight years until maturity for your calculations.
    - Assume the market yield is a constant 4.5%. Calculate the purchase price each year until maturity. Plot the eight purchase prices onto a line chart.
    - Assume the market yield initially is set at 3% when issued and rises by 0.375% each year until maturity. Plot the eight purchase prices onto the same line chart.
    - Assume the market yield initially is set at 6% when issued and decreases by 0.375% each year until maturity. Plot the eight purchase prices onto the same line chart. Looking at the line charts for all three situations, what do you see? Explain the results.

10.5: Application - Inflation, Purchasing Power, and Rates Of Change

Mechanics

Solve the following inflation questions for the unknown variable (indicated with a ?). Note that the inflation values in each question represent an average for an actual time period in Canadian history.

Beginning Value	Ending Value	Annual Inflation and Term
1. $30,000.00	?	2.68% for 20 years
2. $50,288.00	?	2.58% for 3 years; then 0.35% for 2 years
3. ?	$40,000.00	5.77% for 25 years
4. $15,000.00	$48,905.33	?% for 17 years
5. $21,000.00	$78,969.53	7.22% for ? years

For questions 6 and 7, use the information provided to calculate the purchasing power of a dollar from the base year to the end of the specified term. Note that the inflation values in each question are the actual averages for the specified time period in Canadian history.

Base Year	Annual Inflation and Term
6. 1978	9.68% for 5 years
7. 1989	5.33% for 2 years; then 0.91% for 3 years

Solve the following rates of change questions for the unknown variable (indicated with a ?). Round all answers to two decimals.

Beginning Value	Ending Value	Percent Change and # Times in a Row
8. 98,000	?	3.5% for 10 periods
9. 235,000	?	2.3% for 4 periods, then −1.6% for 2 periods
10. ?	47,500	−5% for 5 periods

Applications

11. In 1982, the average price of a new car sold in Canada was $10,668. By 2009, the average price of a new car had increased to $25,683. What average annual rate of change in car prices has been experienced during this time frame?
12. From September 8, 2007, to November 7, 2007, the Canadian dollar experienced one of its fastest periods of appreciation against the US dollar. It started at $0.9482 and rose 0.2514% per day on average. What was the final value of the Canadian dollar rounded to four decimals on November 7, 2007?
13. The average price of a detached home in Calgary in 2005 was $246,308. Prices rose by 10.4834% on average per year for four years, and then changed by −2.7248%. What is the new price of a home in Calgary in 2010 rounded to the nearest thousand dollar?
14. Example 10.5.1 used the average inflation rate of Canada for the past 97 years as an indicator of future inflation. Some critics argue that using “outdated” inflation rates is not a good representation of the future. Recall that you are 20 years old in 2012 and the comfortable retirement income is $40,000. The historical inflation rate over the past 40 years for Canada has averaged 4.5% from 1971 to 2011.
    a. Recalculate your required retirement income at age 65.
    b. What will be the purchasing power of a 2012 dollar in 45 years?
15. In the late 1970s and early 1980s, Canada went through a period of high inflation. How did the purchasing power of a 1975 dollar decrease if the average inflation rate from May 1975 to May 1985 was 8.2% per year?
16. The world population in 2011 was approximately 6.9 billion people. At the current annual population growth rate of 1.1%, how long will it take the world's population to double (round to the nearest year)?

Challenge, Critical Thinking, & Other Applications

17. Cadbury Canada is thinking about launching a new chocolate bar. In the first year, sales are forecast to hit 750,000 units. The marketing department has then forecast sales increases of 50% in each of the next five years, followed by annual increases of 15% in the years following. Cadbury will launch a new chocolate bar only if it forecasts annual sales of 22 million units by the end of the first 15 years. Should Cadbury Canada launch the new chocolate bar?
18. How much money would have been required 20 years ago to have the same purchasing power of $10,000 today if the average rate of inflation per year for the period has been 2%?
19. Measures of life expectancy account for various factors, including smoking and body mass index. Since 1975, when 33.3% of the population smoked, smoking has been declining at a rate of 1.71% per year on average. In the same time frame, the average body mass index (BMI) started at 25.2 and increased by 0.42% per year on average. While reduced smoking prolongs life, the increasing BMI shortens life. The average life expectancy of a Canadian is 80.7 years old. It is forecasted, though, to decrease by 0.0756% per year on average based on the trends in smoking and BMI. Assume these trends continue and calculating the following (rounded to the nearest year):
    a. In what year will half of the population be considered obese (a BMI average of 30)?
    b. In what year will less than 10% of the population be smokers?
    c. In what year will life expectancy fall below 75 years of age?
20. Future inflation rates can only be speculated about. Using the recommended $40,000 of retirement income in 2012 when you are 20 years old, create a range of possible retirement incomes needed when you retire at age 65. The average Canadian lives to 80.7 years old, so your retirement income needs to be regularly updated after your retirement to reflect the increased costs of living.

   a. Start with a low inflation rate of 2%. Forecast the needed retirement income at age 65, 70, and 75 years.
   b. Repeat the process using inflation rates of 3%, 4%, and 5%.
   c. Graph the results on a line chart. For all three retirement ages, specify the range of income required. Calculate the average income needed at each age of retirement.
   d. Why do you think it is important to do these types of calculations?

Review Exercises

For all questions that involve Canada Savings Bonds, refer to the two tables on savings and premium bonds as needed.

Mechanics

1. Guido placed $28,300 into a five-year regular interest GIC with interest of 6.3% compounded semi-annually. Determine the total interest Guido will earn over the term.
2. An eight-year, $35,000 noninterest-bearing promissory note is discounted 6% compounded quarterly and sold to a finance company three years and nine months after issue. What are the proceeds of the sale?
3. A $9,800 Series S100 R-bond CSB was redeemed on March 1, 2012. Calculate the total interest earned on the investment.
4. On April 27, 1990, Graham purchased a $100,000 face value 25-year Government of Quebec strip bond. The market yield on such bonds was 13.3177%. What was the purchase price for the strip bond?
5. In retirement, Bill determined that he could safely invest $50,000 into a nonredeemable five-year GIC with a posted rate of 5.35% compounded semi-annually. Calculate the maturity value and the amount of interest earned on the investment.
6. In 2012, a $10,000 Series P52 C-bond CPB was redeemed. Calculate the total interest earned on the bond.
7. In 1979, Shania earned $10 per hour at her place of employment. From 1979 to 1982, the average annual inflation in Canada was 11.36%. Determine what Shania's hourly wage needed to be in 1982 to keep up with inflation.
8. In September 2004, Google employed 2,688 workers. Over the next two years, the number of employees grew at an average of 86.7843% per year. How many employees did Google have in September 2006?

Applications

9. Entegra Credit Union offers a five-year escalator GIC with annual rates of 1.65%, 2.4%, 2.65%, 2.95%, and 3.3%. Determine the maturity value and total interest earned on an investment of $6,000 along with the equivalent five-year fixed rate.
10. A 21-month $6,779.99 promissory note bearing interest of 7.5% compounded monthly was sold on its date of issue to a finance company at a discount rate of 9.9% compounded monthly. Determine the proceeds of the sale.
11. A $20,500 Series S104 C-bond CSB was redeemed on its maturity date in 2011. Calculate the maturity value and the total interest earned on the bond.
12. On January 19, 2012, a Series P36 C-bond CPB was redeemed for $19,549.90. How much interest is included in the maturity value?
13. On September 30, 2011, a $34,000 Series S93 C-bond CSB was redeemed.
    a. Calculate the maturity value and total interest earned on the bond.
    b. How much more interest could have been earned if the bond was not redeemed until October 1, 2011?
14. On February 7, 1990, a $100,000 face value 25-year Government of Saskatchewan strip bond was purchased for $9,365.85. On February 7, 2005, the investor sold the strip bond for $65,900.
    a. What was the posted yield on strip bonds on the date of issue?
    b. What was the posted yield on strip bonds when it was sold?
    c. What actual semi-annual rate of return did the original investor realize on the strip bond investment?
15. General Motors has watched its market share decline over the past several decades. In 1980, GM held a 46% share of the market. From 1980 to 2003, it declined on average by 2.09% annually. From 2003 to 2007, it declined on average by 9.36% annually. Calculate GM's 2007 market share rounded to one decimal in percentage format.

Challenge, Critical Thinking, & Other Applications

16. Canada suffered a prolonged period of deflation in the early 1930s. From 1930 to 1933, the rate of deflation averaged 8.27% per year. Calculate the purchasing power of a 1930 dollar in 1933.
17. On October 10, 1995, a $100,000 Government of Canada 30-year strip bond was purchased at a posted yield of 8.1258%. The investor sold the strip bond on October 10, 2005, and realized an actual yield of 15.9017% on the investment. What was the prevailing yield on strip bonds when the strip bond was sold on October 10, 2005?
18. From 2004 to 2008, the average family after-tax annual earnings increased from $68,200 by 2.2677% per year. The inflation rate during that time period was 1.71%, 2.43%, 2.19%, and 3.13% in successive years, respectively. Determine the amount that Canadian family after-tax annual earnings have increased or decreased in 2008. Show calculations to support your answers.
19. The research and development department forecasts that it will require $100 million in funding for a project scheduled for implementation four years from today. If the company wants to place $80 million into a $500 million 25-year strip bond, what is the minimum by which the yield in the market needs to change for the department to have sufficient funds when the project is started?
20. On November 1, 2004, Andy invested $50,000 into a Series P40 R-bond CPB. On the 2005, 2007, and 2009 anniversary dates, he took his regular interest payment and acquired a CSB Series C-bond on the same date. On the 2006 and 2008 anniversary dates, he took his regular interest payment and acquired a CPB Series C-bond on the same date. On November 1, 2010, calculate the maturity value of all of his investments. Determine the total interest earned from November 1, 2004, to November 1, 2010.

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CHAPTER OVERVIEW

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11.3: Present Value Of Annuities
11.4: Annuity Payment Amounts
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11.0: Introduction

Payments show up everywhere. You make payments on retail purchases; bank loans require payments; you even make payments to yourself into your RRSP! Payments fall into a few categories subject to important characteristics. Let's explore these categories with a few everyday examples.

- Last week you completed the purchase of your $150,000 starter home and finished all of the paperwork on your mortgage with the bank. You snagged a great mortgage rate of 5% compounded semi-annually, locking it in for a five-year term with monthly payments of $872.41 (for the next 25 years) starting one month after you move in.
- You excitedly cruise to the Honda dealership to pick up your new $22,475 Honda Civic Sport, on which you have a four year lease. Pulling out your iPhone (for which you pay $60 monthly), you ensure that you have enough funds in your chequing account to make the $2,000 down payment today along with the first of your $242.16 monthly lease payments at 2.9% compounded monthly.
- Driving away in your Civic, you stop in at Sleep Country Canada and purchase a queen size mattress for your new bedroom. You recall a TV ad promoting an $895 Sealy mattress plus 12% taxes with no money down and 12 easy, interest-free monthly payments of $83.53. It will be nice to have a comfortable bed to sleep in! Did you notice some of the characteristics of these various types of payments?
- The mattress was low priced and was paid off in a short, one-year time frame, while the house was high priced and required a long, 25-year time frame to extinguish the debt.
- The lease on your new car required a payment up front before you could drive it home, while your mortgage and mattress had payments that would start later.
- Your monthly iPhone payment recurs endlessly until you terminate the contract.

These examples also illustrate how businesses participate in consumer payments. Companies must understand financing for their consumers for a variety of reasons:

- Banks create mortgages for their clients so that consumers can make large purchases through affordable payments. These payment plans are structured to allow the bank to earn profits through the interest charged.
- Car dealerships use similar payment plans that earn interest on the payments in addition to the profits built into the products themselves. It is like double-dipping!
- Sleep Country Canada sells more mattresses by allowing consumers to spread their payments over the course of a year instead of requiring them to hand over one big lump sum, which many people may not be able to do.

And do not forget that regular payments are needed in business-to-business deals too, including mortgages, leases, insurance, utilities, and even product acquisitions.

This chapter explores the concept of making regular payments toward savings goals or debt extinguishment. First it introduces key payment concepts and reviews different types of payment plans. You will then calculate the future and present value of a stream of payments. You extend the application of the concepts further to solve for other variables such as the payment amount, the term, and the nominal interest rate. Chapter 12 discusses commonplace annuity applications such as planning your RRSP and buying a car.

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11.1: Fundamentals of Annuities

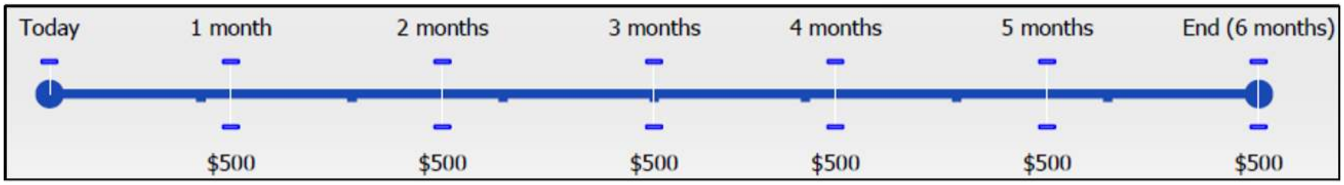
It is impossible to calculate payments accurately until you recognize a few of the key characteristics illustrated in the chapter introduction:

- All of the examples required a payment in the same amount on a regular basis, such as the $872.41 every month for your mortgage.
- The timing of the payments varied. The car lease required the first payment up front (at the beginning of the month), while the mortgage and mattress purchase had the first payment due a month after the purchase (at the end of the month).
- The frequency of the payments and the frequency of the interest rate varied. The mortgage had monthly payments with interest being charged semi-annually, while the car lease had monthly payments and monthly interest.

Unlike single payments (covered in Chapter 9) for which there is only one formula, you solve series of payments by choosing the appropriate formula from four possibilities defined by the financial characteristics of the payments. This section defines the characteristics of four different types of payment series and then contrasts them to the Chapter 9 and Chapter 10 single payment calculations. This section also develops a new, simplified structure for timelines to help you visualize a series of payments.

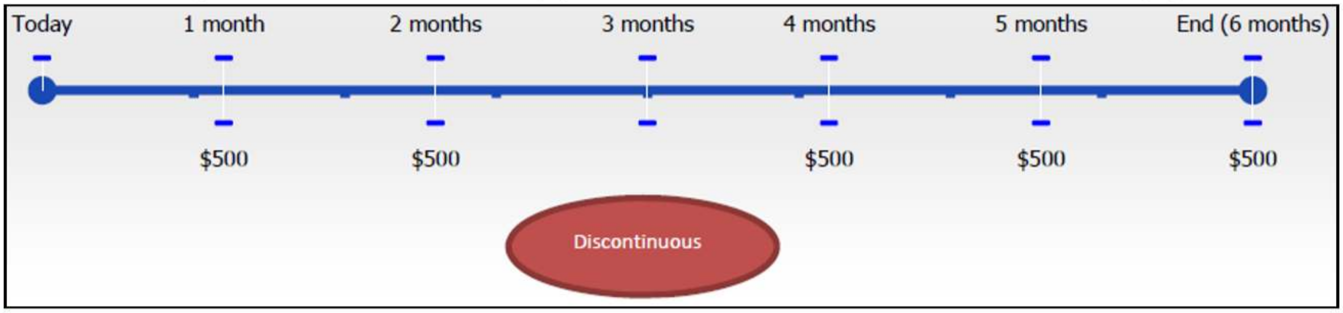
What Are Annuities?

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The figure below illustrates a six-month annuity with monthly payments. Notice that the payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.



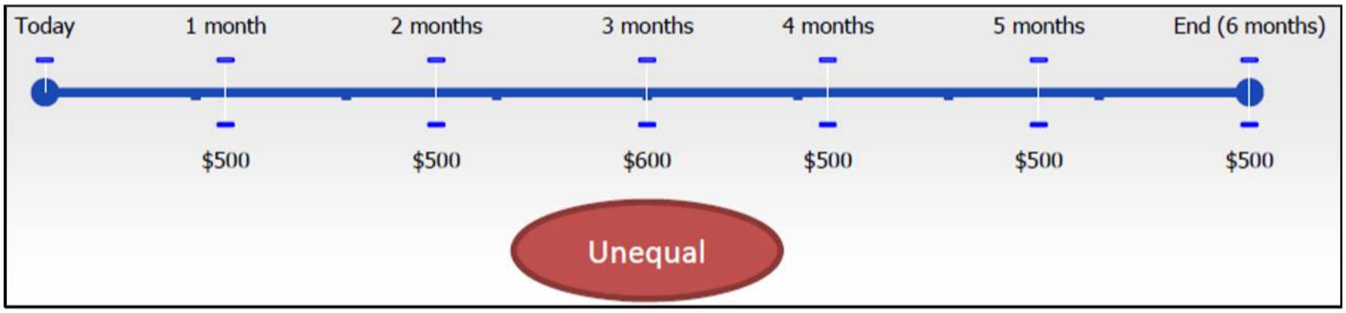
The examples below illustrate four timelines that look similar to the one above, but with one of the characteristics of an annuity violated. This means that none of the following in their entirety are considered an annuity:

1. **Continuous**. Annuity payments are without interruption or breaks from the beginning through to the end of the annuity's term. In the figure above there are no breaks in the annuity since every month has an annuity payment. This next figure is not an annuity because the absence of a payment in the third month makes the series of payments discontinuous.

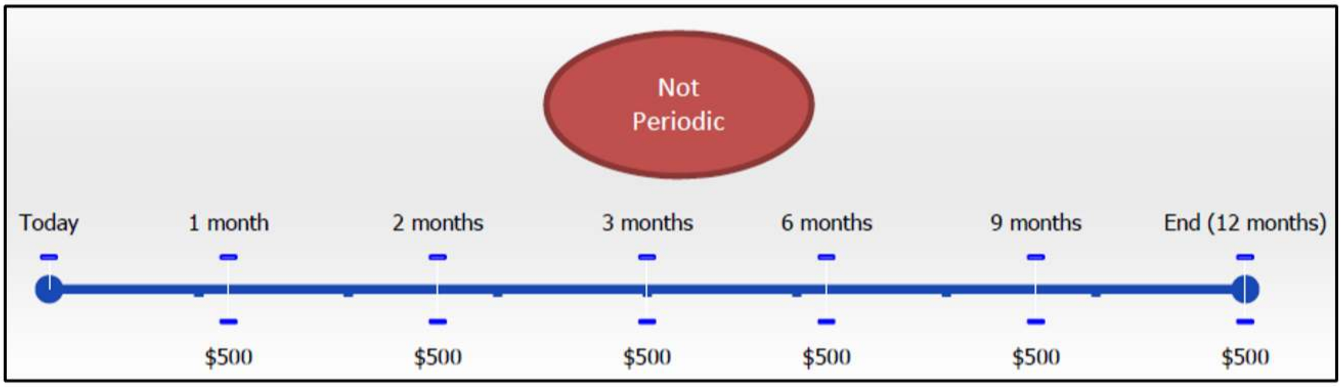


2. **Equal**. The annuity payments must be in the same amount every time from the beginning through to the end of the annuity's term. In the original figure, every monthly annuity is $500. Notice that in this next figure the payment

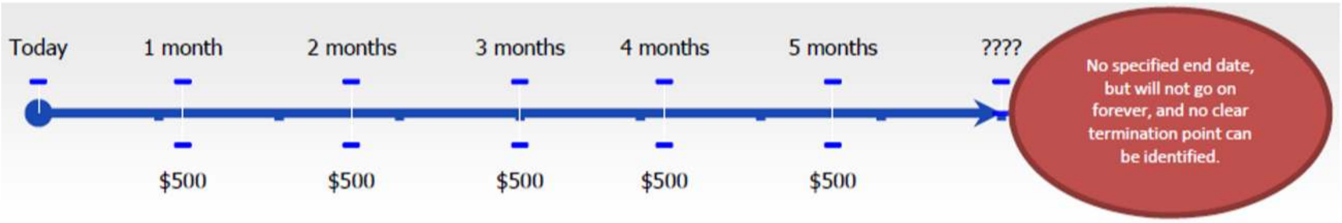
amount varies and includes values of both $500 and $600.


Today
1 month
2 months
3 months
4 months
5 months
End (6 months)
$500
$500
$600
$500
$500
$500
Unequal

3. **Periodic**. The amount of time between each continuous and equal annuity payment is known as the payment interval. Hence, a monthly payment interval means payments have one month between them, whereas a semi-annual payment interval means payments have six months between them. Annuity payments must always have the same payment interval from the beginning through to the end of the annuity's term. In the original figure there is exactly one month between each equal and repeated payment in the annuity. Notice that in this next figure the payments, although equal and continuous, do not occur with the same amount of time between each one. In fact, the first three payments are made monthly while the last three payments are made quarterly. (You may notice that if the first three payments are treated separately from the last three payments, then each separate grouping represents an annuity and therefore there are two separate annuities. However, as a whole this is not an annuity.)


Not Periodic
Today
1 month
2 months
3 months
6 months
9 months
End (12 months)
$500
$500
$500
$500
$500
$500

4. **Specified Period of Time**. The annuity payments must occur within an identifiable time frame that has a specified beginning and a specified end. The annuity's time frame can be (1) known with a defined starting date and defined ending date, such as the annuity illustrated in the original figure, which endures for six periods; (2) known but nonterminating, such as beginning today and continuing forever into the future (hence an infinite period of time); or (3) unknown but having a clear termination point, for example, monthly pension payments that begin when you retire and end when you die—a date that is obviously not known beforehand. In the next figure, the annuity has no defined ending date, does not continue into the future, and no clear termination point is identifiable.


Today
1 month
2 months
3 months
4 months
5 months
????
$500
$500
$500
$500
$500
No specified end date, but will not go on forever, and no clear termination point can be identified.

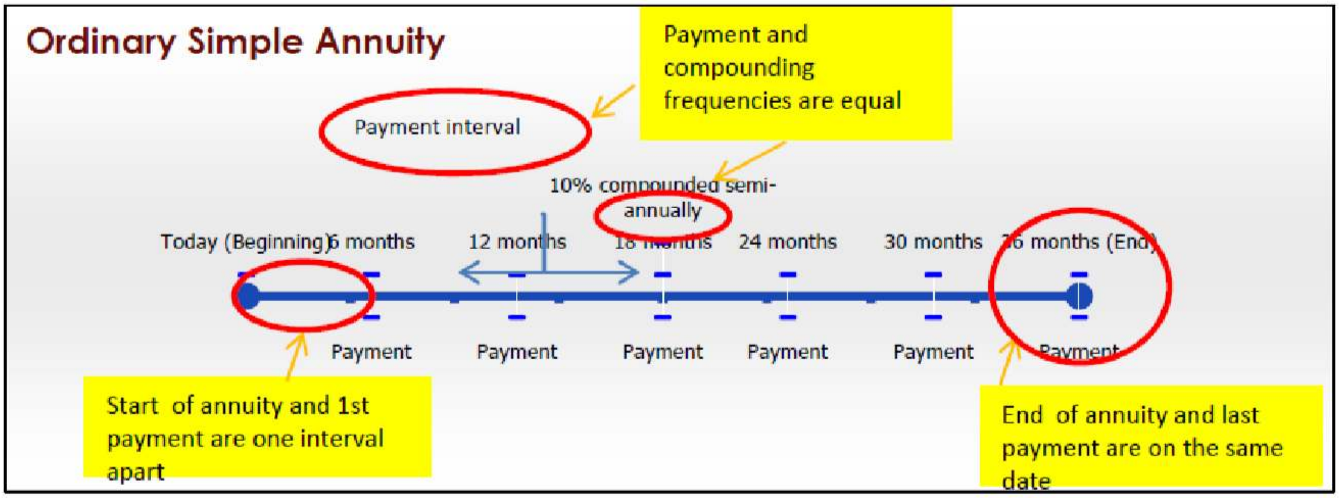
In summary, the first figure is an annuity that adheres to all four characteristics and can be addressed via an annuity formula. The next four figures are not annuities and need other financial techniques or formulas to perform any necessary calculations.

Types of Annuities

There are four types of annuities, which are based on the combination of two key characteristics: timing of payments and frequency. Let's explore these characteristics first, after which we will discuss the different annuity types.

1. **Timing of Payments**. An example best illustrates this characteristic. Assume that you take out a loan today with monthly payments. If you were to make your first annuity payment on the day you take out the loan, the amount of principal owing would be immediately reduced and you would accumulate a smaller amount of interest during your first month. This is called making your annuity payment at the beginning of a payment interval, and this payment is known as a **due**. However, if a month passes before you make your first monthly payment on the loan, your original principal accumulates more interest than if the principal had already been reduced. This is called making your payment at the end of the payment interval, and this payment is known as an **ordinary** because it is the most common form of annuity payment. Depending on when you make your payment, different principal and interest amounts occur.
2. **Frequency**. The frequency of an annuity refers to a comparison between the payment frequency and the compounding frequency. A **payment frequency** is the number of annuity payments that would occur in a complete year. Recall from Chapter 9 that the compounding frequency is the number of compounds per complete year. If the payment frequency is the same as the compounding frequency, this is called a **simple annuity**. When interest is charged to the account monthly and payments are also made monthly, you determine principal and interest using simplified formulas. However, if the payment frequency and the compounding frequency are different, this is called a **general annuity**. If, for example, you make payments monthly while interest is compounded semi-annually, you have to use more complex formulas to determine principal and interest since the paying of principal and charging of interest do not occur simultaneously.

Putting these two characteristics together in their four combinations creates the four types of annuities. Each timeline in these figures assumes a transaction involving six semi-annual payments over a three-year time period.

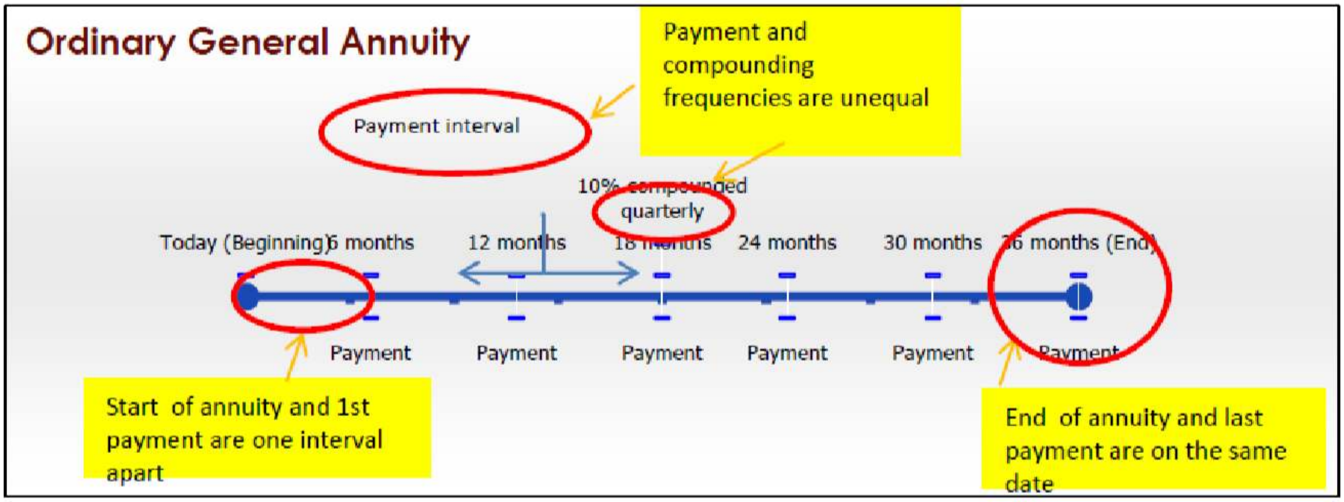

Ordinary Simple Annuity
Payment and compounding frequencies are equal
Payment interval
10% compounded semi-annually
Today (Beginning)
6 months
12 months
18 months
24 months
30 months
36 months (End)
Payment
Payment
Payment
Payment
Payment
Payment
Start of annuity and 1st payment are one interval apart
End of annuity and last payment are on the same date

Ordinary Simple Annuity

An **ordinary simple annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

For example, most car loans are ordinary simple annuities where payments are made monthly and interest rates are compounded monthly. As well, car loans do not require the first monthly payment until the end of the first month.

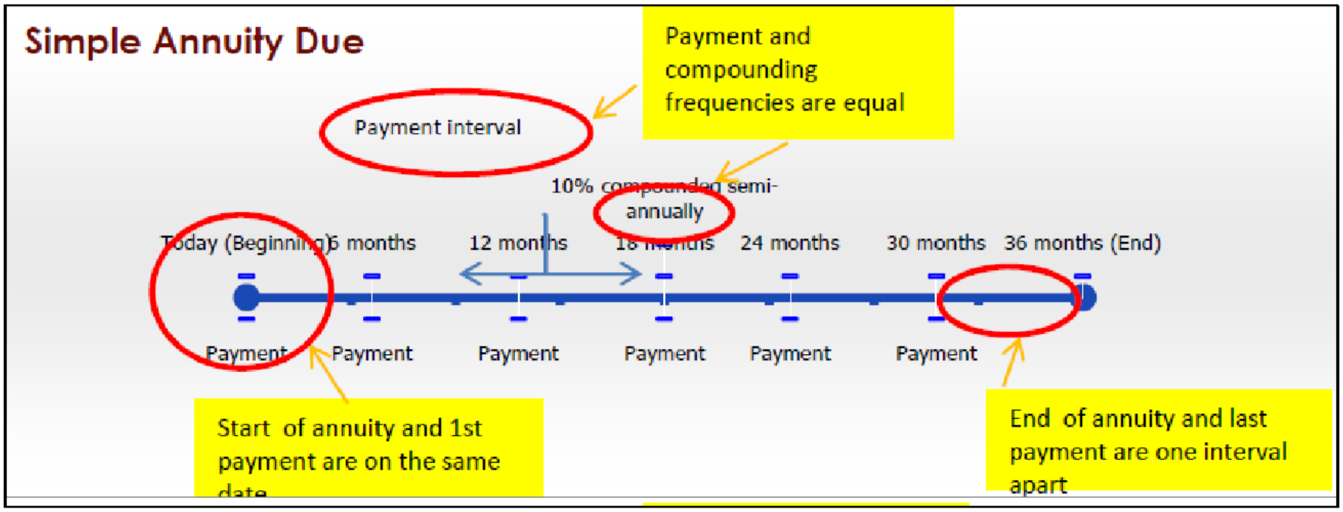

Ordinary General Annuity
Payment and compounding frequencies are unequal
Payment interval
10% compounded quarterly
Today (Beginning)
6 months
12 months
18 months
24 months
30 months
36 months (End)
Payment
Payment
Payment
Payment
Payment
Payment
Start of annuity and 1st payment are one interval apart
End of annuity and last payment are on the same date

Ordinary General Annuity

An ordinary general annuity has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

For example, most mortgages are ordinary general annuities, where payments are made monthly and interest rates are compounded semi-annually. As with car loans, your first monthly payment is not required until one month elapses.

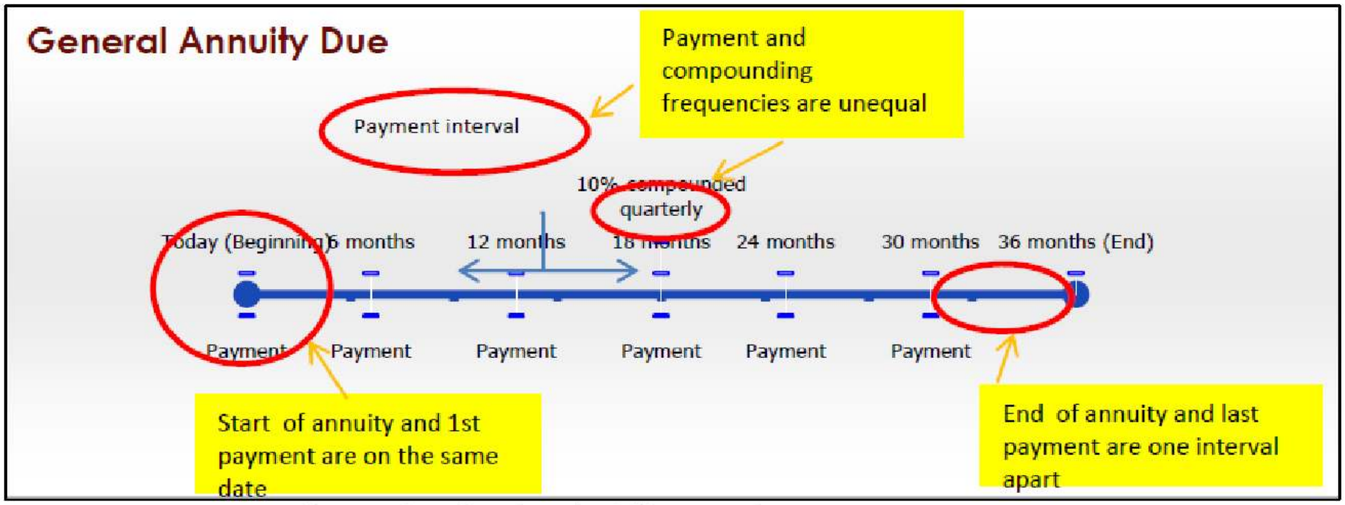

Simple Annuity Due
Payment and compounding frequencies are equal
Payment interval
10% compounded semi-annually
Today (Beginning)
6 months
12 months
18 months
24 months
30 months
36 months (End)
Payment
Payment
Payment
Payment
Payment
Payment
Start of annuity and 1st payment are on the same date
End of annuity and last payment are one interval apart

Simple Annuity Due

A **simple annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, most car leases are simple annuities due, where payments are made monthly and interest rates are compounded monthly. However, the day you sign the lease is when you must make your first monthly payment.


General Annuity Due
Payment and compounding frequencies are unequal
Payment interval
10% compounded quarterly
Today (Beginning)
6 months
12 months
18 months
24 months
30 months
36 months (End)
Payment
Payment
Payment
Payment
Payment
Payment
Start of annuity and 1st payment are on the same date
End of annuity and last payment are one interval apart

General Annuity Due.

A **general annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, many investments, like your RRSP, are general annuities due where payments (contributions) are typically made monthly but the interest compounds in another manner, such as annually. As well, when most people start an RRSP they pay into it on the day they set it up, meaning that their RRSP commences with the first deposited payment.

The table below summarizes the four types of annuities and their characteristics for easy reference.

Annuity Type	Timing of Payments in a Payment Interval	Payment Frequency and Compounding Frequency	Start of Annuity and First Payment Same Date?	End of Annuity and Last Payment Same Date?
Ordinary Simple Annuity	End	Equal	No, first payment one interval later	Yes
Ordinary General Annuity	End	Unequal	No, first payment one interval later	Yes
Simple Annuity Due	Beginning	Equal	Yes	No, last payment one interval earlier
General Annuity Due	Beginning	Unequal	Yes	No, last payment one interval earlier

Paths To Success

One of the most challenging aspects of annuities is recognizing whether the annuity you are working with is ordinary or due. This distinction plays a critical role in formula selection later in this chapter. To help you recognize the difference, the table below summarizes some key words along with common applications in which the annuity may appear.

Type	Key Words or Phrases	Common Applications

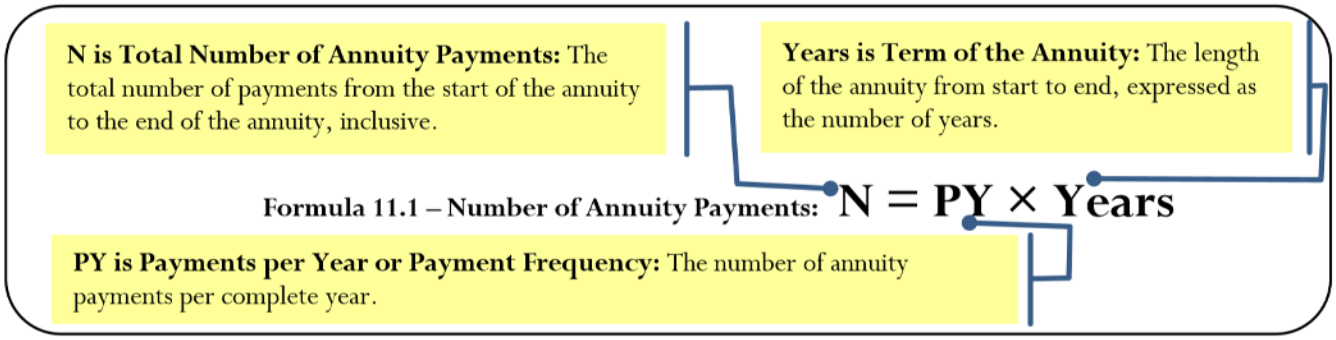
Type	Key Words or Phrases	Common Applications
Ordinary	-... payments are at the end.... -... payments do not start today... -... payments are later... -... first payment next interval...	- bank loans of any type - mortgages - bonds - Canada Pension Plan (CPP)
Due	-...payments are at the beginning... -...payments start today... -... payments are in advance... -...first payment today... -...payments start now...	- any kind of lease - any kind of rental - RRSPs (usually) - membership dues - insurance

Annuities versus Single Payments

To go from single payments in Chapter 9 to annuities in this chapter, you need to make several adaptations:

1. **Annuity Payment Amount (PMT)**. This variable was not used in any of the formulas introduced in Chapter 9 or Chapter 10, though it was briefly introduced when Section 9.2 when demonstrating the calculator functions. Notice that in the previous two chapters this variable was set to zero for every question, since every payment was not part of an annuity. Annuity calculations require you to tie a value to this variable in the formulas and when you use technology such as the BAII+ calculator.
2. **Payment Frequency or Payments per Year (PY)**. For single payments, this variable did not show up in any of the formulas from the previous two chapters. It too was introduced in Section 9.2 as a calculator button with the requirement that it automatically be set as a null variable matching the compounding frequency (CY). When you work with annuities, an actual value for PY is determined by the payment frequency. For simple annuities PY remains the same as CY, whereas the variables are different for general annuities.
3. **Cash Flow Sign Convention on the Calculator**. It now becomes critical to ensure the proper application of the cash flow sign convention on the calculator—failure to do so will result in an incorrect answer. For example, if you borrow money and then make annuity payments on it, you enter the present value (PV) as a positive (you received the money) while you enter the annuity payments as negatives (you paid the money to the bank). This results in future balances getting smaller and you owing less money. If you inadvertently enter the annuity payment as a positive number, this would mean you are borrowing more money from the bank so your future balance would increase and you would owe more money. These two answers are very different!
4. **Definition and Computation of N**. When you worked with single payments, N was defined as the total number of compounds throughout the term of the financial transaction. When you work with annuities, N is defined as the total number of payments throughout the term of the annuity. You calculate it using Formula 11.1.

The Formula



N is Total Number of Annuity Payments: The total number of payments from the start of the annuity to the end of the annuity, inclusive.

Years is Term of the Annuity: The length of the annuity from start to end, expressed as the number of years.

Formula 11.1 – Number of Annuity Payments: $N = PY \times Years$

PY is Payments per Year or Payment Frequency: The number of annuity payments per complete year.

Formula 11.1

How It Works

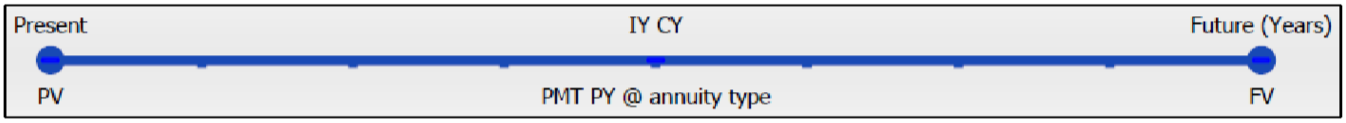
On a two-year loan with monthly payments and semi-annual compounding, the payment frequency is monthly, or 12 times per year. With a term of two years, that makes N = 2 × 12 = 24 payments. Note that the calculation of N for an annuity does not involve the compounding frequency.

Adapting Timelines to Incorporate Annuities

Annuity questions can involve many payments. For example, in a typical 25-year mortgage with monthly payments, that would be 25 × 12 = 300 payments in total. How would you draw a timeline for these? Clearly, it would be impractical to draw 300 payments.

A good annuity timeline should illustrate the present value (PV), future value (FV), number of annuity payments (N), nominal interest rate (IY), compounding frequency (CY), annuity payment (PMT), and the payment frequency (PY). One of these variables will be the unknown.

As well, a good timeline requires a clear distinction between ordinary annuities and annuities due. END is used to represent ordinary annuities, since payments occur at the end of the payment interval. Similarly, BGN is used to represent annuities due, since payments occur at the beginning of the payment interval. The figure below illustrates the adapted annuity timeline format.

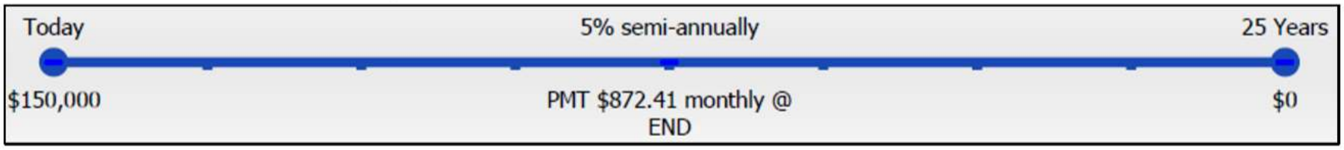


Sometimes a variable will change partway through the period of an annuity, in which case the timeline must be broken up into two or more segments. When you use this structure, in any time segment the annuity payment PMT is interpreted to have the same amount at the same payment interval continuously throughout the entire segment. The number of annuity payments N does not directly appear on the timeline since it is the result of a formula. However, its two components (Years and PY) are drawn on the timeline.

How It Works

A mortgage is used to illustrate this new format. For now, focus strictly on the variables and how to illustrate them in a timeline. Do not focus on any mortgage calculations yet.

- As per the chapter introduction, you purchased a $150,000 home ($PV$) with a 25-year mortgage (Years). The mortgage rate is 5% (IY) compounded semi-annually (CY = 2), and the monthly payments (PY = 12) are $872.41 ($PMT$).
- After 25 years you will own your house and therefore have no balance remaining. This sets the future value to $0 ($FV$).
- Mortgages are ordinary in nature, meaning that payments occur at the end of the payment intervals (END). The figure below places all of these mortgage elements into the new timeline format. Once you have drawn the timeline, you can determine the following:



- The number of payments (N), which you calculate using Formula 11.1. Since both PY and Years are known, you have N = 25 × 12 = 300.
- Whether the annuity is simple or general, which depends on the PY below the timeline and CY above the timeline. If they match, the annuity is simple; if they differ, as in this example, it is general.

Things To Watch Out For

The word "payment" often confuses people because it has two interpretations. It could mean either "single payment," such as in Chapter 9, or "annuity payment," which is meant in this chapter. To correctly interpret this word, recall the characteristics of an annuity payment and determine if the situation at hand matches the criteria. Let's review two examples illustrating this point:

1. Suppose John owes Mary three payments of $200 due two months, five months, and eleven months from now. In this case, the word "payment" means "single payment" because the payments are equal (all $200) and the time periods are known, but they are neither periodic (payment intervals are unequal) nor continuous (since no periods exist, there is no continuity). Working with these payments requires you to apply the formulas and techniques from Chapter 9.
2. Suppose John owes Julia three payments of $200 due three months, six months, and nine months from now. In this case, the word "payment" means "annuity payment" because the payments are equal (all $200), periodic (every quarter), and continuous (occurring every quarter without interruption), and they have a specified time frame (over nine months). Working with these payments allows you to apply the formulas and techniques from this chapter.

Ultimately, when in doubt you can solve any question involving time value of money using the formulas and techniques from Chapter 9. The annuity formulas introduced in the next section just allow you to arrive at the same answer with a lot less calculation.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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11.2: Future Value Of Annuities

Though your retirement is probably still a long way off, the earlier you start investing the more you can take advantage of the power of compounding interest to generate your savings.

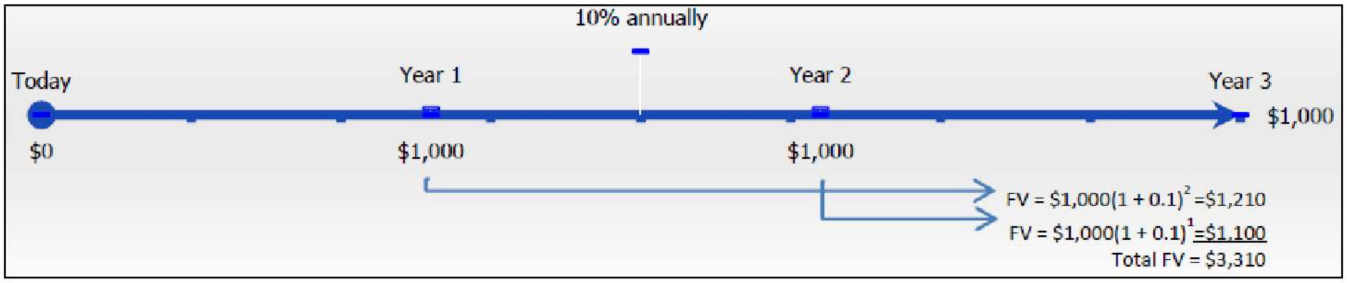
Did it startle you to learn in Section 10.5 that your annual gross retirement income might need to be \$160,000 when you turn 65 years old? How long will you live? The average Canadian life expectancy is around 80 years old; if you live to be that age, you will need 15 years of retirement income. Using a conservative interest rate of 5% compounded annually along with 3% annual inflation, this works out to saving up approximately \$2 million for when you retire. A daunting goal, isn't it? You might well wonder, "If I start saving \$300 per month today, will I have enough?"

Clearly it is important to know how much your annuities are worth in the future. This matters not only for investments but also for debts, since most businesses and individuals pay off their debts through annuity structures. After you have made several annuity payments, can you tell at any time how much you or your company still owes on an outstanding debt?

In the previous section you learned to recognize the fundamental characteristics of annuities, so now you can start to solve any annuity for any unknown variable. There are four annuity formulas. This section covers the first two, which calculate future values for both ordinary annuities and annuities due. These formulas accommodate both simple and general annuities.

Ordinary Annuities

The **future value of any annuity** equals the sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval. For example, assume you will make \$1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. This is an ordinary simple annuity since payments are at the end of the intervals, and the compounding and payment frequencies are the same. If you wanted to know how much money you have in your investment after the three years, the figure below illustrates how you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.



While you could use this technique to solve all annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person instead made quarterly contributions of \$250? That is 12 payments over three years, resulting in 11 separate future value calculations. Or if they made monthly payments, the 36 payments over three years would result in 35 separate future value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error. There must be a better way!

The Formula

The formula for the future value of an ordinary annuity is indeed easier and faster than performing a series of future value calculations for each of the payments. At first glance, though, the formula is pretty complex, so the various parts of the formula are first explored in some detail before we put them all together.

The annuity formula is a more complex version of the rate, portion, and base formula introduced in Chapter 2. Relating Formula 2.2 and the first payment from the figure above gives the following:

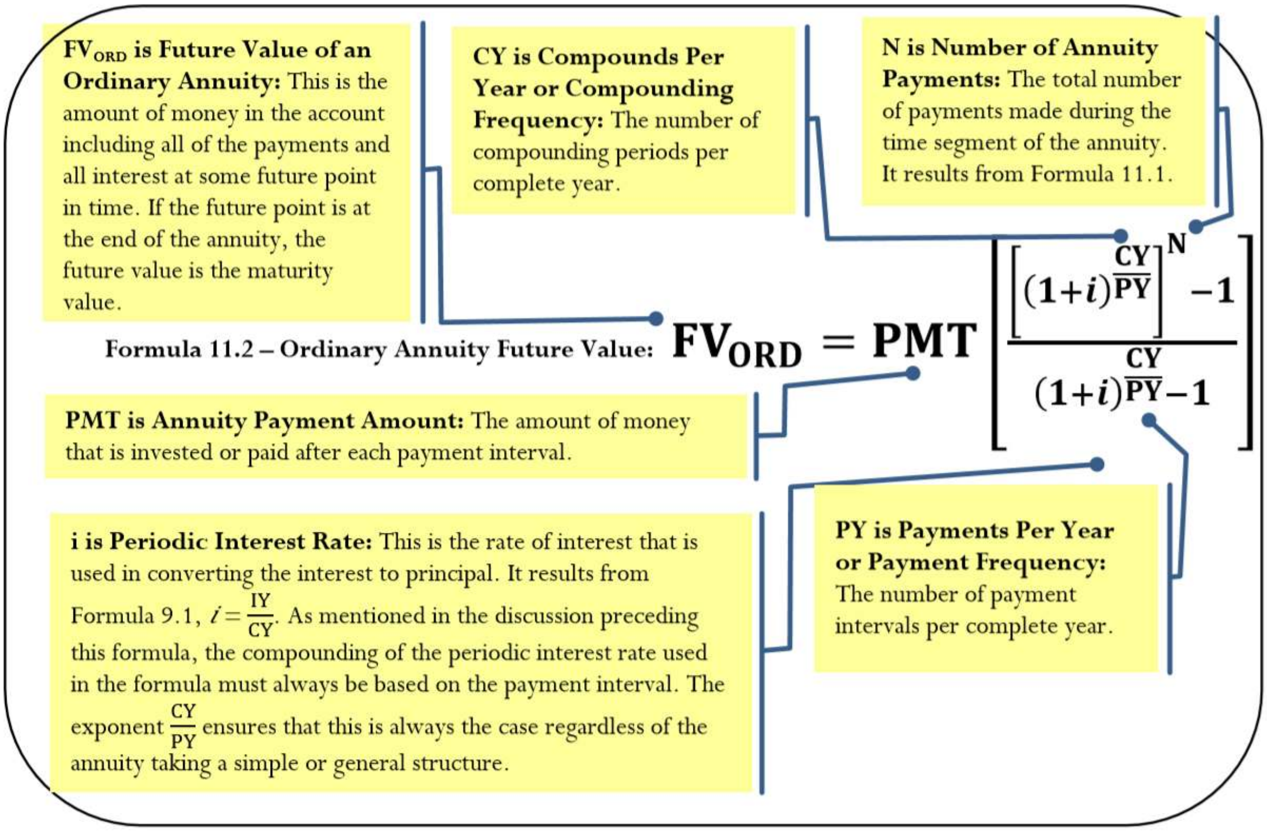
$$\text{Portion} = \text{Base} \times \text{Rate}$$

$$\$1,210 = \$1,000 \times (1 + 0.1)^2$$

The *portion* equals the future value and the base equals the annuity payment amount. The rate is expressed as a formula and written as $(1 + 0.1)^2$. However, notice that each payment in the figure has a different exponent to reflect the compounding required. This requires a mathematical adaptation of $(1 + i)^N$, which permits the determination of an equivalent rate representing all payments in a single calculation.

This equivalent rate turns out to be a pretty complex expression, which is examined in three parts: the percent change overall, the percent change with each payment, and their quotient.

1. **Numerator - The Percent Change Overall**:$\left[(1+i)^{\frac{CY}{PY}}\right]^N - 1$. This part of the formula determines the percent change from the start of the annuity to the end of the annuity. It has three critical elements:
   a. **Interest Rate Conversion** $(1+i)^{\frac{CY}{PY}}$. The rate of interest that occurs with each payment must be known. All annuity calculations require the compounding period to equal the payment interval. If this is not already the case, then you must convert the expressed interest rate into an equivalent interest rate.
      - For simple annuities, no conversion is necessary since the frequencies are the same: $CY = PY$. The exponent of $\dfrac{CY}{PY}$ always equals 1 and has no effect.
      - For general annuities, recall Formula 9.4 for calculating equivalent interest rates. Here the old compounding frequency forms the numerator (CY_{Old}) and the new compounding frequency (which matches the payment frequency) forms the denominator $(CY_{New}) = PY$. Thus $\dfrac{CY_{OLD}}{PY_{New}}$ becomes $\dfrac{CY}{PY}$.
   b. **The Compounding** N. The exponent N compounds the periodic interest rate (which matches the payment interval) in accordance with the number of annuity payments made. For example, assume there are two end-of-year payments at 10% compounded semi-annually. If the interest rate is left semi-annually, there are four compounds over the two years, which does not match the payments. The interest rate is converted within the brackets from 10% compounded semi-annually to its equivalent 10.25% compounded annually rate. The end result is that interest will now compound twice over the two years, matching the number of payments.
   c. **Removing the Starting Point (−1)**. Since you added 1 to perform the compounding, mathematically you now need to remove the 1. The end result is that you now know (in decimal format) how much larger the future value is relative to its starting value.
2. **Denominator - The Percent Change with Each Payment**: $(1+i)^{\frac{CY}{PY}} - 1$. The denominator in the formula shows the percent change in the account with each payment made. It too ensures that the denominator has a periodic rate matching the payment interval.
3. **The Quotient**: By taking the numerator and dividing by the denominator, the percent change overall is divided by the percent change with each payment. This establishes a ratio between what is happening overall in the annuity versus what is happening with each transaction. Thus, this computation determines how much bigger the final value is relative to what happens with each payment. This ratio (the rate) is then multiplied against the annuity payment (the base) to get the final balance (the portion). You are now in a position to see the whole formula by which you calculate the future value of any ordinary annuity. This single formula applies regardless of the number of payments that are made. Formula 11.2 summarizes the variables and calculations required.



FV_{ORD} is Future Value of an Ordinary Annuity: This is the amount of money in the account including all of the payments and all interest at some future point in time. If the future point is at the end of the annuity, the future value is the maturity value.

CY is Compounds Per Year or Compounding Frequency: The number of compounding periods per complete year.

N is Number of Annuity Payments: The total number of payments made during the time segment of the annuity. It results from Formula 11.1.

Formula 11.2 – Ordinary Annuity Future Value: $$FV_{ORD} = PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid after each payment interval.

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal. It results from Formula 9.1, $i=\frac{IY}{CY}$. As mentioned in the discussion preceding this formula, the compounding of the periodic interest rate used in the formula must always be based on the payment interval. The exponent $\frac{CY}{PY}$ ensures that this is always the case regardless of the annuity taking a simple or general structure.

PY is Payments Per Year or Payment Frequency: The number of payment intervals per complete year.

Formula 11.2

How It Works

There is a five-step process for calculating the future value of any ordinary annuity:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the known variables, including PV, IY, CY, PMT, PY, and Years.

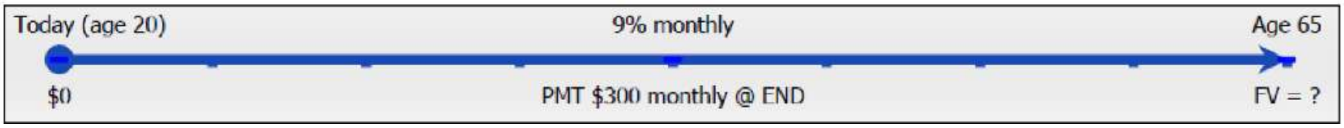
Step 3: Use Formula 9.1 to calculate i.

Step 4: If PV = \$0, proceed to step 5. If there is a nonzero value for PV, treat it like a single payment. Apply Formula 9.2 to determine N since this is not an annuity calculation. Move the present value to the end of the time segment using Formula 9.3.

Step 5: Use Formula 11.1 to calculate N for the annuity. Apply Formula 11.2 to calculate the future value. If you calculated a future value in step 4, combine the future values from steps 4 and 5 to arrive at the total future value.

Revisiting the RRSP scenario from the beginning of this section, assume you are 20 years old and invest \$300 at the end of every month for the next 45 years. You have not started an RRSP previously and have no opening balance. A fixed interest rate of 9% compounded monthly on the RRSP is possible.

Step 1: This is a simple ordinary annuity since the frequencies match and payments are at the end of the payment interval.



Step 2: The known variables are PV = \$0, IY = 9%, CY = 12, PMT = \$300, PY = 12, and Years = 45.

Step 3: The periodic interest rate is i = 9% ÷ 12 = 0.75%.

Step 4: Since PV = \$0, skip this step.

Step 5: The number of payments is N = 12 × 45 = 540. Applying Formula 11.2 gives the following:

$$FV_{ORD} = \$300\left[\frac{\left[(1+0.0075)^{\frac{12}{12}}\right]^{540}-1}{(1+0.0075)^{\frac{12}{12}}-1}\right] = \$2,221,463.54$$

Hence, 540 payments of \$300 at 9% compounded monthly results in a total saving of \$2,221,463.54 by the age of retirement.

Important Notes

Calculating the Interest Amount. For investment annuities, if you are interested in knowing how much of the future value is principal and how much is interest, you can adapt Formula 8.3, where $I = S–P = FV–PV$.

- The FV is the solution to Formula 11.2.
- The PV is the principal (total contributions) calculated by taking $N \times PMT + PV$. In the figure above, $N \times PMT + PV$ = 540 payments × \$300 + \$0 = \$162,000 in principal. Therefore, I =\$2,221,463.54 − \$162,000 = \$2,059,463.54, which is the interest earned.

Your BAII+ Calculator. Adapting your calculator skills to suit annuities requires three important changes:

1. You now have a value for PMT. Be sure to enter it with the correct cash flow sign convention. When you invest, the payment has the same sign as the PV. When you borrow, the sign of the payment is opposite that of PV.
2. The P/Y is no longer automatically set to the same value as C/Y. Enter your values for P/Y and C/Y separately. If the values are the same, as in the case of simple annuities, then taking advantage of the “Copy” feature on your calculator let’s you avoid having to key the value in twice.
3. If a present value (PV) is involved, by formula you need to do two calculations using Formula 9.3 and Formula 11.2. If you input values for both PV and PMT, the calculator does these calculations simultaneously, requiring only one sequence to solve.

Things To Watch Out For

In many annuity situations there might appear to be more than one unknown variable. Usually the extra unknown variables are "unstated" variables that can reasonably be assumed. For example, in the RRSP illustration above, the statement "you have not started an RRSP previously and have no opening balance" could be omitted. If something were saved already, the number would need to be stated. As another example, it is normal to finish a loan with a zero balance. Therefore, in a loan situation you can safely assume that the future value is zero unless otherwise stated.

Paths To Success

The ability to recognize a simple annuity can allow you to simplify Formula 11.2. Since CY = PY, these two variables form a quotient of 1 for the exponent. For a simple annuity, you can simplify any appearances of the following algebraic expressions in any annuity formula (not just Formula 11.2) as follows:

$(1+i)^{\frac{CY}{PY}}$ in the numerator can be simplified to $1+i$

and

$(1+i)^{\frac{CY}{PY}}-1$ in the denominator can be simplified to just i

Thus, for simple annuities only, you simplify Formula 11.2 as

$$FV_{ORD} = PMT\left[\frac{(1+i)^{N}-1}{i}\right]$$

Exercise 11.2.1: Give It Some Thought

Assume you had planned to make 10 annuity payments to an investment. However, before you started paying in to the investment, you changed your mind, doubling your original payment amount while still making 10 payments. What happens to the maturity value of your new investment compared to that of your original plan? Will your new balance be exactly double, more than double, or less than double? Explain and justify your answer.

Answer

Your new balance will be exactly double. Mathematically, you have taken PMT in Formula 11.2 and multiplied it by 2. That is the only difference between your original plan and your new plan. Therefore, the future value will double as well.

Example 11.2.1: Future Value of an Investment Account

A financial adviser is reviewing one of her client's accounts. The client has been investing $1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have today in his account?

Solution

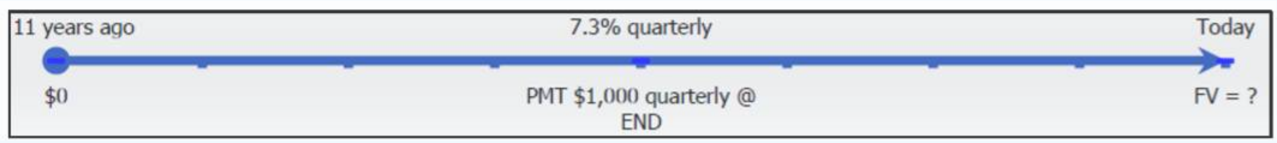
Step 1:

The payments are at the end of the payment intervals, and both the compounding period and the payment intervals are the same. This is an ordinary simple annuity. Calculate its value at the end, which is its future value, or FV_{ORD}.

What You Already Know

Step 1 (continued):

The timeline shows the client's account.



Step 2:

PV = \$0, IY = 7.3%, CY = 4, PMT = \$1,000, PY = 4, Years = 11

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Skip this step since PV = \$0.

Step 5:

Apply Formula 11.1 and Formula 11.2.

Perform

Step 3:

$i = 7.3\% \div 4 = 1.825\%$

Step 5:

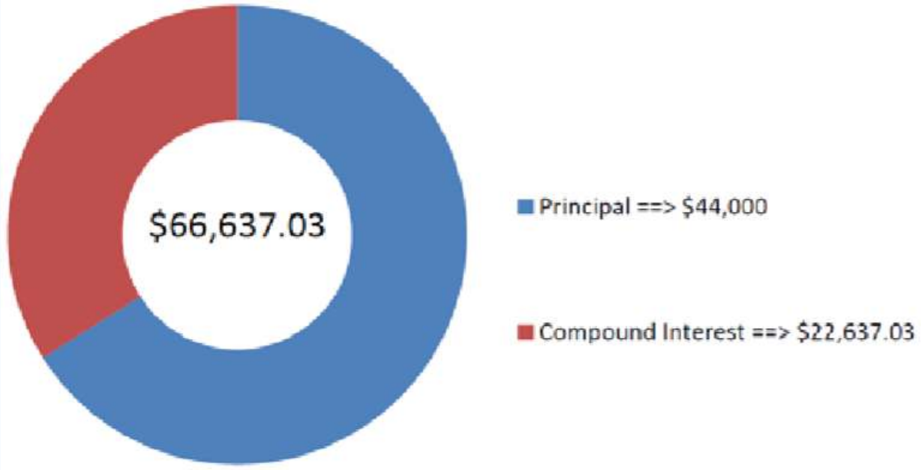
$N = 4 \times 11 = 44$ payments

$$FV_{ORD} = \$1000 \left[\frac{\left[(1+0.01825)^{\frac{4}{4}}\right]^{44} - 1}{(1+0.01825)^{\frac{4}{4}} - 1} \right] = \$66,637.03$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
44	7.3	0	-1000	Answer: 66,637.03449	4	4

The figure shows how much principal and interest make up the final balance. After 11 years of $1,000 quarterly contributions, the client has $66,637.03 in the account.



✓ Example 11.2.2: Future Value of a Savings Annuity

A savings annuity already contains $10,000. If an additional $250 is invested at the end of every month at 9% compounded semi-annually for a term of 20 years, what will be the maturity value of the investment?

Solution

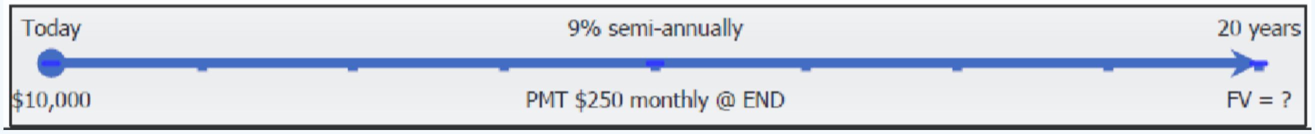
Step 1:

The payments are at the end of the payment intervals, and the compounding period and payment intervals are different. This is an ordinary general annuity. Calculate its value at the end, which is its future value, or FV_{ORD}.

What You Already Know

Step 1 (continued):

The timeline appears below.



Step 2:

PV = $10,000, IY = 9%, CY = 2, PMT = $250, PY = 12, Years = 20

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.1 and Formula 11.2 The final future value is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}).

Step 3:

$i = 9\% \div 2 = 4.5\%$

Step 4:

$N = 2 \times 20 = 40$ compounds

$$FV = \$10,000(1+0.045)^{40} = \$58,163.64538$$

Step 5:

$N = 12 \times 20 = 240$ payments

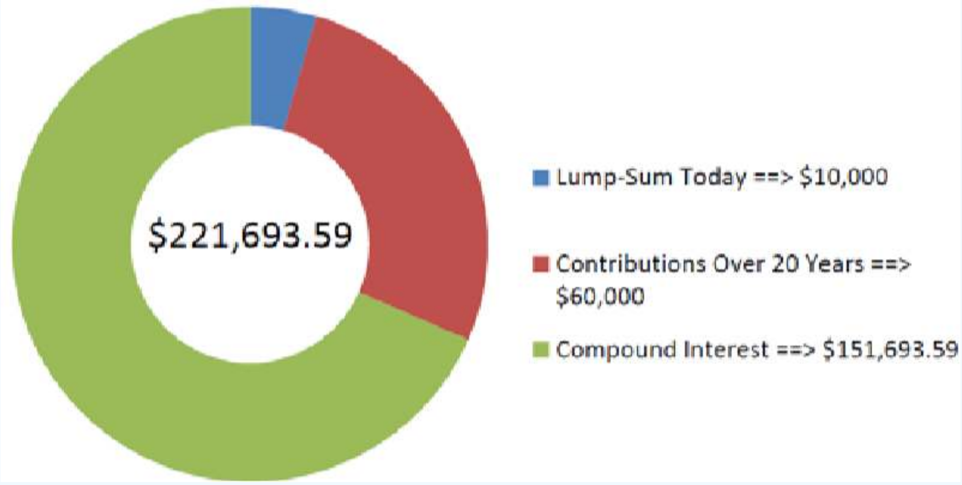
$$FV_{ORD} = \$250\left[\frac{\left[(1+0.045)^{\frac{2}{12}}\right]^{240}-1}{(1+0.045)^{\frac{2}{12}}-1}\right] = \$163,529.9492$$

$$\text{Total } FV = \$58,163.64538 + \$163,529.9492 = \$221,693.59$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
240	9	-10000	-250	Answer: 221,693.5946	12	2

The figure shows how much principal and interest make up the final balance. The savings annuity will have a balance of $221,693.59 after the 20 years.



Important Notes

If any of the variables, including IY, CY, PMT, or PY, change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. There will then be multiple time segments that require you to work left to right by repeating steps 3 through 5 in the procedure. The future value at the end of one time segment becomes the present value in the next time segment. Example 11.2.3 illustrates this concept.

Things To Watch Out For

Pay extra attention when the variable that changes between time segments is the payment frequency (PY). When inputted into a BAII+ calculator, the PY automatically copies across to the compounding frequency (CY). Unless your

CY also changed to the same frequency, this means that you must scroll down to the CY window and re-enter the correct value for this variable, even if it didn't change.

Paths To Success

When working with multiple time segments, it is important that you always start your computations on the side opposite the unknown variable. For future value calculations, this means you start on the left-hand side of your timeline; for present value calculations, start on the right-hand side.

✓ Example 11.2.3: Saving Up for a Vacation

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has \$1,000 saved today. For the first year, she plans on making end-of-month contributions of \$300 and then switching to end-of-quarter contributions of \$1,000 in the second year. If the account can earn 5% compounded semi-annually in the first year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

Solution

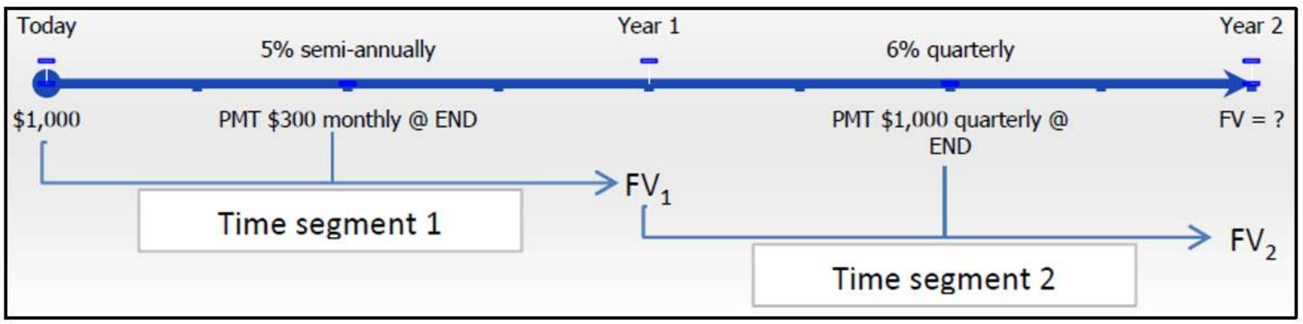
Step 1:

There is a change of variables after one year. As a result, you need a Year 1 time segment and a Year 2 time segment. In both segments, payments are at the end of the period. In Year 1, the compounding period and payment intervals are different. In Year 2, the compounding period and payment intervals are the same. This is an ordinary general annuity followed by an ordinary simple annuity. You aim to calculate the future value, or FV_{ORD}.

What You Already Know

Step 1 (continued):

The timeline for her vacation savings appears below.



Step 2:

Time segment 1: PV = \$1,000, IY = 5%, CY = 2, PMT = \$300, PY = 12, Years = 1

Time segment 2: $PV = FV_1$ of time segment 1, IY = 6%, CY = 4, PMT = \$1,000, PY = 4, Years = 1

How You Will Get There

For each time segment repeat the following steps:

Step 3:

Apply Formula 9.1.

Step 4:

Apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.1 and Formula 11.2. The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}).

Perform

For the first time segment:

Step 3:

$i = 5\% \div 2 = 2.5\%$

Step 4:

$N = 2 \times 1 = 2$ compounds

$$FV_1 = \$1,000(1+0.025)^2 = \$1,050.625$$

Step 5:

$N = 12 \times 1 = 12$ payments

$$FV_{ORD1} = \$300\left[\frac{\left[(1+0.025)^{\frac{2}{12}}\right]^{12}-1}{(1+0.025)^{\frac{2}{12}}-1}\right] = \$3,682.786451$$

$$\text{Total } FV_1 = \$1,050.625 + \$3,682.786451 = \$4,733.411451$$

For the second time segment:

Step 3:

$i = 6\% \div 4 = 1.5\%$

Step 4:

$N = 4 \times 1 = 4$ compounds

$$FV_2 = \$4,733.411451(1+0.015)^4 = \$5,023.870384$$

Step 5:

$N = 4 \times 1 = 4$ payments

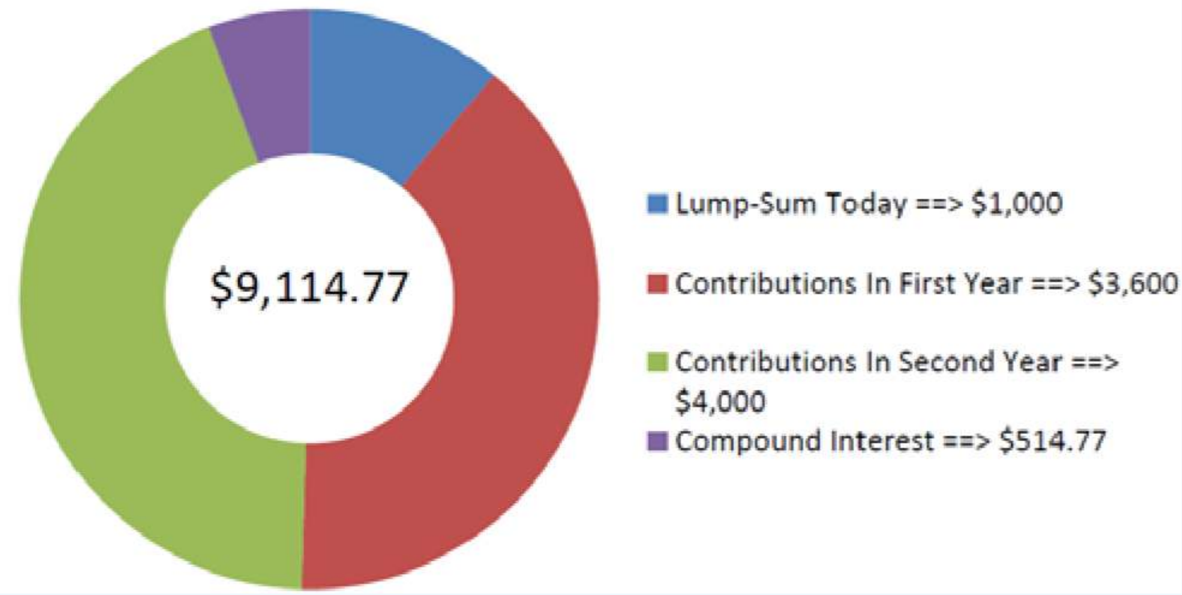
$$FV_{ORD2} = \$1,000\left[\frac{\left((1+0.015)^{\frac{4}{4}}\right]^{4}-1}{(1+0.015)^{\frac{4}{4}}-1}\right] = \$4,090.903375$$

$$\text{Total } FV_2 = \$5,023.870384 + \$4,090.903375 = \$9,114.77$$

Calculator Instructions

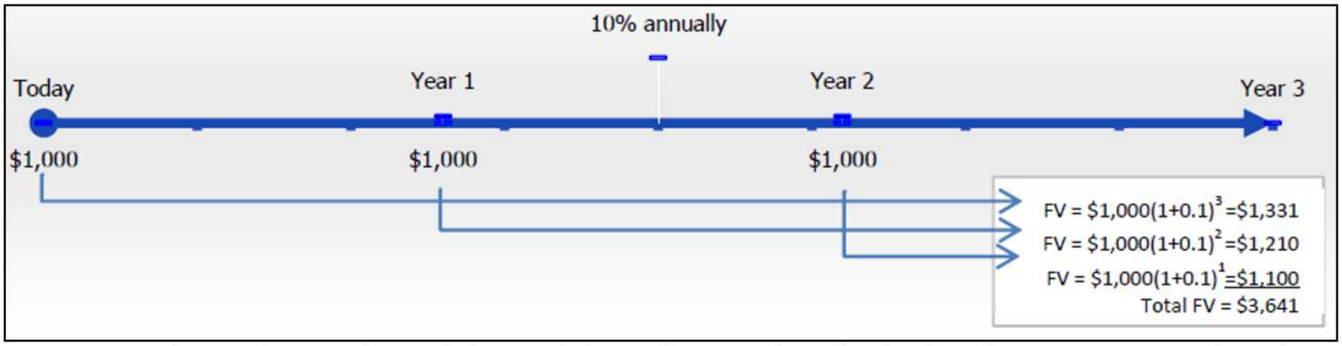
Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	12	5	-1000	-300	Answer: 4,733.411451	12	2
2	4	6	-4,733.411451	-1000	Answer: 9,114.773759	4	4

The figure shows how much principal and interest make up the final balance. When Genevieve graduates she will have saved $9,114.77 toward her vacation.

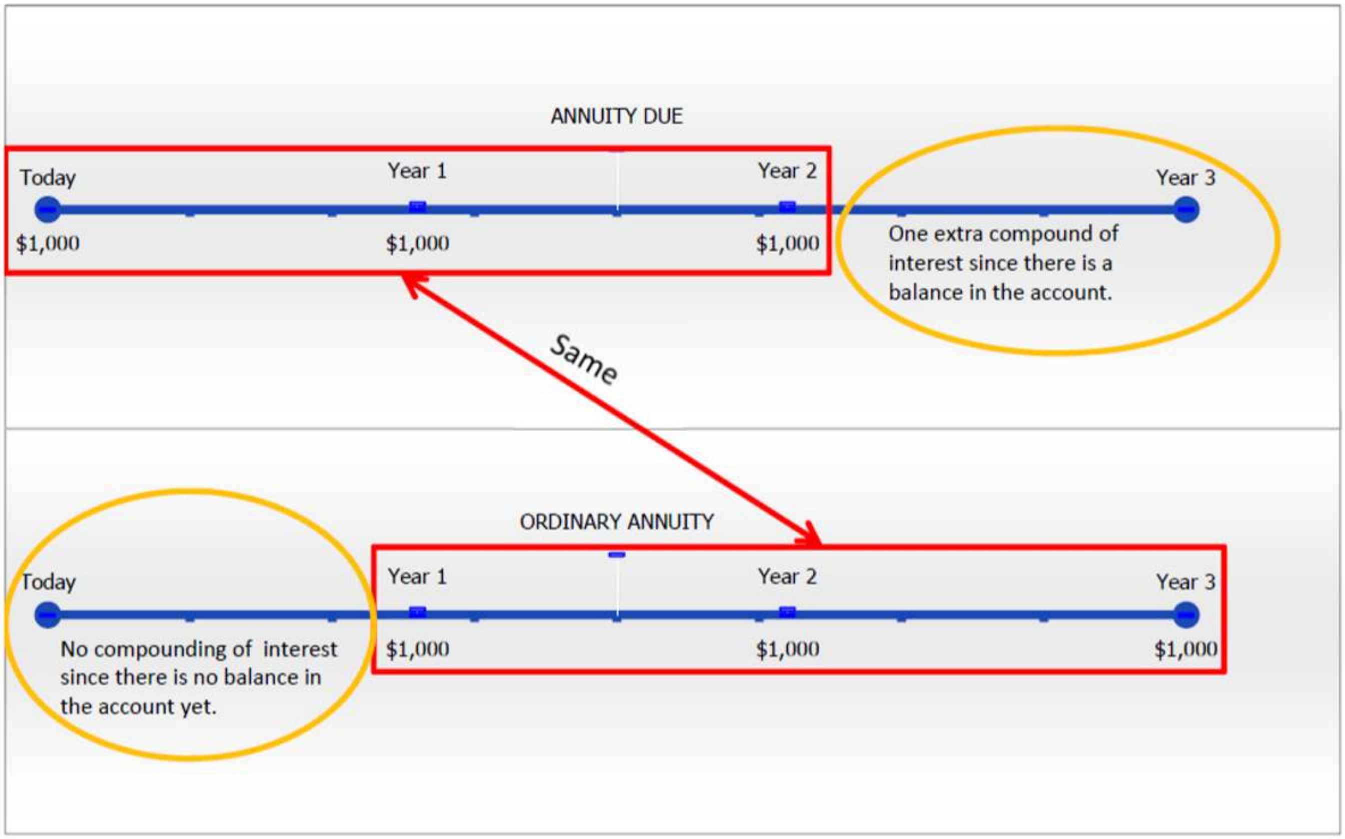


Annuities Due

An annuity due occurs when payments are made at the beginning of the payment interval. To understand the difference this makes to the future value, let's recalculate the RRSP example from earlier in this section, but treat it as an annuity due. You want to know the future value of making $1,000 annual contributions at the beginning of every payment interval for the next three years to an investment earning 10% compounded annually. This forms a simple annuity due. The figure below illustrates how you apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.


10% annually
Today
Year 1
Year 2
Year 3
$1,000
$1,000
$1,000
FV = $1,000(1+0.1)^3 =$1,331
FV = $1,000(1+0.1)^2 =$1,210
FV = $1,000(1+0.1)^1 =$1,100
Total FV = $3,641

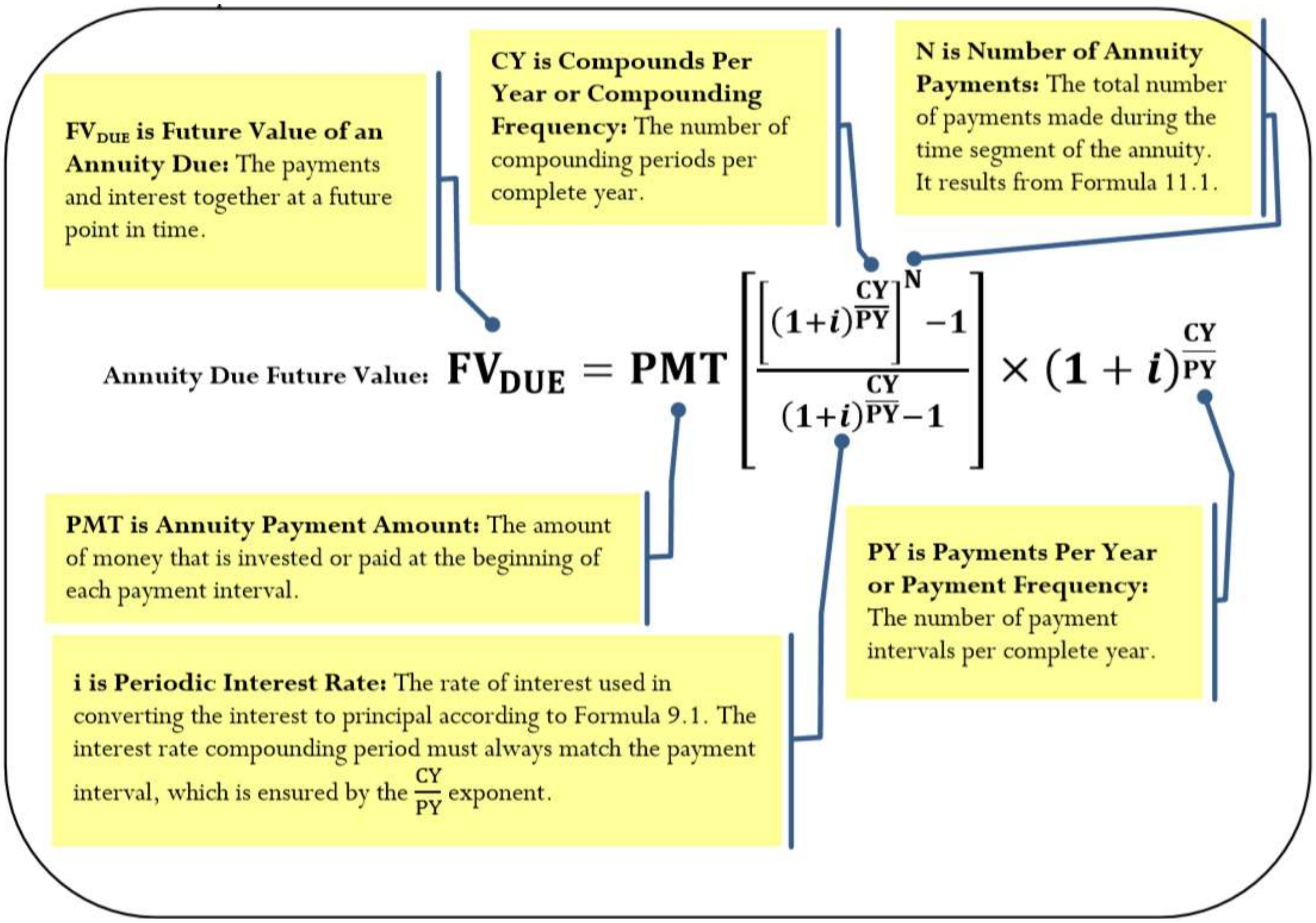
Note that you do not end up with the same balance of $3,310 achieved under the ordinary annuity. Instead, you have a larger balance of $3,641. Why is this? Placing the two types of annuities next to each other in the next figure demonstrates the key difference between the two examples.


ANNUITY DUE
Today
Year 1
Year 2
Year 3
$1,000
$1,000
$1,000
One extra compound of interest since there is a balance in the account.
Same
ORDINARY ANNUITY
Today
Year 1
Year 2
Year 3
No compounding of interest since there is no balance in the account yet.
$1,000
$1,000
$1,000

Both annuities have an identical sequence of three $1,000 payments. However, in the ordinary annuity no interest is earned during the first payment interval since the principal is zero and the payment does not occur until the end of the interval. On the other hand, in the annuity due an extra compound of interest is earned in the last payment interval because of the existing principal at the end of the second year. If you take the ordinary annuity's final balance of $3,310 and give it one extra compound you have FV = $3,310(1 + 0.1) = $3,641. In summary, the key difference between the two types of annuities is that an annuity due always receives one extra compound of interest.

The Formula

Adapting the ordinary annuity future value formula to suit the extra compound creates Formula 11.3. Note that all the variables in the formula remain the same; however, the subscript on the FV symbol is changed to recognize the difference in the calculation required.



FV_{DUE} is Future Value of an Annuity Due: The payments and interest together at a future point in time.

CY is Compounds Per Year or Compounding Frequency: The number of compounding periods per complete year.

N is Number of Annuity Payments: The total number of payments made during the time segment of the annuity. It results from Formula 11.1.

Annuity Due Future Value: $$FV_{DUE}=PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]\times(1+i)^{\frac{CY}{PY}}$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid at the beginning of each payment interval.

PY is Payments Per Year or Payment Frequency: The number of payment intervals per complete year.

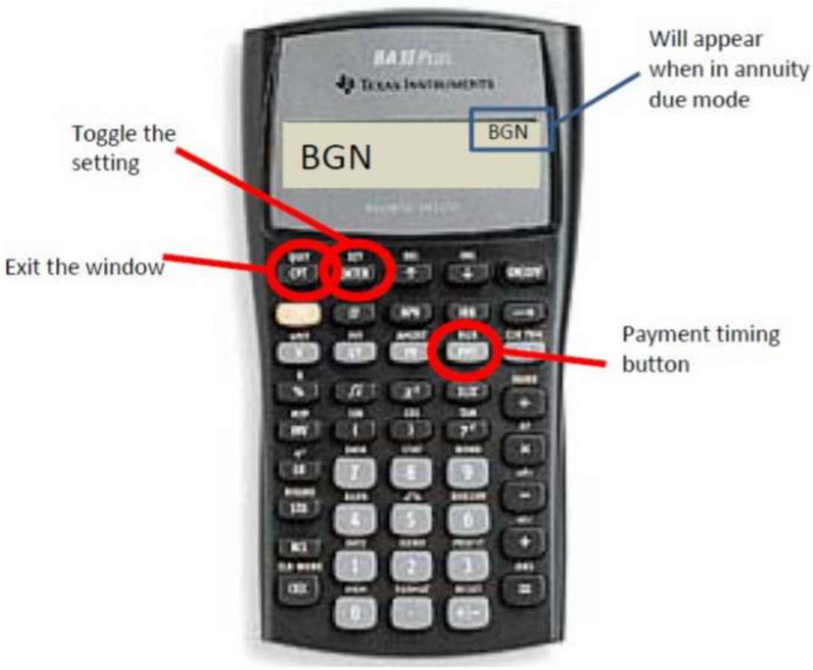
i is Periodic Interest Rate: The rate of interest used in converting the interest to principal according to Formula 9.1. The interest rate compounding period must always match the payment interval, which is ensured by the $\frac{CY}{PY}$ exponent.

Formula 11.3

How It Works

The steps required to solve for the future value of an annuity due are almost identical to those you use for the ordinary annuity. The only difference lies in step 5, where you use Formula 11.3 instead of Formula 11.2. Example 11.2.4 and Example 11.2.5 illustrate the adaptation.

Important Notes



To adapt your calculator to an annuity due, you must toggle the payment timing setting from END to BGN. The calculator default is END, which is the ordinary annuity. The payment timing setting is found on the second shelf above the PMT key (because it is related to the PMT!). To toggle the setting, complete the following sequence:

1. 2nd BGN (the current payment timing of END or BGN is displayed)
2. 2nd SET (it toggles to the other setting)
3. 2nd Quit (to get out of the window)

When the calculator is in annuity due mode, a tiny BGN appears in the upper right-hand corner of your calculator. To return the calculator to ordinary mode, repeat the above

? Exercise 11.2.2: Give It Some Thought

Under equal conditions,

a. For any investment, which will always have the higher future value: an ordinary annuity or an annuity due? Explain.
b. For any debt, which will always have a higher future value: an ordinary annuity or an annuity due? Explain.

Answer

a. The annuity due will have the higher future value, since it always has one extra compound compared to an ordinary annuity.
b. The ordinary annuity will have the higher future value, since the principal in the first payment interval is higher and therefore more interest accrues than in the annuity due.

✓ Example 11.2.4: Lottery Winnings

The Set for Life instant scratch n' win ticket offers players a chance to win $1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

Solution

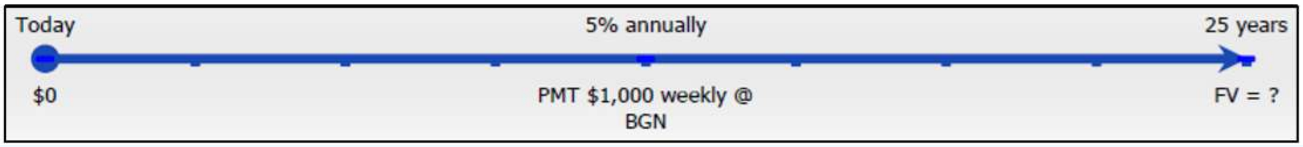
Step 1:

The payments start immediately, and the compounding period and payment intervals are different. Therefore, this a general annuity due. Calculate its value at the end, which is its future value, or FV_{DUE}.

What You Already Know

Step 1 (continued):

The timeline for the lottery savings is below.



Step 2:

PV = \$0, IY = 5%, CY = 1, PMT = \$1,000, PY = 52, Years = 25

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Skip this step since there is no PV.

Step 5:

Apply Formula 11.1 and Formula 11.3.

Perform

Step 3:

$i = 5\% \div 1 = 5\%$

Step 5:

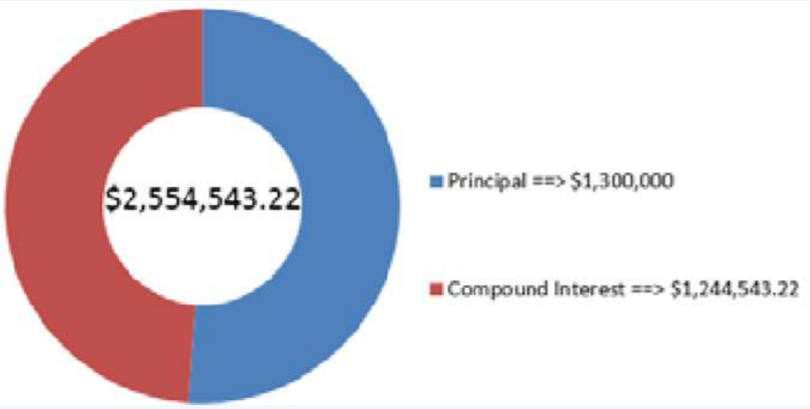
$N = 52 \times 25 = 1,300$ payments

$$\begin{aligned} FV_{DUE} &= \$1,000\left[\frac{\left[(1+0.05)^{\frac{1}{52}}\right]^{1300}-1}{(1+0.05)^{\frac{1}{52}}-1} \times (1+0.05)^{\frac{1}{52}}\right] \\ &= \$2,544,543.22 \end{aligned}$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	1300	5	0	-1000	Answer: 2,544,543.218	52	1

The figure shows how much principal and interest make up the final balance. If the winner was to invest all of his lottery prize money, he would have \$2,544,543.22 after 25 years.



✓ Example 11.2.5: Saving into a Trust Fund with a Variable Change

When Roberto's son was born, Roberto started making payments of $1,000 at the beginning of every six months to a trust fund earning 5.75% compounded monthly. After five years, he changed his contributions and started depositing $500 at the beginning of every quarter. How much money will be in his son's trust fund when his son turns 18?

Solution

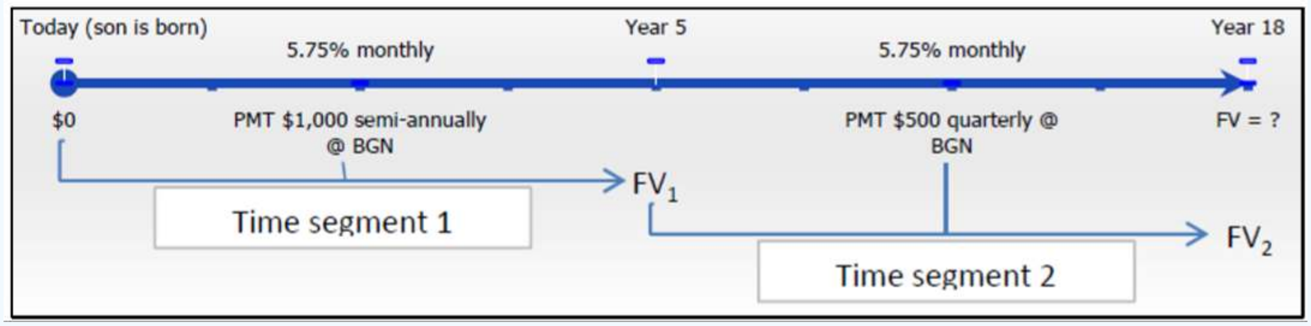
Step 1:

There is a change of variables after five years. As a result, you need two time segments. In both segments, payments are at the beginning of the period and the compounding periods and payment intervals are different. Therefore, Roberto has two consecutive general annuities due. Combined, calculate the future value, or FV_{DUE}.

What You Already Know

Step 1 (continued):

The timeline for the trust fund is below.


Today (son is born)
5.75% monthly
Year 5
5.75% monthly
Year 18
$0
PMT $1,000 semi-annually @ BGN
PMT $500 quarterly @ BGN
FV = ?
FV1
Time segment 1
FV2
Time segment 2

Step 2:

Time segment 1: PV = $0, IY = 5.75%, CY = 12, PMT = $1,000, PY = 2, Years = 5

Time segment 2: $PV = FV_1$ of time segment 1, IY = 5.75%, CY = 12, PMT = $500, PY = 4, Years = 13

How You Will Get There

For each time segment, repeat the following steps:

Step 3:

Apply Formula 9.1.

Step 4:

If there is a PV, apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.1 and Formula 11.3. The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}).

Perform

Step 3:

$i = 5.75\% \div 12 = 0.47196\overline{6}\%$

Step 4:

No PV, so skip this step.

Step 5:

$N = 2 \times 5 = 10$ payments

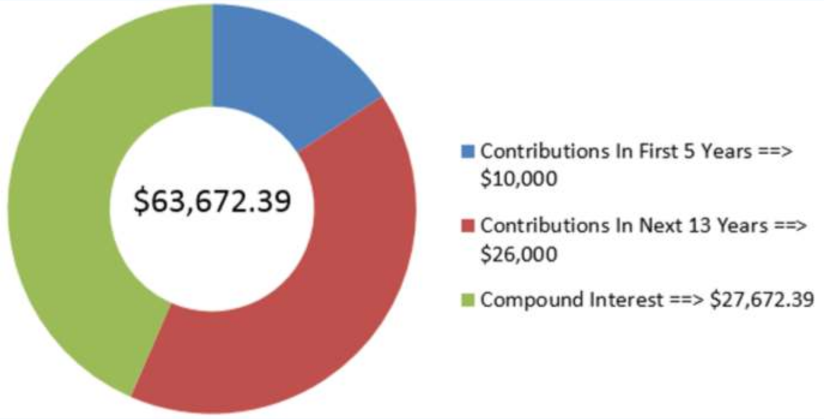
$$FV_{DUE1} = \$1,000 \left[\frac{\left[(1+0.0047196\overline{6})^{\frac{12}{2}}\right]^{10} - 1}{(1+0.0047196\overline{6})^{\frac{12}{2}} - 1} \times (1+0.0047196\overline{6})^{\frac{12}{2}} \right] = \$11,748.47466 = FV_1$$

$$\text{Total } FV_2 = \$24,765.17 + \$38,907.21529 = \$63,672.39$$

Calculator Instructions

Time segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	10	5.75	0	-1000	Answer: 11,748.47466	2	12
2	√	52	√	-11,748.47466	-500	Answer: 63,672.385.29	4	12

The figure shows how much principal and interest make up the final balance. When Roberto’s son turns 18, the trust fund will have a balance of $63,672.39.



Contributors and Attributions

- Template:ContribBusiMathOlivier

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11.3: Present Value Of Annuities

Have you ever noticed that the prices of expensive products are generally not advertised? Instead, companies that market these high-ticket products promote the payment amounts of the annuity, not the actual sticker price. In fact, locating the actual price for these products in the advertisements usually requires a magnifying glass. For example, a Mitsubishi Outlander was recently advertised for only $193 biweekly. That does not sound so bad, and perhaps you might be tempted to head on down to the car dealership to buy one of those vehicles. However, the fine print indicates that you need to make 182 payments, which then total almost $32,000. Why is the vehicle advertised this way? Numerically, $193 sounds a lot better than $32,000!

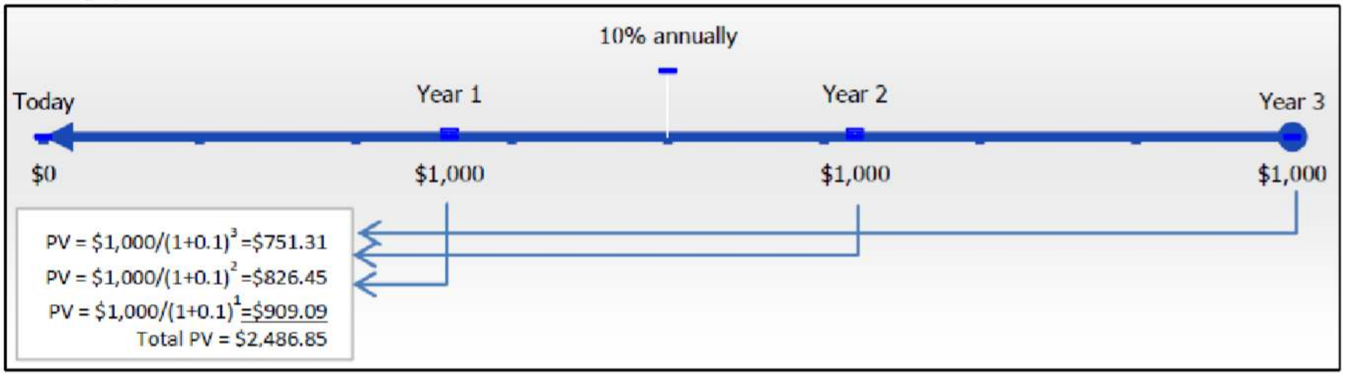
In business, whether you are setting up consumers with payment plans or buying and selling loan contracts, you need to calculate present values. As a consumer, you encounter present value calculations in many ways:

- How to take advertised payment amounts and convert them to the actual price that you must pay.
- Meeting financial goals by planning your RRSP, which requires knowing how much money you need at the start.

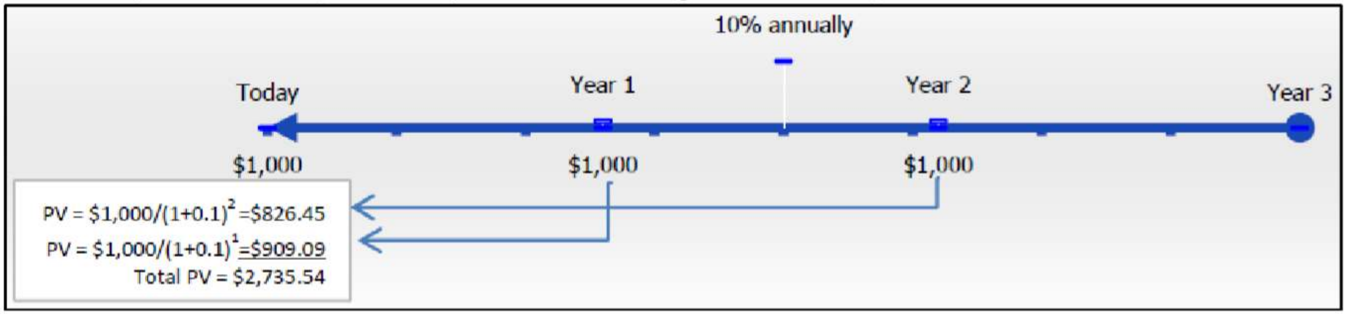
This section develops present value formulas for both ordinary annuities and annuities due. Like future value calculations, these formulas accommodate both simple and general annuities as needed. From investments, we will then extend annuity calculations to loans as well.

Ordinary Annuities and Annuities Due

The present value of any annuity is equal to the sum of all of the present values of all of the annuity payments when they are moved to the beginning of the first payment interval. For example, assume you will receive $1,000 annual payments at the end of every payment interval for the next three years from an investment earning 10% compounded annually. How much money needs to be in the annuity at the start to make this happen? In this case, you have an ordinary simple annuity. The figure below illustrates the fundamental concept of the time value of money and shows the calculations in moving all of the payments to the focal date at the start of the timeline.

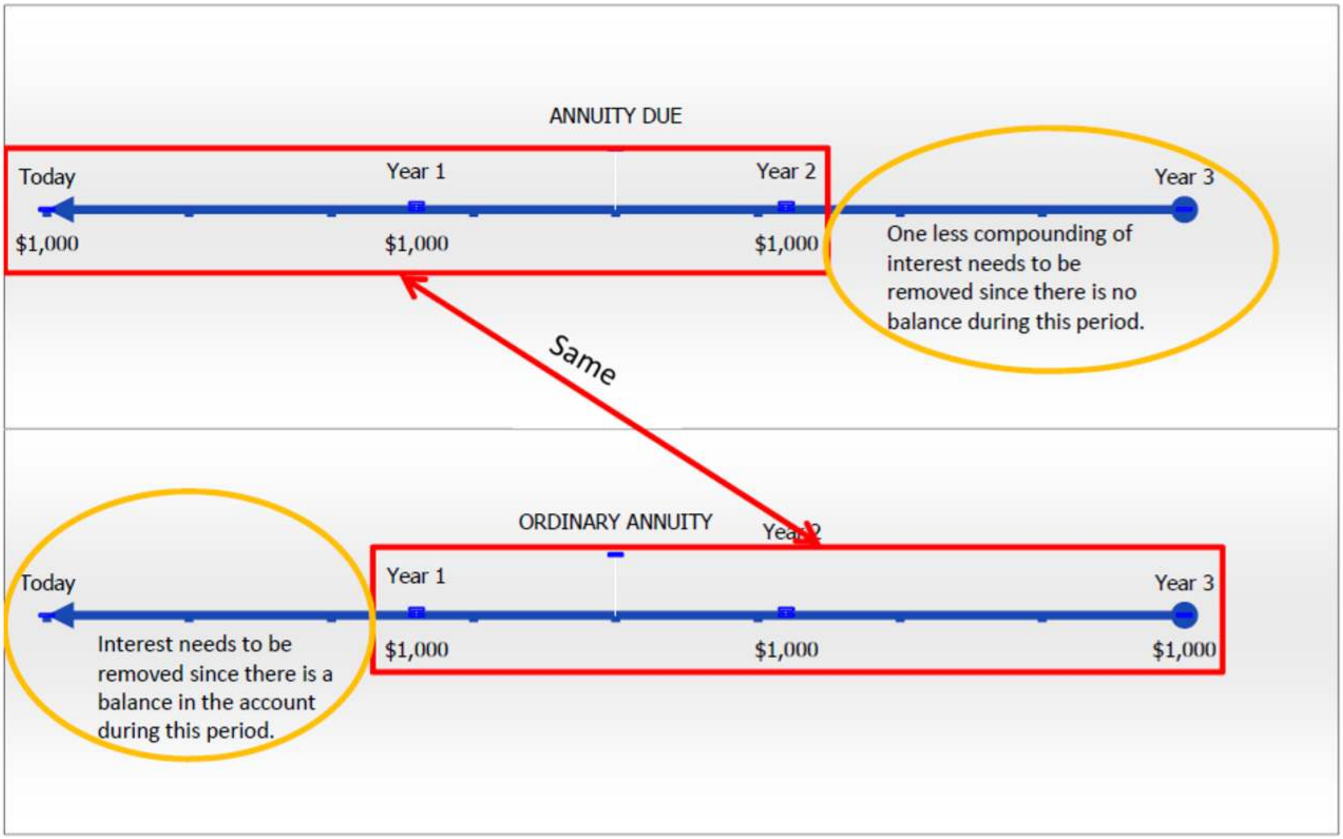

10% annually
Today
Year 1
Year 2
Year 3
$0
$1,000
$1,000
$1,000
PV = $1,000/(1+0.1)^3 =$751.31
PV = $1,000/(1+0.1)^2 =$826.45
PV = $1,000/(1+0.1)^1=$909.09
Total PV = $2,486.85

Observe that all three payments are present valued to your focal date, requiring an investment of $2,486.85 today. In contrast, what happens to your timeline and calculations if those payments are made at the beginning of every payment interval? In this case, you have a simple annuity due. The next figure illustrates your timeline and calculations.



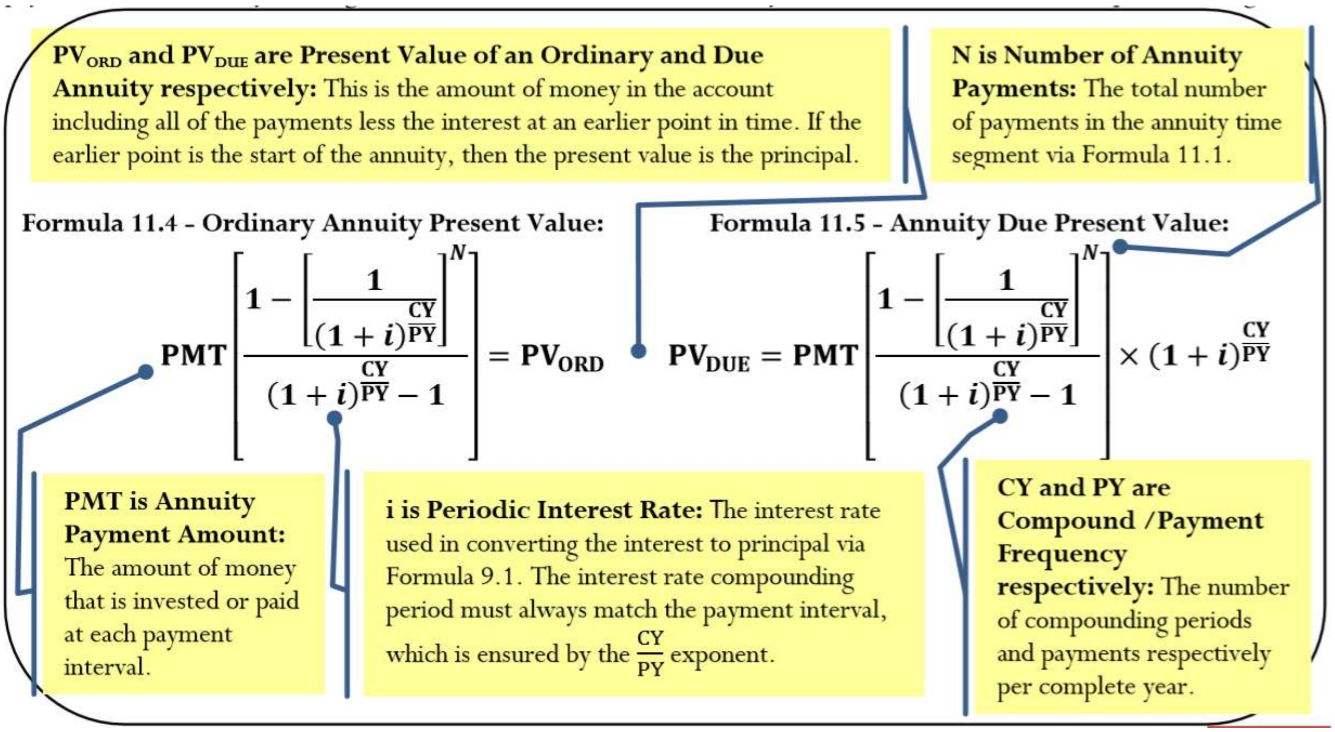
Observe that only two of the three payments need to be present valued to your focal date since the first payment is already on the focal date. The total investment for an annuity due is higher at $2,735.54 because the first payment is withdrawn immediately, so a smaller principal earns less interest than does the ordinary annuity.

The next figure below contrasts the two annuity types. Working from right to left on the timeline, the key difference is that the annuity due has one less compound of interest to remove. Its first time segment (from the right) contains a zero balance, while the ordinary annuity does contain a balance that needs to have interest discounted. Note that if you take the annuity due and remove one extra compound of interest, you arrive at $2,735.54 ÷ (1 + 0.1) = $2,486.85, which is the present value of the corresponding ordinary annuity.


ANNUITY DUE
Today
Year 1
Year 2
Year 3
$1,000
$1,000
$1,000
One less compounding of interest needs to be removed since there is no balance during this period.
Same
ORDINARY ANNUITY
Year 2
Today
Year 1
Year 3
Interest needs to be removed since there is a balance in the account during this period.
$1,000
$1,000
$1,000

The Formula

As with future value calculations, calculating present values by manually moving each payment to its present value is extremely time consuming when there are more than a few payments. Similarly, annuity formulas allow you to move all payments simultaneously in a single calculation. The formulas for ordinary annuities and annuities due are presented together.



PV_{ORD} and PV_{DUE} are Present Value of an Ordinary and Due Annuity respectively: This is the amount of money in the account including all of the payments less the interest at an earlier point in time. If the earlier point is the start of the annuity, then the present value is the principal.

N is Number of Annuity Payments: The total number of payments in the annuity time segment via Formula 11.1.

Formula 11.4 - Ordinary Annuity Present Value:

$$PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right]=PV_{ORD}$$

Formula 11.5 - Annuity Due Present Value:

$$PV_{DUE}=PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right]\times(1+i)^{\frac{CY}{PY}}$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid at each payment interval.

i is Periodic Interest Rate: The interest rate used in converting the interest to principal via Formula 9.1. The interest rate compounding period must always match the payment interval, which is ensured by the $\frac{CY}{PY}$ exponent.

CY and PY are Compound / Payment Frequency respectively: The number of compounding periods and payments respectively per complete year.

Formula 11.4 and Formula 11.5

The following observations are made about these two formulas:

1. The formulas adapt to both simple and general annuities. In the case of simple annuities, the compounding frequency already matches the payment frequency, so it requires no conversion; numerically, the exponent of $\frac{CY}{PY}$ produces a quotient of 1 and removes itself from the formula. In the case of general annuities, the exponent converts the compounding frequency of the interest rate to match the payment frequency.
2. Identical in structure to the future value formulas, these two formulas for present value are also complex equations involving rate, portion, and base. Here the present value (PV_{ORD} or PV_{DUE}) = portion, PMT = base, and everything else = rate. The numerator, $1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}$, produces the overall percent decrease in the annuity; the denominator, $(1+i)^{\frac{CY}{PY}}-1$, produces the percent change with each payment; the division of these two percent changes creates a ratio by which present value relates to the annuity payment itself. To illustrate, assume that i = 5%, N = 2, and $CY=PY$ = 1. Substituting these numbers into the formula gives the following:

   $$\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}=\frac{1-\left[\frac{1}{(1+.05)^{\frac{1}{1}}}\right]^{2}}{(1+.05)^{\frac{1}{1}}-1}=\frac{0.092970}{0.05}=1.859410$$

   From these numbers, it is evident that the present value amount represents a decrease of 9.2970% (the numerator) overall. With each payment, the decrease is 5% (the denominator). Therefore, the ratio of the present value overall to each payment (the PMT) is 1.859410. If the annuity payment amount is PMT = \$1,000, then the value of the $PV_{ORD}=\$1,000(1.859410)$ $=\$1,859.41$.
3. The only difference from an ordinary annuity is the multiplication of the extra term $(1+i)^{\frac{CY}{PY}}$. The multiplication results in the removal of one fewer compounds of interest than the ordinary annuity.

How It Works

Follow these steps to calculate the present value of any ordinary annuity or annuity due:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that you know, including FV, IY, CY, PMT, PY, and Years.

Step 3: Use Formula 9.1 to calculate i.

Step 4: If FV = \$0, proceed to step 5. If there is a nonzero value for FV, treat it like a single payment. Apply Formula 9.2 to determine N since this is not an annuity calculation. Move the future value to the beginning of the time segment using Formula 9.3, rearranging for PV.

Step 5: Use Formula 11.1 to calculate N. Apply either Formula 11.4 or Formula 11.5 based on the annuity type. If you calculated a present value in step 4, combine the present values from steps 4 and 5 to arrive at the total present value.

Important Notes

Calculating the Interest Amount. If you are interested in knowing how much interest was removed in the calculation of the present value, adapt Formula 8.3, where \(I = S – P = FV − PV\). The present value (PV) is the solution to either Formula 11.4 or Formula 11.5. The FV in Formula 8.3 is expanded to include the sum of all future monies, so it is replaced by $N \times PMT + FV$. Therefore, you rewrite Formula 8.3 as $I = (N \times PMT + FV) - PV$.

Your BAII+ Calculator. If a single payment future value (FV) is involved in a present value calculation, then you require two formula calculations using Formula 9.3 and either Formula 11.4 or Formula 11.5. The calculator performs both of these calculations simultaneously if you input values obeying the cash flow sign convention for both FV and PMT.

? Exercise 11.3.1: Give It Some Thought

For two equal investment annuities, will the present value of an ordinary annuity and annuity due be the same or different?

Answer

> They will be different. The annuity due always has the larger present value since it removes one fewer compound of interest than the ordinary annuity.

✓ Example 11.3.1: Amount Needed at Time of Retirement

Rodriguez is planning on having an annual gross income of \$50,000 at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78, which is the average life expectancy for a Canadian man. If the account earns 5.1% compounded annually, what amount of funds needs to be in the account when he retires?

Solution

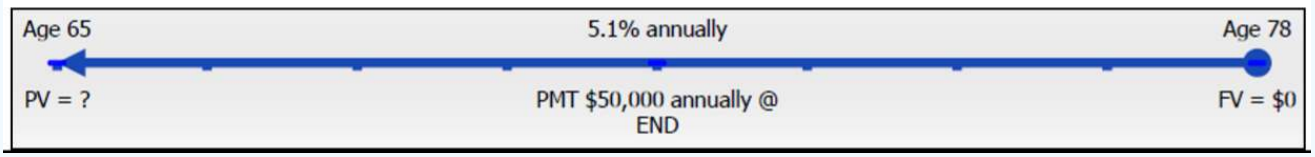
Step 1:

The payments are at the end of the payment intervals, and the compounding period and payment intervals are the same. This is, therefore, an ordinary simple annuity. Calculate its value at the start, which is its present value, or PV_{ORD}.

What You Already Know

Step 1 (continued):

The timeline for the client's account appears below.



Step 2:

FV = \$0, IY = 5.1%, CY = 1, PMT = \$50,000, PY = 1, Years = 13

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Since FV = \$0, skip this step.

Step 5:

Apply Formula 11.4 and Formula 11.5.

Perform

Step 3:

$i = 5.1\% \div 1 = 5.1\%$

Step 5:

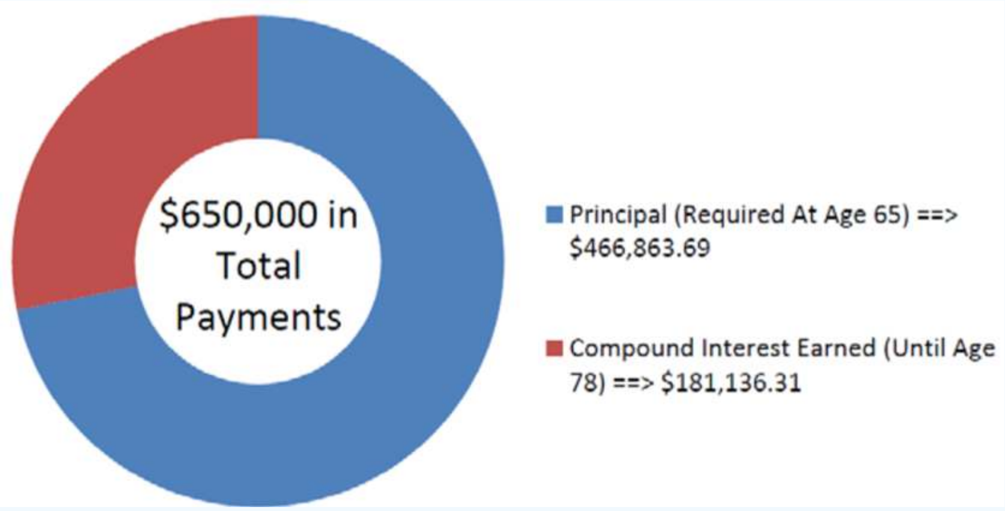
$N = 1 \times 13 = 13$ payments

$$PV_{ORD} = \$50,000\left[\frac{1-\left[\frac{1}{(1+0.051)^{\frac{1}{1}}}\right]^{13}}{(1+0.051)^{\frac{1}{1}}-1}\right] = \$466,863.69$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
13	5.1	Answer: -466,863.694	50000	0	1	1

The figure shows how much principal and interest make up the payments. Rodriguez will need to have \$466,863.69 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments.



✓ Example 11.3.2: Leaving an Inheritance

Recalculate Example 11.3.1 applying three changes:

a. Rodriguez wants to leave a \$100,000 inheritance for his children (assuming he dies at age 78).
b. Payments are at the beginning of the year.
c. His interest rate is 5.1% compounded semi-annually.

Calculate the present value and the amount of interest.

Solution

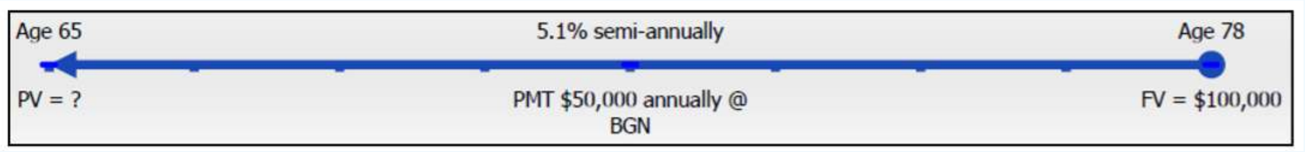
Step 1:

The payments are made at the beginning of the payment intervals, and the compounding period (semiannually) and payment intervals (annually) are different. This is now a general annuity due. Calculate its value at the start, which is its present value, or PV_{DUE}.

What You Already Know

Step 1 (continued):

The timeline for the client's account appears below.



Step 2:

FV = $100,000, IY = 5.1%, CY = 2, PMT = $50,000, PY = 1, Years = 13

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Since there is a future value, apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.4 and Formula 11.5.

Step 6:

To calculate the interest, apply and adapt Formula 8.3, where $FV = N \times PMT + FV$ and $I = FV - PV$

Perform

Step 3:

$i = 5.1\% \div 2 = 2.55\%$

Step 4:

$N = 2 \times 13 = 26$ compounds

$$\begin{aligned} \$100,000 &= PV(1+0.0255)^{26} \\ PV &= \$100,000 \div 1.0255^{26} \\ &= \$51,960.42776 \end{aligned}$$

Step 5:

$N = 1 \times 13 = 13$ payments

$$\begin{aligned} PV_{DUE} &= \$50,000\left[\frac{1-\left[\frac{1}{(1+0.0255)^{\frac{2}{1}}}\right]^{13}}{(1+0.0255)^{\frac{2}{1}}-1}\right] \times (1+0.0255)^{\frac{2}{1}} \\ &= \$489,066.6372 \end{aligned}$$

$$PV = \$51,960.42776 + \$489,066.6372 = \$541,027.07$$

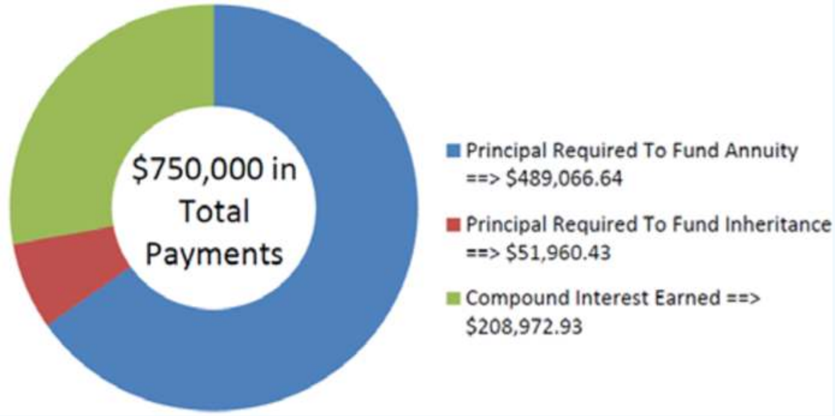
Step 6:

$$\begin{aligned} I &= (13 \times \$50,000 + \$100,000) - \$541,027.07 \\ I &= \$750,000 - \$541,027.07 \\ &= \$208,972.93 \end{aligned}$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	13	5.1	Answer: -541,027.065	50000	100000	1	2

The figure shows how much principal and interest make up the payments.Rodriguez will require more money, needing to have \$541,027.07 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments while leaving a \$100,000 inheritance for his children. His account will earn \$208,972.93 over the time frame.



Important Notes

If any of the variables, including IY, CY, PMT, or PY, change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, this creates a new time segment that you must treat separately. There will then be multiple time segments that require you to work from right to left by repeating steps 3 through 5 in the procedure. Remember that the present value at the beginning of one time segment becomes the future value at the end of the next time segment. Example $11.3.3$ illustrates this concept.

✓ Example $11.3.3$: Adjusting for Inflation

Continuing with the previous two examples, Rodriguez realizes that during his retirement he needs to make some type of adjustment to his annual gross income to account for the rising cost of living. Consequently, he will take \$50,000 at the beginning of each year for six years, then increase it to \$60,000 for the balance. Assume his interest rate is still 5.1% semiannually and that he still wants to leave a \$100,000 inheritance for his children. How much money needs to be in his retirement fund at age 65?

Solution

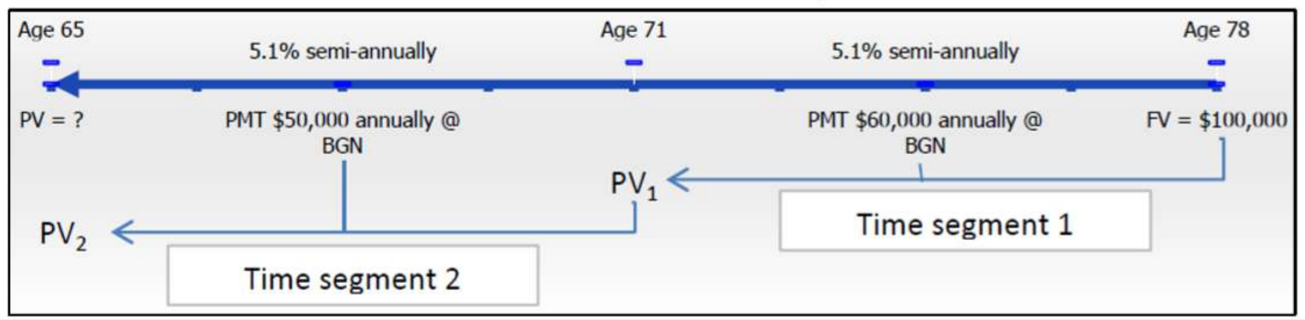
Step 1:

There is a change of variables after six years. As a result, you need two time segments. In both segments, payments are made at the beginning of the period, and the compounding periods and payment intervals are different. These are two consecutive general annuities due. You need to calculate the resulting present value, or PV_{DUE}.

What You Already Know

Step 1 (continued):

The timeline for the client's account appears below.


Age 65
5.1% semi-annually
Age 71
5.1% semi-annually
Age 78
PV = ?
PMT $50,000 annually @ BGN
PMT $60,000 annually @ BGN
FV = $100,000
PV_1
PV_2
Time segment 1
Time segment 2

Step 2:

Time segment 1 (starting from the right side): FV = \$100,000, IY = 5.1%, CY = 2, PMT = \$60,000, PY = 1, Years = 7

Time segment 2: $FV = PV_1$ of time segment 1, IY = 5.1%, CY = 2, PMT = \$50 000, PY = 1, Years = 6

How You Will Get

There For each time segment repeat the following steps:

Step 3:

Apply Formula 9.1.

Step 4:

Apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.1 and Formula 11.5.

Perform

For the first time segment (starting at the right side):

Step 3:

$i = 5.1\% \div 2 = 2.55\%$

Step 4:

$N = 2 \times 7 = 14$ compounds

$$\$100,000 = PV(1+0.0255)^{14}$$

$$PV_1 = 5100,000 \div 1.0255^{14} = \$70,291.15736$$

Step 5:

$N = 1 \times 7 = 7$ payments

$$PV_{DUE1} = \$60,000\left[\frac{1-\left[\frac{1}{(1+0.0255)^{\frac{2}{1}}}\right]^{7}}{(1+0.0255)^{\frac{2}{1}}-1}\right] \times (1+0.0255)^{\frac{2}{1}} = \$362,940.8778$$

$$\text{Total } PV_1 = \$70,291.15736 + \$362,940.8778 = \$433,232.0352$$

For the second time segment:

Step 3:

i remains the same = 2.55%

Step 4:

$N = 2 \times 6 = 12$ compounds

$$\$433,232.0352 = PV(1+0.0255)^{12}$$

$$\$433,232.0352 = PV(1+0.0255)^{12}$$

$$PV_2 = \$433,232.0352 \div 1.0255^{12} = \$320,252.5426$$

Step 5:

$N = 1 \times 6 = 6$ payments

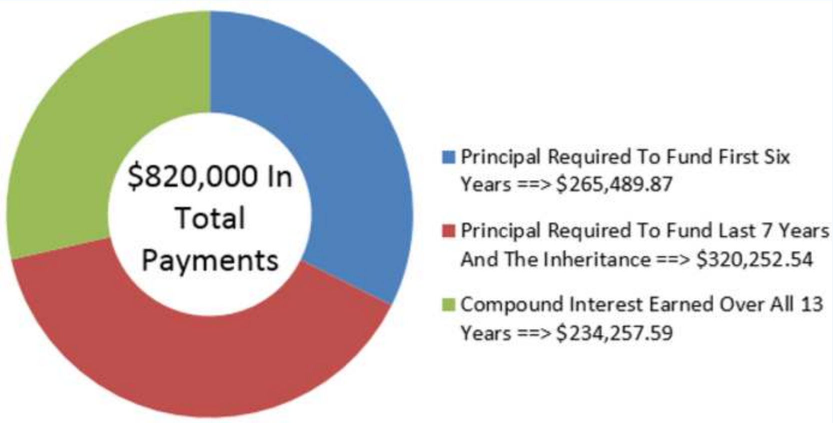
$$PV_{DUE2} = \$50,000\left[\frac{1-\left[\frac{1}{(1+0.0255)^{\frac{2}{1}}}\right]^{6}}{(1+0.0255)^{\frac{2}{1}}-1}\right] \times (1+0.0255)^{\frac{2}{1}} = \$265,489.8749$$

$$\text{Total } PV_2 = \$320,252.5426 + \$265,489.8749 = \$585,742.42$$

Calculator Instructions

Time Segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	7	5.1	Answer: -433,232.0352	60000	100000	1	2
2	✓	6	✓	Answer: -585,742.4175	50000	433,232.0352	✓	✓

The figure shows how much principal and interest make up the payments. To have his retirement income increased by $10,000 after six years, Rodriguez needs to have $585,742.42 invested in his retirement fund at age 65.



Working with Loans

At the start of this chapter, you purchased your first home and started your $150,000 mortgage at 5% compounded semi-annually. Assume your mortgage has a five-year term. Before that term expires, you have to start shopping around at different financial institutions to find the best rate for your next mortgage term. These other financial institutions will have one burning question: How much do you still owe on your mortgage and, thus, how much do you need to borrow from them?

Up to this point, this chapter has addressed only the concept of investment annuities. But what about debts? All annuity concepts also apply to the borrowing of money. When you work with loans, both future value and present value calculations may be required, which is why this topic has been delayed to this point.

As a consumer, you are probably most interested in the balance owing on any of your debts at any given point. Today's technology has made it easy to know your current balance by visiting your online bank account; however, the bank account does not assist you in identifying your future balance at a given point in time. To figure this out, you need annuity calculations.

Similarly, businesses apply annuity calculations all the time. Ongoing financial reporting has to be accurate. To provide insight into the company's true financial health, balance sheets need to reflect not only monies payable or receivable today, but also all future cash flows such as those arising from annuities. The purchase and sale of business contracts, such as the sale of a consumer payment plan to a financial institution, requires working with future payments and discounting those payments to the contract's date of sale.

In this section, you will calculate loan balances at any given point in time throughout the loan's term. As well, you will explore how to buy and sell loan contracts.

The Balance Owing on Any Loan Contract

To determine accurately the balance owing on any loan at any point in time, always start with the loan's starting principal and then deduct the payments made. This means a future value calculation using the loan's interest rate.

Some may ask why they can't figure out the loan balance by starting at the end of the loan (where the future value is zero, since no balance remains) and calculating a present value of the outstanding payments? The answer is because the annuity payment PMT is almost always a rounded number (this characteristic is explored in greater depth in Section 11.4, when you learn how to calculate the annuity payment). Every payment therefore has a slight discrepancy from its true value, which accumulates with each subsequent payment. For example, assume you calculate your payment mathematically as $500.0045. Since you can't pay more than two decimals, your actual payments are $500.00. The $0.0045 is dropped, which means every payment you make is $0.0045 short of what is mathematically required. If you make 100 such payments, the $0.0045 error accumulates to $0.45 plus interest. This means your very last payment must be increased by $0.45 or more to pay off your loan. Thus, the last payment is a different amount than every other payment.

To complicate matters further, the last payment amount may be unknown and incalculable, particularly if interest rates are variable. You can't calculate a present value from an unknown number nor can you use an annuity formula where a payment is in a different amount. Chapter 13 provides much more detail about these concepts of loan payments, loan balances, and final payment differences. For now, you can conclude that an accurate calculation of a loan balance is achieved through a future value annuity formula.

The Formula

Loans are most commonly ordinary annuities requiring the application of Formula 11.2 (ordinary annuity future value) to calculate the future balance, FV_{ORD}. This is the basic assumption in performing loan calculations unless otherwise specified. In the rare instance of a loan structured as an annuity due, you apply Formula 11.3 (annuity due future value) to calculate the future value, \(FV_{DUE}\).

Calculating the total amount of interest paid on a loan (in whole or for any time segment) once again requires the adaptation of Formula 8.3 ($I = S – P = FV - PV$), where:

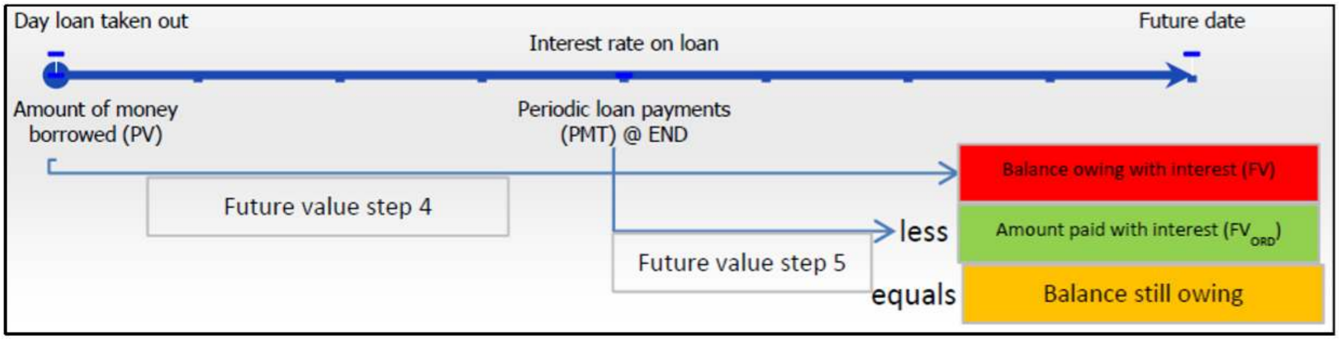
- The future value (FV) term in the formula represents the total principal and interest combined. In loan annuities, the annuity payment incorporates both of these elements. As well, any future principal remaining at the end of the loan, or a future balance outstanding, must also be factored into the calculation. Hence, the FV at any time interval in the formula is expanded to include both of these elements and replaced by $N \times PMT + FV$.
- The present value (PV) in the formula is what you started with. It is the opening amount of the loan. Therefore, the adaptation of Formula 8.3 remains the same as previously discussed and is written as:

$$I = (N \times PMT + FV) - PV$$

.

How It Works

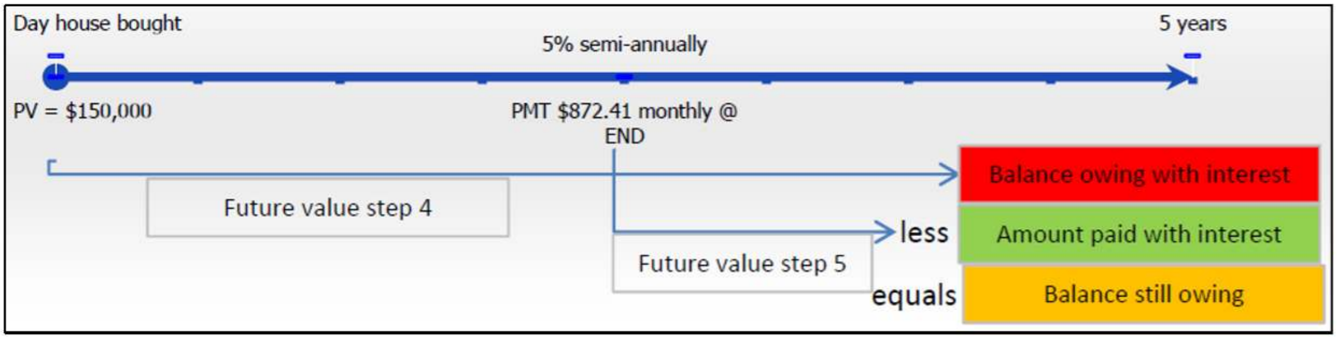
Solving for a future loan balance is a future value annuity calculation. Therefore, you use the same steps as discussed in Section 11.2. However, you need to modify your interpretation of these steps for loan balances. The figure below helps you understand these differences.


Day loan taken out
Future date
Interest rate on loan
Amount of money borrowed (PV)
Periodic loan payments (PMT) @ END
Balance owing with interest (FV)
Future value step 4
less
Amount paid with interest (FV_ORD)
Future value step 5
equals
Balance still owing

- The principal of the loan forms the present value, or PV. In step 4, when you move the present value to the future, your answer (FV) represents the total amount owing on the loan with interest as if no payments had been made.
- In step 5, the future value of the annuity (FV_{ORD}) represents the total amount paid against the loan with interest. With both the FV and FV_{ORD} on the same focal date, the fundamental concept of the time value of money allows you to then take the FV and subtract the FV_{ORD} to produce the balance owing on the loan.

Let’s calculate what you still owe on your $150,000 new home after five years of making $872.41 monthly payments at 5% compounded semi-annually:

Step 1: The structure of your mortgage is depicted in the Figure above. Since payments are at the end of the period, and the payment interval and compounding frequency are different, you have an ordinary general annuity.


Day house bought
5 years
5% semi-annually
PV = $150,000
PMT $872.41 monthly @ END
Balance owing with interest
Future value step 4
less
Amount paid with interest
Future value step 5
equals
Balance still owing

Step 2: The known variables are PV = $150,000, IY = 5%, CY = 2, PMT = $872.41, PY = 12, and Years = 5.

Step 3: The periodic interest rate is i = 5% ÷ 2 = 2.5%.

Step 4: N = 2 × 5 = 10 compounds, FV = $150,000(1 + 0.025)10 = $192,012.6816

Step 5: N = 12 × 5 = 60 payments

$$FV_{ORD} = \$872.41\left[\frac{\left[(1+0.025)^{\frac{2}{12}}\right]^{60}-1}{(1+0.025)^{\frac{2}{12}}-1}\right] = \$59,251.59215$$

Therefore, if the unpaid loan is worth $192,012.6816 and your payments are worth $59,251.59215, then your balance still owing is $192,012.6816 − $59,251.59215 = $132,761.09. The amount of interest paid is I = (60 × $872.41 + $132,761.09) − $150,000.00 = $35,105.69. With payments totaling $872.41 × 60 = $52,344.60, that means only $17,238.91 actually went towards the principal!

Important Notes

Present Value Method of Arriving at a Balance Owing. In the rare circumstance where the final payment is exactly equal to all other annuity payments, you can arrive at the balance owing through a present value annuity calculation.

However, this strict condition must be met. In this instance, since you are starting at the end of the loan, the future value is always zero, so to bring all payments back to the focal date you only need Formula 11.4.

Your BAII+ Calculator. Proper application of the cash flow sign convention for the present value and annuity payment will automatically result in a future value that nets out the loan principal and the payments. Assuming you are the borrower, you enter the present value (PV) as a positive number since you are receiving the money. You enter the annuity payment (PMT) as a negative number since you are paying the money. When you calculate the future value (FV), it displays a negative number, indicating that it is a balance owing.

Exercise 11.3.2: Give It Some Thought

On any interest-bearing loan at any point in time, will the principal be reduced by an amount equal to the payments made, more than the payments made, or less than the payments made?

Answer

> The principal will be reduced by an amount less than the payments. A portion of the payments always goes toward the interest that is being charged on the loan.

Example 11.3.4: Balance Owing on a New Truck

Two years ago, Jillian purchased a new Ford F-250 for $71,482.08 with a $5,000 down payment and the remainder financed through her Ford dealership at 5.9% compounded monthly. She has been making monthly payments of $1,282.20. What is her balance owing today? How much interest has she paid to date?

Solution

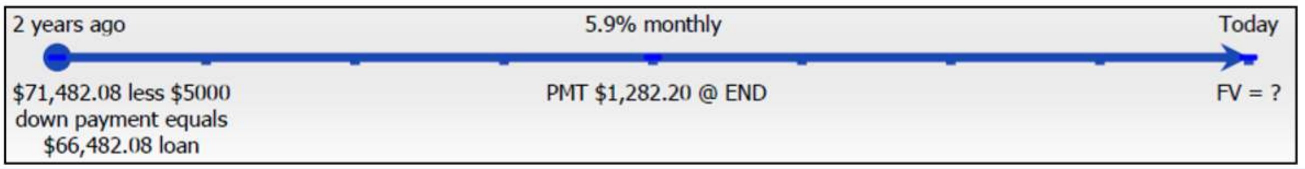
Step 1:

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple ordinary annuity. Calculate its value two years after its start, which is its future value, or FV_{ORD}. Once you know the FV_{ORD}, you can determine the amount of interest, or I.

What You Already Know

Step 1 (continued):

The timeline for the savings annuity appears below.



Step 2:

PV = $71,482.08 − $5,000 = $66,482.08, IY = 5.9%, CY = 12, PMT = $1,282.20, PY = 12, Years = 2

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Apply Formula 9.2 and Formula 9.3.

Step 5:

Apply Formula 11.1 and Formula 11.2. The final future value is the difference between the answers to step 4 and step 5.

Step 6:

To calculate the interest, apply the adapted Formula 8.3, where $I=(N \times PMT+FV)-PV$.

Step 3:

$$i=5.9\% \div 12=0.491\overline{6}\%$$

Step 4:

$N=12 \times 2=24$ compounds

$$FV=\$66,482.08(1+0.004916\overline{6})^{24}=\$74,786.94231$$

Step 5:

$N=12 \times 2=24$ payments

$$FV_{ORD}=\$1,282.20\left[\frac{\left[(1+0.00491\overline{6})^{\frac{12}{12}}\right]^{24}-1}{(1+0.00491\overline{6})^{\frac{12}{12}}-1}\right]=\$32,577.13179$$

$$FV=\$74,786.94231-\$32,577.13179=\$42,209.81$$

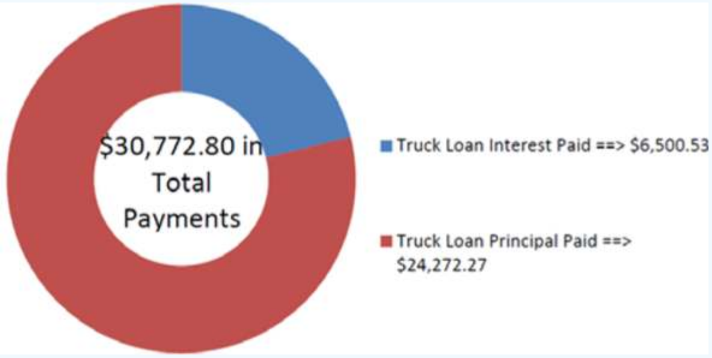
Step 6:

$$\begin{aligned}I&=(24 \times \$1,282.20+\$42,209.81)-\$66,482.08\\&=\$72,982.61-\$66,482.08\\&=\$6,500.53\end{aligned}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	5.9	66482.08	-1282.2	Answer: -42,209.81052	12	12

The figure shows how much principal and interest make up the payments. After two years of making monthly payments, Jillian has a balance owing on the Ford F-250 of $42,209.81. Altogether, she has made $30,772.80 in payments, of which $6,500.53 went toward the interest on her loan.

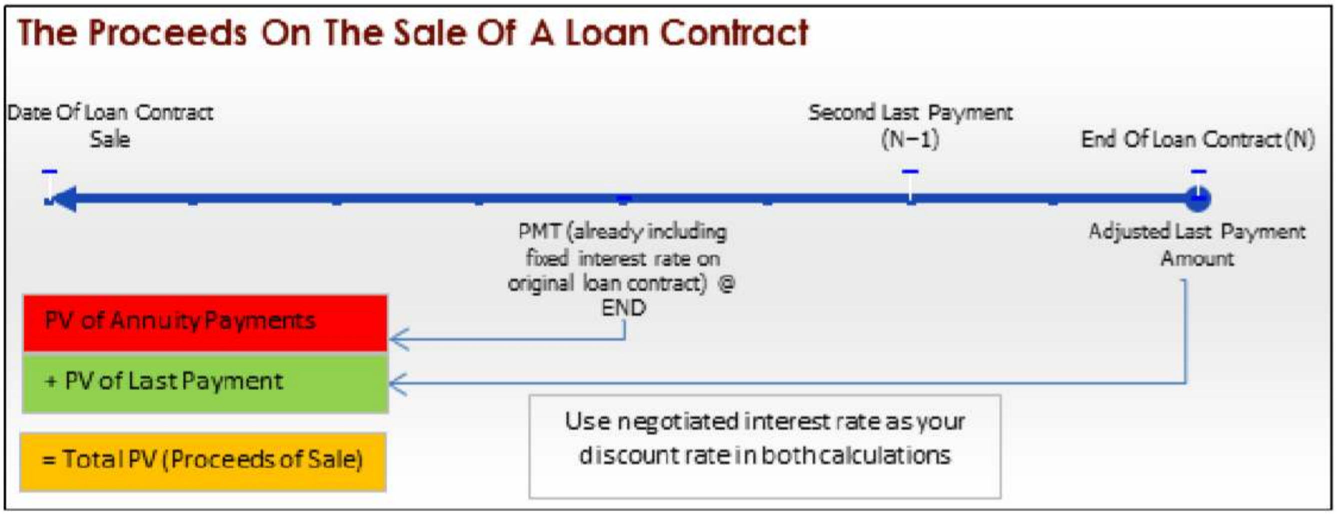


Selling a Loan Contract

It is common for loan contracts to be sold from retailers to financial institutions. For example, when a consumer makes a purchase from Sleep Country Canada on its payment plan, the financing is actually performed through its partner Citi Financial. In most retail situations, this would then mean that Sleep Country Canada receives the money right away by selling the contract to Citi Financial, whereas Citi Financial assumes the financial responsibility of collecting the payments.

When a finance company purchases a loan contract from another organization, it is essentially investing in the future payments of the loan contract. The two companies commonly agree on a profitable interest rate for the finance company and use it to determine the amount, known as the sale price, paid by the finance company to the other organization to purchase the contract. This textbook covers only fixed interest rate calculations with known final payment amounts.

Previously, it was discussed how the last payment in a loan almost always differs from every other payment in the annuity because of the rounding discrepancy in the annuity payment amount. Thus, the selling of a loan contract needs to calculate the present value of all remaining annuity payments (except for the last one) plus the present value of the adjusted single final payment as shown in this figure.


The Proceeds On The Sale Of A Loan Contract
Date Of Loan Contract Sale
Second Last Payment (N−1)
End Of Loan Contract (N)
PMT (already including fixed interest rate on original loan contract) @ END
Adjusted Last Payment Amount
PV of Annuity Payments
+ PV of Last Payment
= Total PV (Proceeds of Sale)
Use negotiated interest rate as your discount rate in both calculations

The Formula

As evident in the figure, two calculations are required. The first involves a present value annuity calculation using Formula 11.4. Note that the annuity stops one payment short of the end of the loan contract, so you need to use $N-1$ rather than N. The second calculation involves a present-value single payment calculation at a fixed rate using Formula 9.3 rearranged for PV. Thus, no new formulas are required to complete this calculation.

How It Works

The steps involved in selling any loan contract are almost identical to any present value annuity calculation with only minor differences as noted below.

Step 1: No changes.

Step 2: Identify the known variables, including PMT, PY, and Years, along with the newly negotiated IY and CY. Also identify the amount of the last payment, which is the FV.

Step 3: Use Formula 9.1 to calculate i.

Step 4: The last payment is the FV, which you treat like a single payment. Apply Formula 9.2 to determine N since this step is not an annuity calculation. Move the future value to the beginning of the time segment using Formula 9.3, rearranging for PV.

Step 5: Use Formula 11.1 to calculate N and subtract 1 to remove the final payment (since it is accounted for in step 4). Apply Formula 11.4 (or Formula 11.5 if it is an annuity due) to calculate the present value. Add both of the present values from steps 4 and 5 together to arrive at the total present value, which is known as the total proceeds of the sale.

✓ Example $11.3.5$: Ford Sells the Truck Contract

Continuing with Jillian's Ford F-250 purchase, recall that Jillian's monthly payments are fixed at $1,282.20 for five years. Assume that after two years Ford wants to sell the contract to another finance company, which agrees to a discount rate of 10.8% compounded semi-annually. Jillian's final payment is known at $1,282.49. What are the proceeds of the sale?

Solution

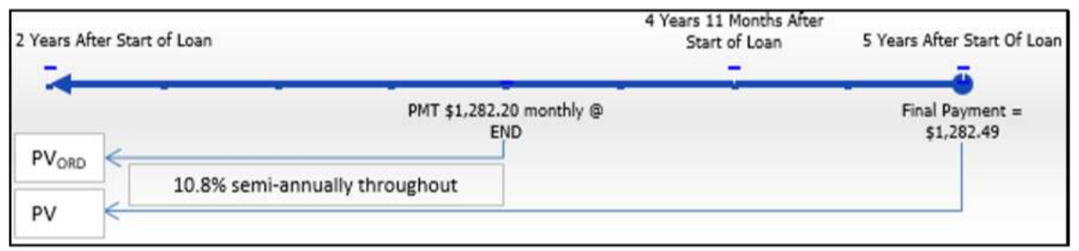
Step 1:

The payments are made at the end of the payment intervals, and the compounding period (semi-annually) and payment intervals (monthly) are different. Therefore, this is an ordinary general annuity. Calculate its value on the date of sale, which is its present value, or PV_{DUE}, plus the present value of the final payment, or PV.

What You Already Know

Step 1 (continued):

The timeline for the sale of the loan contract appears below.


2 Years After Start of Loan
4 Years 11 Months After Start of Loan
5 Years After Start Of Loan
PMT $1,282.20 monthly @ END
Final Payment = $1,282.49
PVORD
PV
10.8% semi-annually throughout

Step 2: FV = $1,282.49, IY = 10.8%, CY = 2, PMT = $1,282.20, PY = 12, Years = 3

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

For the final payment amount, apply Formula 9.2 and Formula 9.3, rearranging for PV.

Step 5:

For the annuity, apply Formula 11.1 (taking away the last payment) and Formula 11.4.

Perform

Step 3:

$$i=10.8\%\div 2=5.4\%$$

Step 4:

$N=2\times 3=6$ compounds

$$\$1,282.49=PV(1+0.054)^{6}$$

$$PV=\$1,282.49\div 1.054^{6}=\$935.427906$$

Step 5:

$N=12\times 3-1=35$ payments

$$PV_{ORD}=\$1,282.20\left[\frac{1-\left[\frac{1}{(1+0.054)^{\frac{2}{12}}}\right]^{35}}{(1+0.054)^{\frac{2}{12}}-1}\right]=\$38,477.10711$$

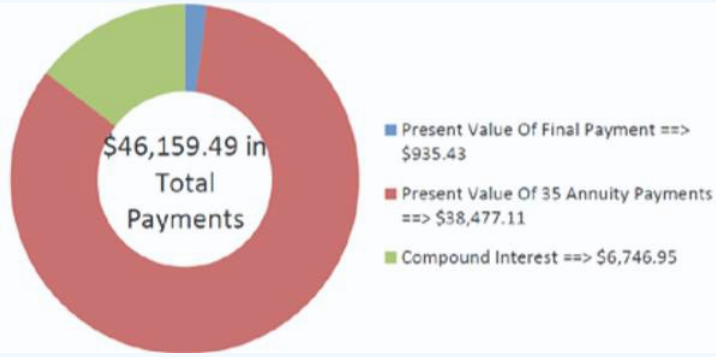
$$PV=\$935.427906+\$38,477.10711=\$39,412.54$$

Calculator Instructions

Element	N	I/Y	PV	PMT	FV	P/Y	C/Y
Final Payment	6	10.8	Answer: -935.427906	0	1282.49	2	2

Element	N	I/Y	PV	PMT	FV	P/Y	C/Y
Annuity	35	√	Answer: -38,477.10711	1282.2	0	12	2

The figure shows the present value and interest amounts in the transaction. The finance company will pay $39,412.54 for the contract. In return, it receives 35 payments of $1,282.20 and one payment of $1,282.49 for a nominal total of $46,159.49.



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11.4: Annuity Payment Amounts

Whether you are acquiring merchandise and property or saving up toward some future goal, you will deal with annuity payments. When you graduate college and land that promising entry-level position with your employer, a lot of demands are going to get placed on your limited income. If you do not already own a place of your own, perhaps you will get one. This means the purchase of a starter home for which you will make monthly mortgage payments. To fill that home, acquiring some furniture and electronics might take you to The Brick, Sleep Country Canada, Best Buy, or The Home Depot. Then you may be staggered by all the home maintenance items you need. If you make a lot of purchases all at once, you will probably take advantage of various payment plans. These place even more demands on your monthly income. Do not forget that you will need some wheels too. You can either lease or purchase a car. Great, another payment to make! Finally, you remember what your math instructor taught you about the importance of getting started early on your RRSP, so you should begin making those monthly contributions soon, too.

Will you have enough income to cover all of your payments? Unlike the many consumers who must rely on retailers and banks to figure out their payments, your study of annuity payments will allow you to calculate the amounts yourself.

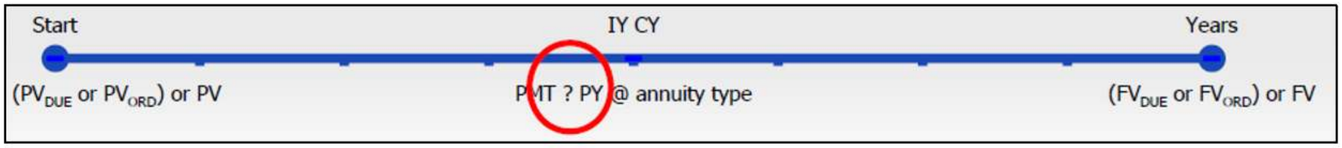
Businesses also make annuity payments for a wide variety of purposes. Whether saving up for future corporate goals or acquiring products and property, businesses have regular bills, too. Marketers develop payment plans for their consumers. Financial agents make investments involving periodic payments. Companies issue marketable bonds that require regular interest payments to investors. Human resource personnel look after employee benefits, including RRSP contributions and pension plan payments. Production departments need expensive machinery, so they must keep payment plans within operating budgets. No matter your choice of profession, as a business manager you will encounter annuity payment calculations.

Ordinary Annuities and Annuities Due

You need to calculate an annuity payment in many situations:

- Figuring out loan or mortgage payments
- Determining membership or product payment plans
- Calculating lease payments
- Determining the periodic payment necessary to achieve a savings goal
- Determining the maximum payment that an investment annuity can sustain over a period of time

A typical timeline for solving for an annuity payment amount appears in the next figure. All variables except for the payment amount (PMT) are known. While the figure shows all the possible variables and their location on the timeline, the following should be noted at each end of the timeline:


Start
IY CY
Years
(PV$_{DUE}$ or PV$_{ORD}$) or PV
PMT ? PY @ annuity type
(FV$_{DUE}$ or FV$_{ORD}$) or FV

- If the PV_{DUE} or PV_{ORD} is known on the left, then FV is the only variable that may appear on the right. FV_{DUE}, FV_{ORD}, and PV are variables that will not appear on the timeline.
- If the FV_{DUE} or FV_{ORD} is known on the right, then PV is the only variable that may appear on the left. PV_{DUE}, PV_{ORD}, and FV are variables that will not appear on the timeline.

The Formula

Recall that the annuity payment amount, PMT, is one of the variables in Formula 11.2, Formula 11.3, Formula 11.4, and Formula 11.5. Calculating this amount then requires you to substitute the known variables and rearrange the formula for PMT. The most difficult part of this process is figuring out which of the four formulas to use. Your selection depends on the two factors summarized in this table and discussed afterwards.

Formula To Use	Annuity Payment Timing	Annuity Value Known
11.2	End	FV_{ORD}
11.4	End	PV_{ORD}
11.3	Beginning	FV_{DUE}
11.5	Beginning	PV_{DUE}

1. Are the annuity payments to be made at the beginning or at the end of the payment interval? In other words, do you have an annuity due or an ordinary annuity?
2. Do you know the amount that the annuity starts or ends with? In other words, do you know the present value or future value of the annuity?

How It Works

Follow these steps to solve for any annuity payment amount:

- **Step 1**: Identify the annuity type. Draw a timeline to visualize the question.
- **Step 2**: Identify the variables that you know, including IY, CY, PY, and Years. You must also identify a value for one of PV_{ORD}, PV_{DUE}, FV_{ORD}, or FV_{DUE}. You may or may not have a value for FV or PV.
- **Step 3**: Use Formula 9.1 to calculate i.
- **Step 4**: If a single payment PV or FV is known, move it to the other end of the time segment using Formula 9.3. To determine N, apply Formula 9.2 since this is for a single payment, not an annuity. When you move the amount to the same focal date as the present or future value of the annuity, either add this number to the annuity value or subtract it as the situation demands. Example 11.4.3 later in this section will illustrate this practice.
- **Step 5**: Use Formula 11.1 to calculate N. Apply the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formula 11.2, Formula 11.3, Formula 11.4, or Formula 11.5 then rearrange for the annuity payment amount, PMT. If you performed step 4 above, be sure to use the adjusted future or present value of your annuity in the formula.

Important Notes

When calculating loan payments, recall that the last payment is slightly different from all other payments. Chapter 13 explores how to calculate the last payment precisely. For the purposes of this section, treat the last payment like any other payment and assume it is equal to all the other payments when you make any statement about loan payment amounts or totals.

Things To Watch Out For

Should you grow or shrink a balance? Give some thought to the relationships between the present value, annuity payments, and future value.

1. If somebody is contributing to an investment, the future value of the annuity should be larger than the total of all annuity payments.
2. If somebody is receiving from an investment or making debt payments, the total of all annuity payments should exceed the present value.

? Exercise 11.4.1: Give It Some Thought

a. An investor has been making $1,000 annual contributions to his account for five years. In this problem, should the future value be more than, less than, or equal to $5,000?
b. A debtor needs to make $1,000 annual payments on her loan for five years to extinguish her debt. In this problem, should the present value be more than, less than, or equal to $5,000?
c. A retirement fund can make five annual payments of $1,000 before being extinguished. In this problem, should the present value be more than, less than, or equal to $5,000?

Answer

a. The future value will be more than $5,000. The payments earn interest.

b. The present value will be less than \$5,000. The payments represent the principal and interest together.
c. The present value will be less than \$5,000. The principal earns interest while paying out.

✓ Example 11.4.1: Payments on a Loan

Morgan wants to consolidate a lot of smaller debts into a single three-year loan for \$25,000. If the loan is charged interest at 7.8% compounded monthly, what is her payment amount at the end of every month?

Solution

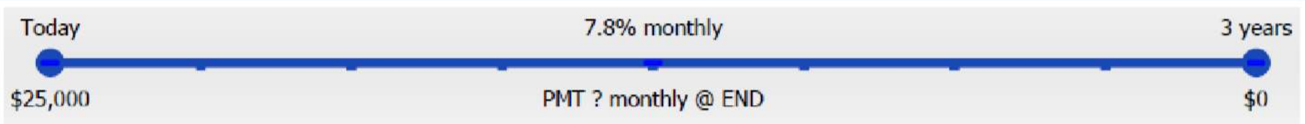
Step 1:

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount of her loan payment, or PMT.

What You Already Know

Step 1 (continued):

The timeline for the loan appears below.



Step 2:

PV_{ORD} = \$25,000, IY = 7.8%, CY = 12, PY = 12, Years = 3, FV = \$0

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Since FV = \$0, skip this step.

Step 5:

Apply Formula 11.1 and Formula 11.4, rearranging for PMT.

Perform

Step 3:

$$i = 7.8\% \div 12 = 0.65\%$$

Step 5:

$N = 12 \times 3 = 36$ payments

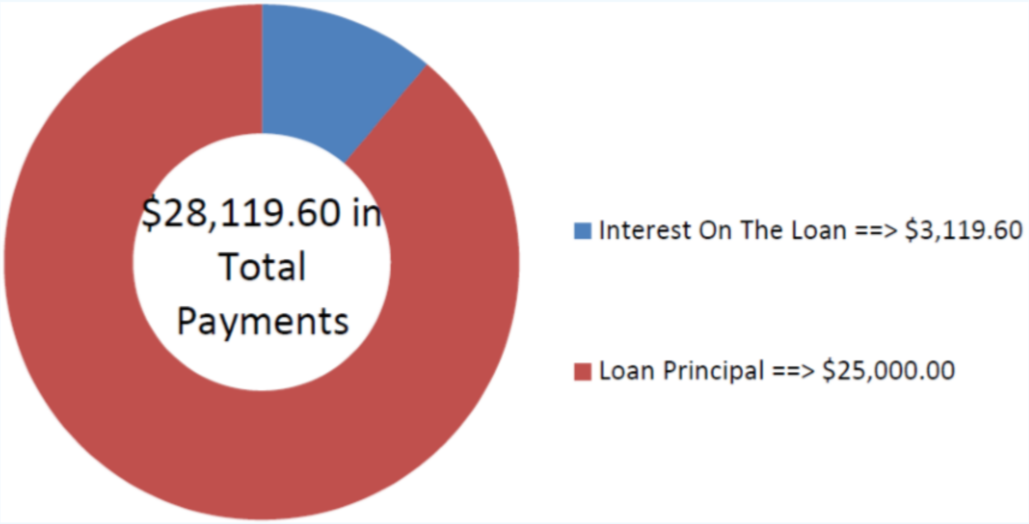
$$\$25,000 = PMT\left[\frac{1-\left[\frac{1}{(1+0.0065)^{\frac{12}{12}}}\right]^{36}}{(1+0.0065)^{\frac{12}{12}}-1}\right]$$

$$PMT = \frac{\$25,000}{32.005957} = \$781.10$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
36	7.8	25000	Answer: -781.104587	0	12	12

The figure shows how much principal and interest make up the payments. To pay off her consolidated loan, Morgan's month-end payments for the next three years will be $781.10.



✓ Example $11.4.2$: Funding a Backpack Trip across Europe

Franco has placed $10,000 into an investment fund with the goal of receiving equal amounts at the beginning of every month for the next year while he backpacks across Europe. If the investment fund can earn 5.25% compounded quarterly, how much money can Franco expect to receive each month?

Solution

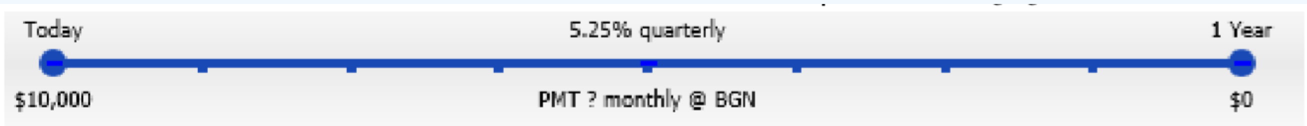
Step 1:

The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount he can receive, or PMT.

What You Already Know

Step 1 (continued):

The timeline for the vacation money appears below.



Step 2:

PV_{DUE} = $10,000, IY= 5.25%, CY= 4, PY= 12, Years=1, FV = $0

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Since FV = $0, skip this step.

Step 5:

Apply Formula 11.1 and Formula 11.3, rearranging for PMT.

Perform

Step 3:

$$i = 5.25\% \div 4 = 1.3125\%$$

Step 5:

$N = 12 \times 1 = 12$ payments

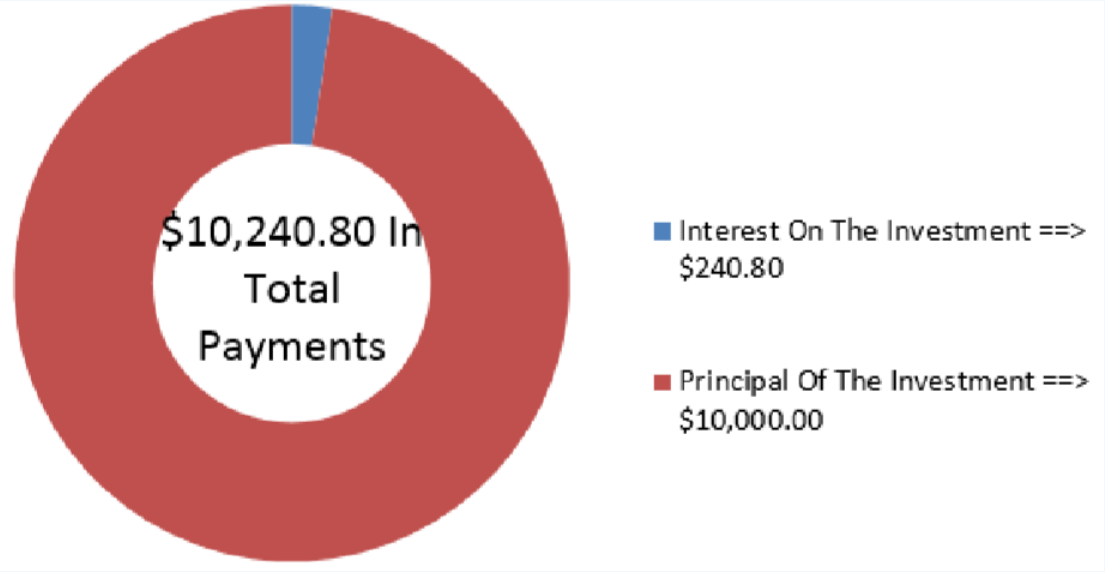
$$\$10,000 = PMT\left[\frac{1-\left[\frac{1}{(1+0.013125)^{\frac{4}{12}}}\right]^{12}}{(1+0.013125)^{\frac{4}{12}}-1}\right] \times (1+0.013125)^{\frac{4}{12}}$$

$$PMT = \frac{\$10,000}{11.717849} = \$853.40$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	12	5.25	-10000	Answer: 853.398928	0	12	4

The figure shows how much principal and interest make up the payments. While backpacking across Europe, Franco will have his annuity pay him $853.40 at the beginning of every month.



✓ Example 11.4.3: Planning RRSP Contributions

Kingsley's financial adviser has determined that when he reaches age 65, he will need $1.7 million in his RRSP to fund his retirement. Kingsley is currently 22 years old and has saved up $10,000 already. His adviser thinks that his RRSP will average 9% compounded annually throughout the years. To meet his RRSP goal, how much does Kingsley need to invest every month starting today?

Solution

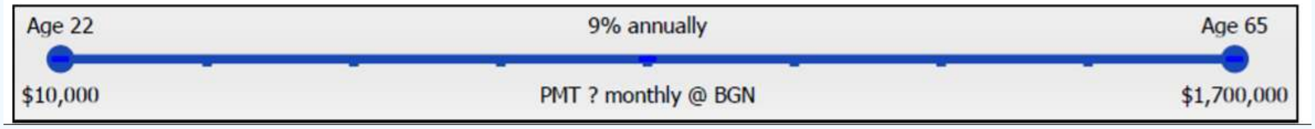
Step 1:

The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount Kingsley needs to contribute, or PMT.

What You Already Know

Step 1 (continued):

The timeline for Kingsley’s RRSP contributions appears below.



Step 2:

PV = \$10,000, FV_{DUE} = \$1,700,000, IY = 9%, CY = 1, PY = 12, Years = 43

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

You need to move the present value to Kingsley's age 65, the same date as FV_{DUE}. Apply Formula 9.2 and Formula 9.3. The FV is money the annuity does not have to save, so you subtract it from FV_{DUE} to arrive at the amount the annuity must generate.

Step 5:

Apply Formula 11.1 and Formula 11.3, rearranging for PMT, using the adjusted FV_{DUE}.

Step 3:

$$i = 9\% \div 1 = 9\%$$

Step 4:

$N = 1 \times 43 = 43$ compounds

$$FV = \$10,000(1+0.09)^{43} = \$406,761.0984$$

$$\text{New } FV_{DUE} = \$1,700,000 - \$406,761.0984 = \$1,293,238.902$$

Step 5:

$N = 12 \times 43 = 516$ payments

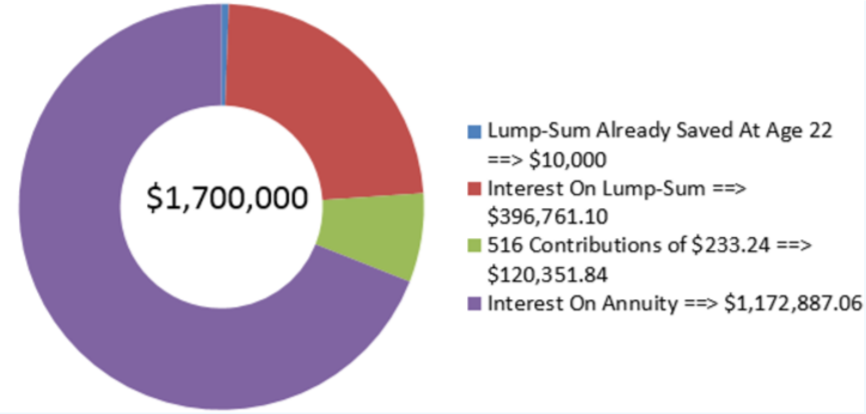
$$\$1,293,238.902 = PMT\left[\frac{\left((1+0.09)^{\frac{1}{12}}\right]^{516}-1}{(1+0.09)^{\frac{1}{12}}-1}\right) \times (1+0.09)^{\frac{1}{12}}$$

$$PMT = \frac{\$1,293,238.902}{5,544.647665} = \$233.24$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	516	9	-10000	Answer: -233.240952	1700000	12	1

The figure shows how much principal and interest make up the final balance. To meet his retirement goals, Kingsley needs to invest \$233.24 at the beginning of every month for the next 43 years. In doing so, he will achieve a \$1.7 million balance in his account at age 65. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly higher amount since the annuity payment was rounded downwards. However, the last payment is treated equally at this time for purposes of all calculations.)



✓ Example 11.4.4: Purchasing New Production Line Machinery

The production department just informed the finance department that in five years' time the robotic systems on the production line will need to be replaced. The estimated cost of the replacement is $10 million. To prepare for this purchase, the finance department immediately deposits $1,000,000 into a savings annuity earning 6.15% compounded semi-annually, and it plans to make semi-annual contributions starting in six months. How large do those contributions need to be?

Solution

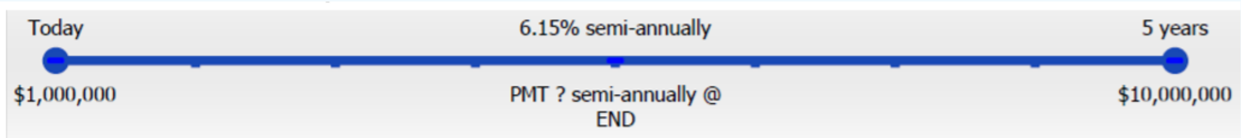
Step 1:

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount the finance department needs to contribute, or PMT.

What You Already Know

Step 1 (continued):

The timeline for the machinery fund appears below.



Step 2:

PV = $1,000,000, FV_{ORD} = $10,000,000, IY= 6.15%, CY= 2, PY= 2, Years= 5

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

The present value must be moved to the five-year date, the same date as FV_{ORD}. Apply Formula 9.2 and Formula 9.3.This FV is money the annuity does not have to save, so it is subtracted from FV_{ORD} to arrive at the amount the annuity must generate.

Step 5:

Apply Formula 11.1 and Formula 11.2, rearranging for PMT, using the adjusted FV_{ORD}.

Step 3:

$$i = 6.15\% \div 2 = 3.075\%$$

Step 4:

$N = 2 \times 5 = 10$ compounds

$$FV = \$1,000,000(1+0.03075)^{10} = \$1,353,734.306$$

$$\text{New } FV_{ORD} = \$10,000,000 - \$1,353,734.306 = \$8,646,265.694$$

Step 5:

$N = 2 \times 5 = 10$ payments

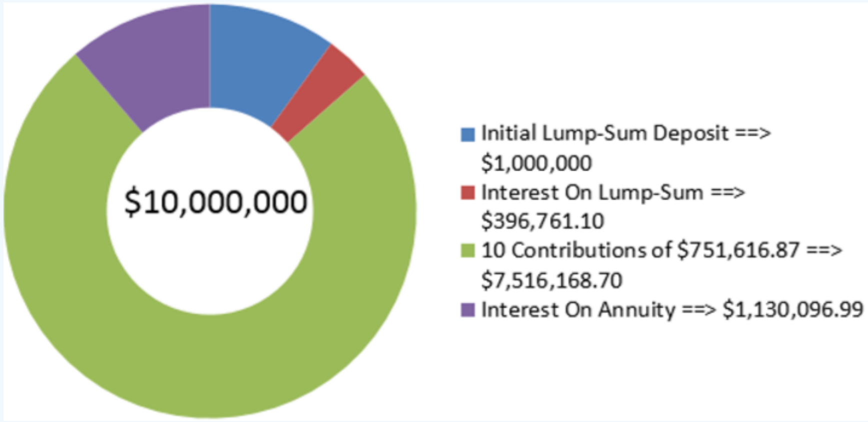
$$\$8,646,265.694 = PMT\left[\frac{\left[(1+0.03075)^{\frac{2}{2}}\right]^{10}-1}{(1+0.03075)^{\frac{2}{2}}-1}\right]$$

$$PMT = \frac{\$8,646,265.694}{11.503554} = \$751,616.87$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
10	6.15	-1000000	Answer: -751,616.8656	10000000	2	2

The figure shows how much principal and interest make up the final balance. To have adequate funding for the production line machinery replacement five years from now, the finance department needs to deposit $751,616.87 into the fund every six months. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly lower amount since the annuity payment was rounded upwards. However, the last payment is treated equally at this time for purposes of all calculations.)



Contributors and Attributions

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11.5: Number Of Annuity Payments

How long do you require to fulfill the goal of your annuity? It all depends on your annuity payment, interest rate, and the amount of money involved.

When saving up for future goals, many people and businesses simply determine how much they can afford to invest each time period and then try to be patient until they meet their savings goal. What they do not know is how long it will take them. If you can put $75 per month into your vacation fund, how long will it take to save up the $1,000 needed for a spring break vacation in Puerto Vallarta?

Credit cards require a small minimum payment each month. A lot of Canadians view this minimum payment as a benefit and pay it without understanding what their decision implies. But interest rates on credit cards are around 21% compounded daily! So you can hardly afford to delay in paying off your credit card debt. Yet if you always make only the minimum monthly payment, extinguishing a $5,000 balance will take approximately 50 years!

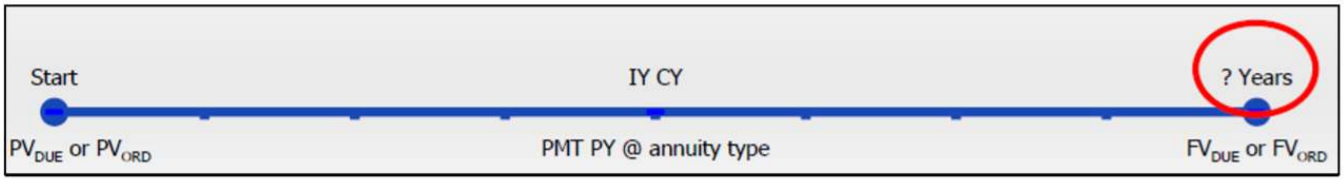
Evidently the number of annuity payments is critical to financial transactions. In this section, you calculate the term of annuities by figuring out the number of annuity payments required.

Ordinary Annuities and Annuities Due

You must calculate the number of annuity payments in a variety of scenarios:

- Savings planning
- Debt extinguishment
- Sustaining withdrawals from an investment

A typical timeline used in calculating the term of an annuity appears below. Recall that N is not illustrated in the timeline but is calculated through Formula 11.1 using Years and PY, which do appear on the timeline. The payment frequency, PY, is a choice determined by the parties involved in the transaction and therefore is always known. That leaves Years, or the term of the annuity, as the unknown variable on the timeline. If you calculate N then you can also calculate Years using a rearrangement of Formula 11.1.



At either end of the timeline, only one of PV_{DUE}, PV_{ORD}, FV_{DUE}, or FV_{ORD} will be known.

The Formula

Recall that the number of annuity payments, N, is one of the variables in Formula 11.2, Formula 11.3, Formula 11.4, and Formula 11.5. Calculating this amount then requires you to substitute the known variables and rearrange the formula to solve for N. Since N is located in the exponent, the rearrangement and isolation demands the usage of natural logarithms (see Section 2.6). The most difficult part is figuring out which formula you need to use. Choose the formula using the same decision criteria explained in Section 11.4 (under "The Formula").

How It Works

Follow these steps, to solve for the number of annuity payments or the annuity term:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that always appear, including PMT, IY, CY, and PY. You must also identify one of the known values of PV_{ORD}, PV_{DUE}, FV_{ORD}, or FV_{DUE}.

Step 3: Use Formula 9.1 to calculate i.

Step 4: Substitute into the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formula 11.2, Formula 11.3, Formula 11.4, or Formula 11.5. Rearrange and solve for N.

Step 5: To convert N back to a more commonly expressed format, such as years and months, take Formula 11.1 and rearrange it for Years. If Years is a decimal number, recall the steps required for converting to common language from "Integer Compounding Periods" on page xx in Section 9.7.

Important Notes

Dealing with Decimals in N

At the end of step 4, the calculated value of N may have decimals. Start by applying the same rounding rules involving the computation of N for single payments, mentally rounding the decimal off to the third decimal place.

1. If the three decimals are zeroes, then the decimals are most likely a result of a rounded FV, PV, or PMT, so treat the N like an integer (ignoring the decimals).
2. If the three decimals are not all zeroes, then N must always be rounded up to the next integer regardless of the decimal value. You must never round it down, because the calculated value of N represents the minimum payments required. The interpretation and implications of this rounding are as follows:
    - **Payments on a Debt**. Assume your loan payments are \$100 and you calculate N = 9.4, which rounds up to 10 payments. The payments must exactly reduce the loan to a balance of zero. The first nine payments are all \$100. The last payment is not a complete payment, hence the 0.4 in the calculation. The precise calculation of this last payment amount is explained in Chapter 13; however, at this point it is sufficient to treat the decimal as an approximate percentage of what the final payment might be. The 0.4 is approximately 40% of \$100, or \$40. As a result, there are nine \$100 payments, and one final payment approximating \$40. Keep in mind that this approximation is rough at best and should only be used to get a feel for what the final payment amount could be. The important point here is that it requires 10 payments to pay off the loan completely, regardless of the amount of the payment.
    - **Payments toward an Investment**. Keeping the numbers the same as above, your investment payments are \$100 with N = 9.4. Again, this is 10 payments. If you only make nine full payments of \$100, your investment will be short of your intended future value. However, if you let it grow and make the tenth full payment, you will have more than your investment goal. What the N = 9.4 is telling you is that a final tenth payment is needed roughly 40% of the way through the next payment interval that is approximately 0.4 of the regular payment amount. However, most investors don't care about having too much money saved and exceeding their savings goal, so usually a full tenth payment is made.

Things To Watch Out For

Two mistakes are common in calculations of the number of annuity payments:

1. **Misinterpreting N**. Confusing the compounding frequency (CY) and payment frequency (PY) results in the misinterpretation of N and the incorrect calculation of the term. Since N represents the number of annuity payments, to assign its meaning you need to look at the PY, not the CY. Then use the value of PY in Formula 11.1 when expressing N in more common terms. For example, if N = 8, PY = 4, and CY =2, a common mistake is to say that there are eight semi-annual payments (using the CY); ensure you look instead at the PY, identifying the eight quarterly payments, which means the term is two years.
2. **Confusing the Term and the Last Payment on Annuities Due**. One of the characteristics of an annuity due is that the last payment occurs one payment interval before the end of the term of the annuity. When you calculate N, you have calculated the term of the annuity. The last payment occurs $N-1$ intervals from the start of the annuity. For example, assume payments are monthly and you calculate $N=12$ months. This means the term of the annuity is 12 months. The last payment on the annuity due is 12 – 1 = 11 months from the start. In your “Plan” for any problem, be sure to recognize whether you are looking for the end of the term or when the last payment is made. This problem does not occur for ordinary annuities since the last payment and the end of the term are on the same date.

? Exercise 11.5.1: Give It Some Thought

Holding all other variables constant, what happens mathematically to the number of payments (the precise calculated value of N produced by rearranging any of the annuity formulas) if the following occur:

a. The variable interest rate increases (and the payment itself remains unchanged)?
b. The annuity payments are doubled?

Answer

a. The number of payments increases if the variable interest rate increases since more interest is charged but the payment has not increased to cover the higher interest charges.
b. The number of payments is normally cut at least by half if the annuity payments are doubled since with each payment the principal becomes smaller at an accelerated rate and less interest is charged.

✓ Example 11.5.1: How Long Until Retirement Savings Are Depleted?

Samia has $500,000 accumulated in her retirement savings when she decides to retire at age 60. If she wants to receive beginning-of-month payments of $3,000 and her retirement annuity can earn 5.2% compounded monthly, how old is Samia when the fund is depleted?

Solution

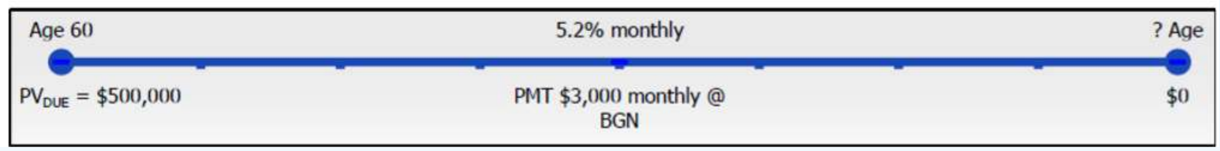
Step 1:

The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. In looking for how old Samia will be when the fund is depleted, calculate the number of annuity payments, or N, that her retirement annuity can sustain.

What You Already Know

Step 1 (continued):

The timeline for the retirement annuity appears below.



Step 2:

PV_{DUE} = $500,000, IY = 5.2%, CY = 12, PMT = $3,000, PY = 12, FV = 0

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Substitute into Formula 11.5, rearranging for N.

Step 5:

Substitute into Formula 11.1 and solve for Years. Add this term to Samia's current age to figure out how old she is when the annuity is depleted.

Perform

Step 3:

$$i = 5.2\% \div 12 = 0.4\overline{3}\%$$

Step 4:

$$\$500,000=\$3,000\left[\frac{1-\left[\frac{1}{(1+0.0043)^{\frac{12}{12}}}\right]^{N}}{(1+0.004\overline{3})^{\frac{12}{12}}-1}\right]\times(1+0.004\overline{3})^{\frac{12}{12}}$$

$$\begin{aligned}165.947560&=\frac{1-0.995685^{N}}{0.0043}\\0.995685^{N}&=0.280893\\N\times\ln(0.995685)&=\ln(0.280893)\\N&=\frac{-1.2697323}{-0.004323}\\N&=293.660180\end{aligned}$$

Rounding up to 294 monthly payments consisting of 293 regular payments plus one additional smaller payment.

Step 5:

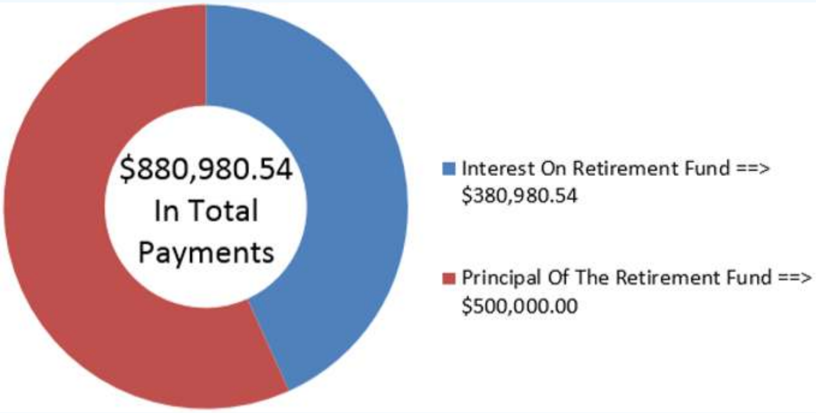
$$294=12\times\text{ Years}$$

$$\text{Years }=\frac{294}{12}=24.5=24\text{ years, }6\text{ months}$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	Answer: 293.660180	5.2	-500000	300	0	12	12

The figure shows how much principal and interest make up the payments. If Samia is currently 60 years old and the annuity endures for 24 years and six months, then she will be 84.5 years old when the annuity is depleted. Note that Samia will receive 293 payments of $3,000 along with a smaller final payment that is approximated by taking 66.018% × $3,000 = $1,980.54



✓ Example 11.5.2: How Long to Pay Off Your Car?

Brendan is purchasing a brand new Mazda MX-5 GT. Including all options, accessories, and fees, the total amount he needs to finance is $47,604.41 at the dealer's special interest financing of 2.4% compounded monthly. If he makes payments of $1,000 at the end of every month, how long will it take to pay off his car loan?

Solution

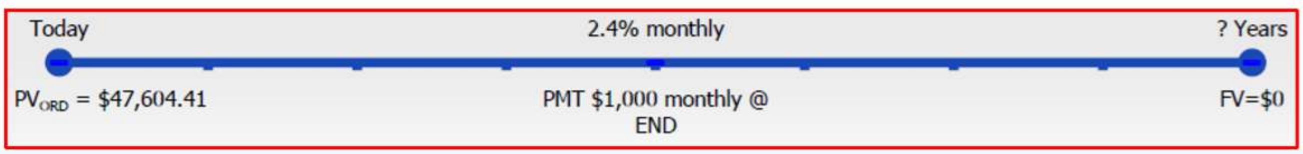
Step 1:

The payments are at the end of the payment intervals with a monthly compounding period and monthly payment intervals. Therefore, this is an ordinary simple annuity. Calculate the number of monthly payments, or N, to figure out the length of time required to pay off the loan.

What You Already Know

Step 1 (continued):

The timeline for the car payments appears below.



Step 2:

PV_{ORD} = \$47,604.41, IY = 2.4%, CY = 4, PMT = \$1,000, PY = 12, FV = 0

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Substitute into Formula 11.4, rearranging for N.

Step 5:

Substitute into Formula 11.1 and solve for Years.

Perform

Step 3:

$$i = 2.4\% \div 12 = 0.2\%$$

Step 4:

$$\begin{aligned} \$47,604.41 &= \$1,000\left[\frac{1-\left[\frac{1}{(1+0.002)^{\frac{12}{12}}}\right]^{N}}{(1+0.002)^{\frac{12}{12}-1}}\right] \\ 0.095208 &= 1-0.998003^{N} \\ 0.998003^{N} &= 0.904791 \\ N \times \ln(0.998003) &= \ln(0.904791) \\ N &= \frac{-0.100051}{-0.001998} \\ &= 50.075560 \end{aligned}$$

Rounded up to 51 monthly payments.

Step 5:

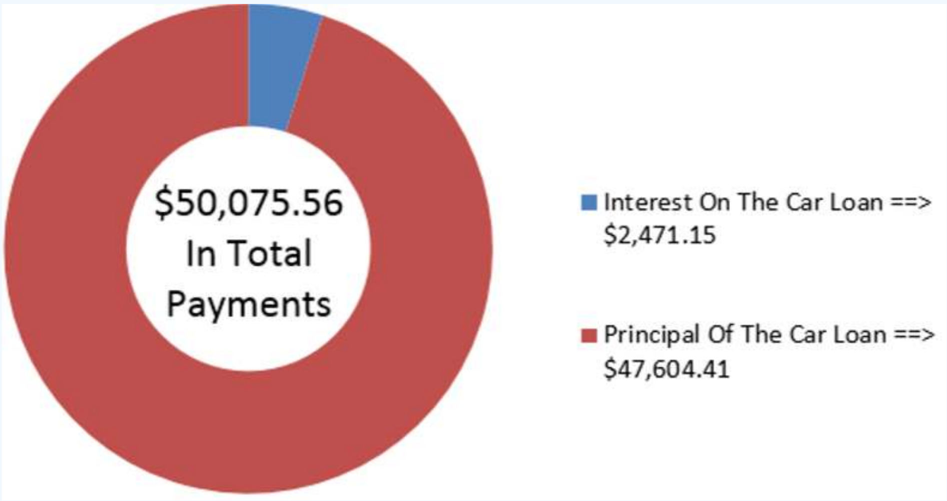
$$51 = 12 \times \text{ Years}$$

$$\text{Years } = \frac{51}{12} = 4.25 = 4 \text{ years, } 3 \text{ months}$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 50.075560	2.4	47604.41	-1000	0	12	12

The figure shows how much principal and interest make up the payments. To own his vehicle, Brendan will make payments for four-and-a-quarter years. This consists of 50 payments of $1,000 and a smaller final payment that is approximated by taking 7.556% × $1,000 = $75.56.



✓ Example 11.5.3: The Importance of the Annuity Type

Trevor wants to save up $1,000,000. He will contribute $5,000 annually to an investment earning 10% compounded monthly. What is the time difference between his last payments (not the end of the annuities) if he makes his contributions at the end of the year instead of at the beginning of the year?

Solution

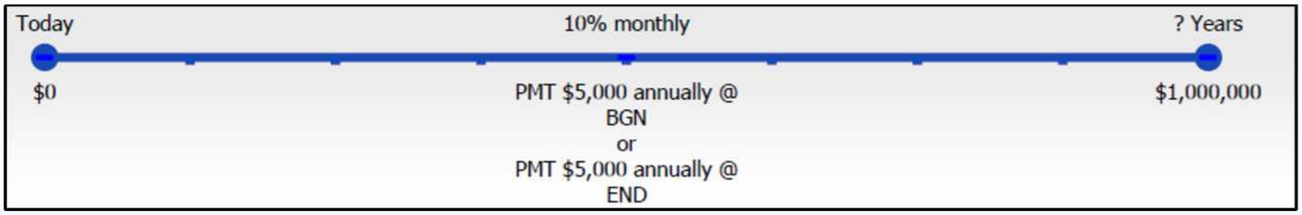
Step 1:

In this question you are being asked to compare two annuities that differ in their payment intervals and their compounding periods. One annuity makes contributions at the beginning of the interval, while the other makes contributions at the end. Therefore, you must contrast one general annuity due with one ordinary general annuity. To determine the time difference, calculate N for each annuity and compare when the last payment is made.

What You Already Know

Step 1 (continued):

A combined timeline for the two annuities appears below.



Step 2:

Ordinary general annuity: FV_{ORD} = $1,000,000, IY = 10%, CY = 12, PMT = $5,000, PY = 1

General annuity due: FV_{DUE} = $1,000,000, IY = 10%, CY = 12, PMT = $5,000, PY = 1

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Apply Formula 11.2 and Formula 11.3, rearranging for N.

Step 5:

If payments are annual, then N = Years. However, for the general annuity due the last payment is made $N - 1$ years from today. The difference between the two payments can then be calculated.

Step 3:

$$i = 10\% \div 12 = 0.8\overline{3}\%$$

Step 4:

Ordinary general annuity:

$$\begin{aligned}
\$1,000,000 &= \$5,000\left[\frac{\left[(1+0.008\overline{3})^{\frac{12}{1}}\right]^{N}-1}{(1+0.008\overline{3})^{\frac{12}{1}}-1}\right]\\
20.942613 &= 1.104713^{N}-1\\
21.942613 &= 1.104713^{N}\\
\ln(21.942613) &= N \times \ln(1.104713)\\
\frac{\ln(21.942613)}{\ln(1.104713)} &= N\\
N &= 31.012812,\ \text{or } 32 \text{ years}
\end{aligned}$$

General annuity due:

$$\begin{aligned}
\$1,000,000 &= \$5,000\left[\frac{\left[(1+0.008\overline{3})^{\frac{12}{1}}\right]^{N}-1}{(1+0.008\overline{3})^{\frac{12}{1}}-1}\right] \times (1+0.008\overline{3})^{\frac{12}{1}}\\
18.957514 &= 1.104713^{N}-1\\
19.957514 &= 1.104713^{N}\\
\ln(19.957514) &= N \times \ln(1.104713)\\
\frac{\ln(19.957514)}{\ln(1.104713)} &= N\\
N &= 30.060618,\ \text{or } 31 \text{ years}
\end{aligned}$$

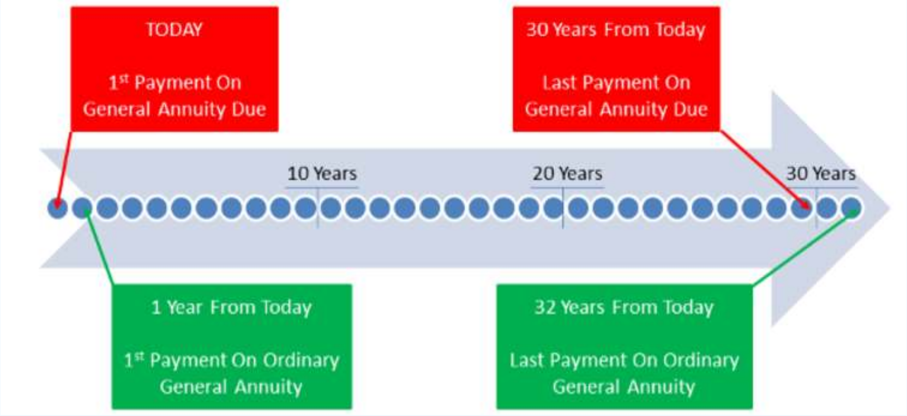
Step 5:

The ordinary general annuity last payment is 32 years from today. The general annuity due has a term of 31 years, but the last payment is 31 − 1 = 30 years from today. The difference between the last payments is 32 − 30 = 2 years sooner. (See the figure below and count the dots!)

Calculator Instructions

Type	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Ordinary		Answer: 30.060618	10	0	-5000	1000000	1	12
Due	BGN	Answer: 31.012812	√	√	√	√	√	√

In order for Trevor to reach his goal, if he were to make his $5,000 contributions at the beginning of the year (that is, under the annuity due) instead of the end of the year (under the ordinary annuity), his last payment would be two years sooner. Do not confuse this with the terms of the two annuities, which end only one year apart (31 years from now and 32 years from now).


TODAY
1st Payment On General Annuity Due
30 Years From Today
Last Payment On General Annuity Due
10 Years
20 Years
30 Years
1 Year From Today
1st Payment On Ordinary General Annuity
32 Years From Today
Last Payment On Ordinary General Annuity

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11.6: Annuity Interest Rates

Canada One RV advertises that you can purchase a Residence 40' Park Model trailer, valued at $32,999, for $181.84 biweekly over the next 10 years. Toronto Hyundai offers a "Smokin' Hot Deal" for a $9,900 Hyundai with $69 biweekly payments for seven years. Going online, you visit the 407 Express Toll Route (407 ETR) website to look at paying your tolls on this Toronto highway and see that you can pay $21.50 annually or $3.25 per month. Checking your vehicle registration, you see on the insurance website that you can pay your motorcycle insurance in a single payment of $2,667 or five beginning-of-month payments of $538. The question in all of these situations is, should you pay all at once or in installments?

Notice in all of these cases that the total of the payments is higher than the purchase price of the item. For example, the 407 ETR is billing $3.25 per month, totalling $39.00 per year! From a strictly financial perspective, the difference from the $21.50 can be treated like an amount of interest charged for not paying the full annual amount up front. But what interest rate are you paying?

In investing, you should choose the highest effective rate that carries a level of risk you can accept. If you make fixed contributions to an investment, what interest rate must you obtain to meet your future goal? On debts, do you know what you are being charged? That is your hard-earned money you are spending, so perhaps you can find a better offer at a lower rate from someone else? In business, debt interest is an expense that affects the pricing of products and is deductible from income tax. To calculate the interest, the business needs to know the interest rate it is being charged.

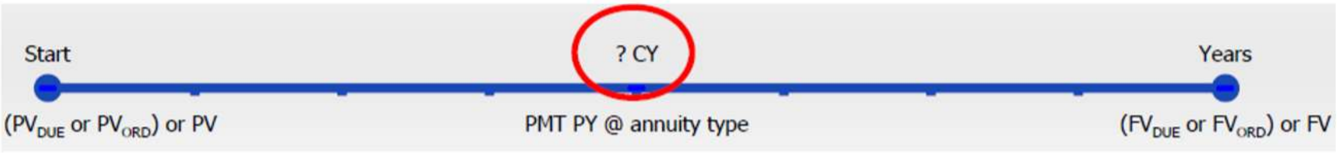
You have many reasons to calculate an annuity's interest rate. In this last section on annuities, you will make this calculation for both ordinary annuities and annuities due.

Ordinary Annuities and Annuities Due

You must calculate the interest rate on an annuity in a variety of situations:

- To determine the interest rate being charged on any debt
- To determine the interest rate that an investment is earning
- To calculate the required interest rate for savings to reach a goal within a certain time period
- To calculate the required interest rate needed for a series of payments to be sustained over a certain time period

A typical timeline for the unknown interest rate on an annuity appears in the figure below.


Start
? CY
Years
(PV_{DUE} or PV_{ORD}) or PV
PMT PY @ annuity type
(FV_{DUE} or FV_{ORD}) or FV

All variables must be known except for the nominal interest rate, IY. At either end of the timeline:

- If you know PV_{DUE} or PV_{ORD}, there might be an FV on the other side as well. FV_{DUE}, FV_{ORD}, and PV will not appear.
- If you know FV_{DUE} or FV_{ORD}, there might be a PV on the other side. PV_{DUE}, PV_{ORD}, and FV will not appear.

The Formula

A most interesting circumstance arises when you attempt to solve any of the future value or present value annuity formulas, both ordinary and due, for the interest rate. Formula 11.2 is reprinted below for illustration; however, the same point holds for Formulas 11.3 to 11.5.

$$\text{Formula 11.2 : } FV_{ORD} = PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]$$

In this formula, the FV_{ORD}, PMT, or N each appears only once. This allows you to easily manipulate the formula to solve for these variables, as we have done in previous sections. However, the periodic interest rate, i, appears in the formula twice. This normally is not a problem because you can apply basic rules of algebra to isolate and combine like terms, but not so in this instance.

To illustrate the difficulty, assign PMT = \$1 and make $CY = PY$. That makes the equation look like this:

$$FV_{ORD} = \frac{(1+i)^N - 1}{i}$$

If you rearrange the formula you have

$$iFV_{ORD} + 1 = (1+i)^N$$

The i on the left-hand side is free to be isolated, but the i on the right-hand side is part of the base for an exponent. To cancel the N exponent, you need to raise both sides of the equation to the exponent of $1/N$, arriving at the following:

$$(iFV_{ORD} + 1)^{\frac{1}{N}} = 1 + i$$

Note the result: Now the left -hand side has i being part of a base for an exponent, while the i on the right-hand side is free to be isolated. Not much of an improvement, is it? To fix this, you will need to take both sides of the equation to the exponent of N, which puts you back in your original dilemma. It is an endless loop with no solution.

The bottom line is that there is no algebraic way to isolate the i in this equation. Thus, no rearrangement of Formulas 11.2 to 11.5 to isolate this variable is possible. So how do you solve for it? There are two ways:

1. **Trial and Error**. In this manual method, you simply keep plugging in various values for i and attempt to home in on the actual value for the variable. Perhaps you start with i = 1% and find out that your FV_{ORD} is too low. Then you try i = 2% and find out that your FV_{ORD} is too high. Now you know that i is between 1% and 2%, so perhaps you try 1.5%. You repeat this process until you end up with a precise enough number. As you can imagine, this process is extremely time consuming, tedious, and prone to errors, so no one performs this process by hand any more.
2. **Use Technology**. Whether you use calculators or spreadsheets, these tools are pre-programmed with the trial-and-error technique and can perform millions of iterations in less than a second to find the solution. All problems in this section, therefore, assume you have access to technology, so no algebraic solutions appear in the formula calculations.

How It Works

Follow these steps to solve for any nominal interest rate:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that you know, including CY, PMT, PY, and Years. You must also identify a value for one of PV_{ORD}, PV_{DUE}, FV_{ORD}, or FV_{DUE}. You may or may not have a value for FV or PV.

Step 3: Use Formula 11.1 to calculate N. Input all six of the known variables into your technology and solve for the interest rate.

Important Notes

Solving for the nominal interest rate on your BAII+ requires you to enter all six of the other variables excluding the I/Y. Then press CPT I/Y to solve the problem. Because of the trial-and-error technique, it may take your calculator a few seconds to compute the answer. If your screen goes blank and your calculator hesitates, this is normal! The values you enter in the present value (PV), future value (FV), and annuity payment (PMT) must adhere to the cash flow sign convention. Remember that on the BAII+, the I/Y represents the nominal interest rate, which is compounded according to the value you entered into the C/Y variable.

Things To Watch Out For

When solving for the interest rate, pay careful attention to which form of interest rate is being sought:

1. Periodic (i): What is the interest rate per month?
2. Nominal (IY with $CY \neq 1$): What is the interest rate compounded monthly?
3. Effective (IY with $CY = 1$): What is the annually compounded or effective rate of interest?

Paths To Success

If you want to know both the nominal and effective interest rate in any situation, there are two methods to produce the answers. The first step in both methods is to solve the problem for the nominal interest rate using the appropriate value for the compounding frequency (CY). Then to calculate the effective rate you can do one of the following:

1. Change the CY to a value of one ($CY = 1$) and recalculate the IY. This book prefers this method as it is the easier and less error-prone method.
2. Take your calculated nominal interest rate and convert it to an effective interest rate by using Formula 9.4 (interest rate conversion), calculator (I Conv function), or the Chapter 9 Excel template (which uses the EFFECT function of Excel). This technique produces the same basic solution; however, it has a higher chance of error and may lack precision if not enough decimals are used.

? Exercise 11.6.1: Give It Some Thought

Is it possible in financial situations for the calculated value of the nominal interest rate, IY, to be negative?

Answer

Yes, it is possible. Many investments such as stocks fluctuate based on market conditions. It is possible for an investment to lose its value after it is purchased, which produces a negative rate of return.

✓ Example 11.6.1: Rent to Own

Smartchoice, a rent-to-own store, offers a Dell 10" Mini Inspiron Netbook for a cash n' carry price of $399. Alternatively, under its rent-to-own plan you could make $59.88 monthly payments in advance and own the laptop after one year. What interest rate is effectively being charged on the rent-to-own plan?

Solution

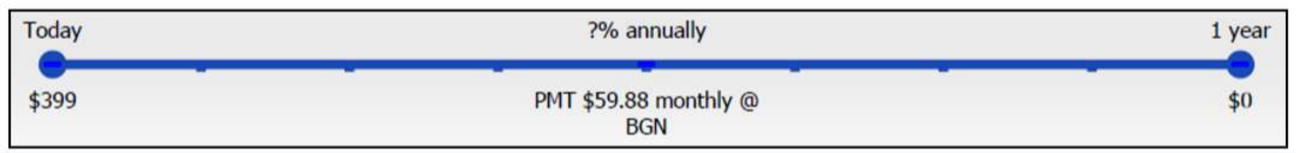
Step 1:

The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Solve for the effective interest rate, which is the nominal interest rate, IY, that has a compounding frequency of one.

What You Already Know

Step 1 (continued):

The timeline for the retirement annuity appears below.



Step 2:

PV_{DUE} = $399, CY = 1, PMT = $59.88, PY = 12, Years = 1

How You Will Get There

Step 3:

Apply Formula 11.1 and Formula 11.5. Enter the information into the calculator to solve for IY (which automatically applies Formula 9.1 to convert i to IY).

Perform

Step 3:

$N = 12 \times 1 = 12$ payments

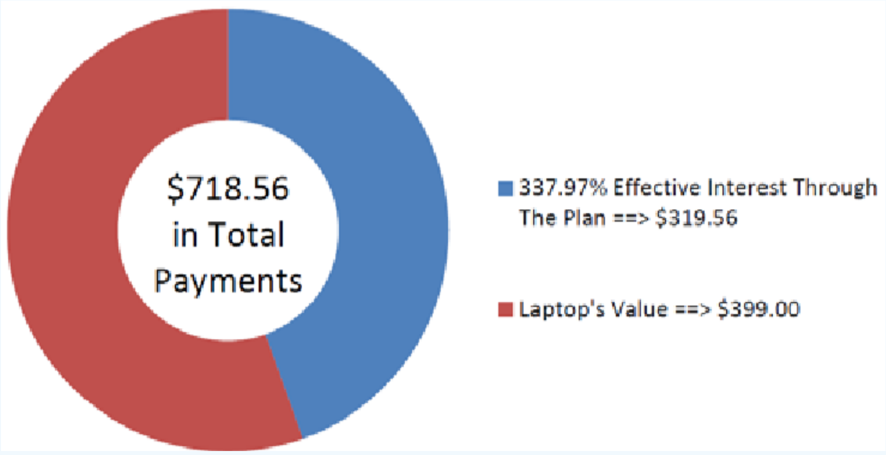
$$\$399 = \$59.88\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{12}}}\right]^{12}}{(1+i)^{\frac{1}{12}}-1}\right]\times(1+i)^{\frac{1}{12}}$$

See calculator instructions for the solution.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	12	Answer: 337.975924	399	-59.88	0	12	1

The figure shows the effective interest and the laptop’s value that make up the payments. The effective interest rate being charged under the rent-to-own payment plan is 337.9759%.



✓ Example 11.6.2: Interest Rate Required to Allow for Capital Project Savings

Cubonic Industries deposits $30,000 at the end of every quarter to save up $550,000 for a capital project in four years. To achieve its goal, what nominal interest rate compounded quarterly does Cubonic Industries require on its investment? What is the effective rate?

Solution

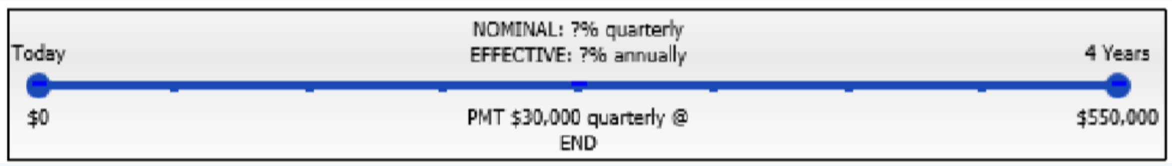
Step 1:

There are two questions here. The first question about the nominal interest rate involves an ordinary simple annuity. Solve this for IY when CY = 4. The second question about the effective rate involves an ordinary general annuity. Solve this for IY when CY = 1.

What You Already Know

Step 1 (continued):

The timeline for both questions appears below.



Step 2:

Ordinary simple annuity: FV_{ORD} = $550,000, CY = 4, PMT = $30,000, PY = 4, Years = 4

Ordinary general annuity: All the same except CY = 1

How You Will Get There

Step 3:

Apply Formula 11.1 and Formula 11.2.

Ordinary simple annuity: Enter the information into the calculator and solve for IY.

Ordinary general annuity: Change the CY to 1 (for the effective rate) and recalculate IY. Note that for both annuities the calculator automatically applies Formula 9.1 to convert i to IY.

Perform

Step 3:

$N = 4 \times 4 = 16$ payments

Ordinary Simple Annuity:

$$\$550,000 = \$30,000\left[\frac{\left[(1+i)^{\frac{4}{4}}\right]^{16}-1}{(1+i)^{\frac{4}{4}}-1}\right]$$

Ordinary General Annuity :

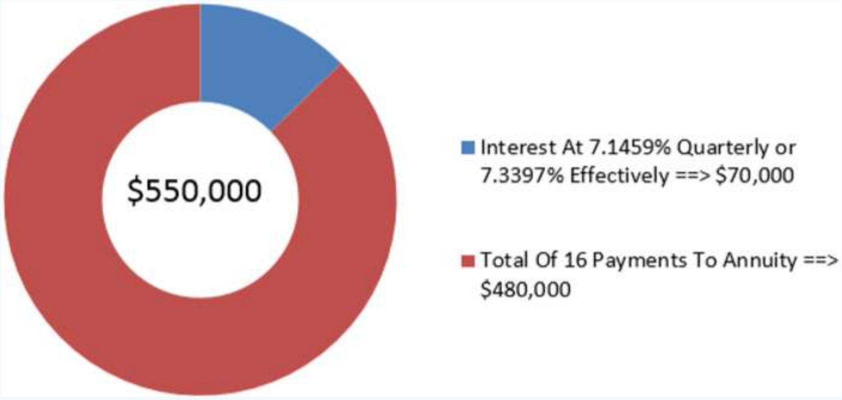
$$\$550,000 = \$30,000\left[\frac{\left[(1+i)^{\frac{1}{4}}\right]^{16}-1}{(1+i)^{\frac{1}{4}}-1}\right]$$

See the calculator instructions below for the solution to each question.

Calculator Instructions

Type	N	I/Y	PV	PMT	FV	P/Y	C/Y
Simple	16	Answer: 7.145908	0	-30000	550000	4	4
General	√	Answer: 7.339689	√	√	√	√	1

The figure shows how much principal and interest make up the final balance. For Cubonic Industries to achieve its savings goal, the savings annuity must earn 7.1459% compounded quarterly, or 7.3397% effectively.



Example 11.6.3: Interest Rate Required to Achieve RRSP Goal

Amadeus has already saved \$5,000 in his RRSP today. Suppose he continues to make \$250 contributions at the beginning of each month for the next 14 years. For him to achieve his goal of having \$100,000, what monthly nominal rate of return must his investment earn?

Solution

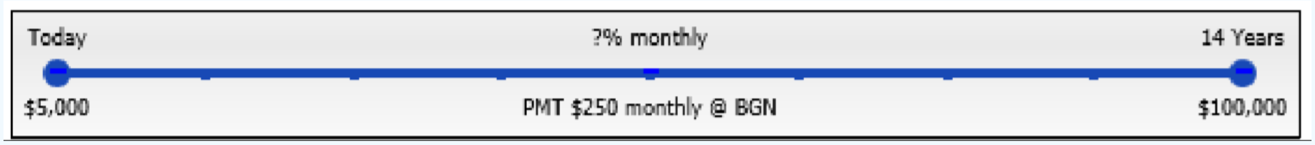
Step 1:

The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. Solve for the monthly nominal interest rate, IY.

What You Already Know

Step 1 (continued):

The timeline for RRSP contributions appears below.



Step 2:

FV_{DUE} = \$100,000, PV = \$5,000, CY =12, PMT = \$250, PY = 12, Years = 14

How You Will Get There

Step 3:

Apply Formula 11.1 and the calculator simultaneously solves Formulas 9.3 and 11.3. The annuity is simple, so N is the same number for both the number of payments and compounds. Enter the information into the calculator and solve for IY.

Perform

Step 3:

$N = 12 \times 14 = 168$ payments

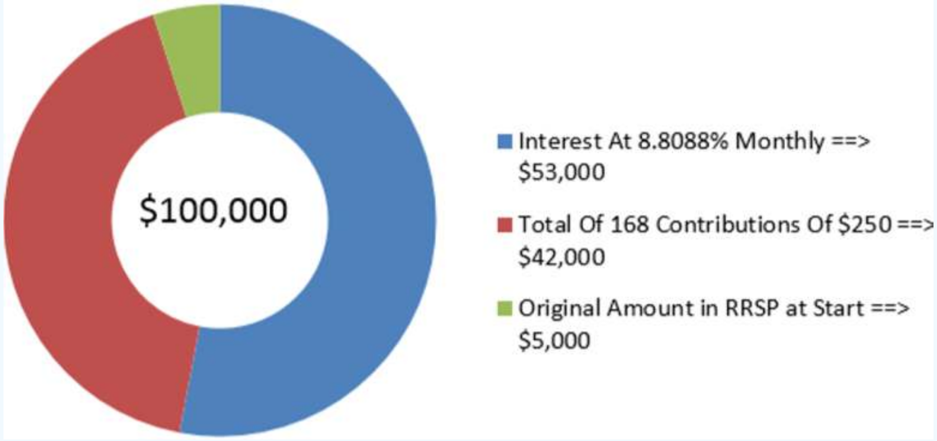
$$\$100,000 - \$5,000(1+i)^{168} = \$250\left[\frac{\left[(1+i)^{\frac{12}{12}}\right]^{168} - 1}{(1+i)^{\frac{12}{12}} - 1}\right] \times (1+i)^{\frac{12}{12}}$$

See the calculator instructions for the solution.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	168	Answer: 8.808808	-5000	-250	100000	12	12

The figure shows how much principal and interest make up the final balance. The nominal rate of interest that Amadeus must earn on his investment is 8.8088% compounded monthly.



✓ Example $11.6.4$: Ordinary General Annuity with Present Value and a Balance Owing in Future

Gemma is looking to purchase a new Nissan Pathfinder for $54,904.64 including all fees and sales taxes. She can afford to pay no more than $1,500 at the end of every month, and she wants to have the balance owing reduced to $30,000 after two years, when she can pay off the vehicle with her trust fund. What is the maximum effective rate of interest she could be charged on the car loan to meet her goals?

Solution

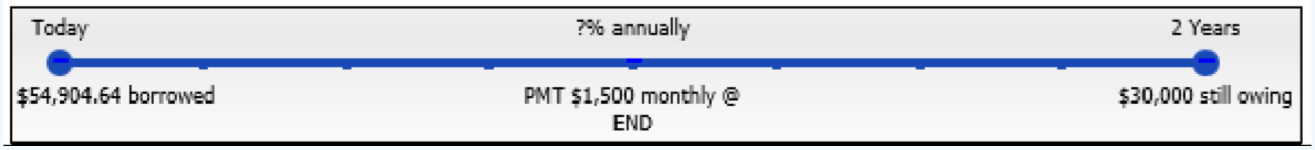
Step 1:

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is an ordinary general annuity. Solve for the effective interest rate, IY.

What You Already Know

Step 1 (continued):

The timeline for the car loan appears below.



Step 2:

PV_{ORD} = $54,904.64, FV = $30,000, CY =1, PMT = $1,500, PY = 12, Years = 2

How You Will Get There

Step 3:

Apply Formula 11.1, and the calculator simultaneously solves Formulas 9.3 and 11.4. *Note that since the annuity is general, the N compounds and N payments are different numbers. Putting these two elements into the same equation requires that N take on a single value equaling the number of payments (the annuity requirement). In this regard, the periodic interest rate used in Formula 9.3 must be converted to match the payment frequency. You accomplish this using the same $\frac{CY}{PY}$ exponent from the annuities formulas. Enter the information into the calculator and solve for IY.

Perform

Step 3:

$N = 12 \times 2 = 24$ payments

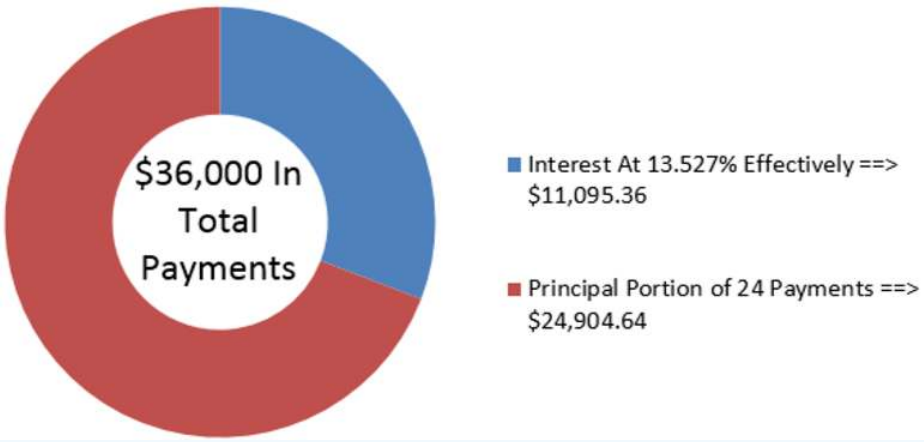
$$\$54,904.64 - \frac{\$30,000}{\left[(1+i)^{\frac{1}{12}}\right]^{24}} = \$1,500\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{12}}}\right]^{24}}{(1+i)^{\frac{1}{12}}-1}\right]$$

See the calculator instructions for the solution.

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	Answer: 13.527019	54904.64	-1500	-30000	12	1

The figure shows how much principal and interest make up the payments. Gemma will be able to purchase the car if she can obtain a car loan that has an effective interest rate lower than 13.527%.



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11.S: Summary

Key Concepts

11.1: Fundamentals of Annuities

- Understanding what an annuity is
- The four different types of annuities
- The difference between annuities and single payments
- The annuity timeline format

11.2: Future Value of Annuities

- The future value of ordinary annuities
- Variable changes in future value annuity calculations
- The future value of annuities due

11.3: Present Value of Annuities

- The present value of both ordinary annuities and annuities due
- Variable changes in present value annuity calculations
- Applying both future value and present value calculations to loans
- Determining loan balances
- Selling loan contracts between companies

11.4: Annuity Payment Amounts

- Calculating the annuity payment amount for both ordinary annuities and annuities due

11.5: Number of Annuity Payments

- Calculating the number of annuity payments (term) for both ordinary annuities and annuities due
- What to do when N has decimals

11.6: Annuity Interest Rates

- Calculating the interest rate for both ordinary annuities and annuities due

The Language of Business Mathematics

annuity

A continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation.

annuity payment

The dollar amount of the equal periodic payment in an annuity environment.

due

Annuity payments that are each made at the beginning of a payment interval.

future value of any annuity

The sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval.

general annuity

An annuity in which the payment frequency and compounding frequency are unequal.

general annuity due

An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

ordinary

Annuity payments that are each made at the end of a payment interval; this is the most common form of an annuity payment.

ordinary general annuity

An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs one interval after the beginning of the annuity, while the last payment is on the same date as the end of the annuity.

ordinary simple annuity

An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs one interval after the beginning of the annuity while the last payment is on the same date as the end of the annuity.

payment interval

The amount of time between each continuous and equal annuity payment.

payment frequency

The number of annuity payments in a complete year.

present value of any annuity

The sum of all the present values for all of the annuity payments when they are moved to the beginning of the first payment interval.

simple annuity

An annuity in which the payment frequency and compounding frequency are equal.

simple annuity due

An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

The Formulas You Need to Know

Symbols Used

CY = compounding per year or compounding frequency

FV_{DUE} = future value of annuity due

FV_{ORD} = future value of an ordinary annuity

IY = nominal interest rate

i = periodic interest rate

N = number of annuity payments

PY = payments per year or payment frequency

PMT = annuity payment amount

PV_{DUE} = present value of annuity due

PV_{ORD} = present value of ordinary annuity

Years = the term of the annuity

Formulas Introduced

Formula 11.1 Number of Annuity Payments:

$$N=PY\times\text{ Years}$$

Formula 11.2 Ordinary Annuity Future Value:

$$FV_{ORD}=PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]$$

Formula 11.3 Annuity Due Future Value:

$$FV_{DUE}=PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]\times(1+i)^{\frac{CY}{PY}}$$

Formula 11.4 Ordinary Annuity Present Value:

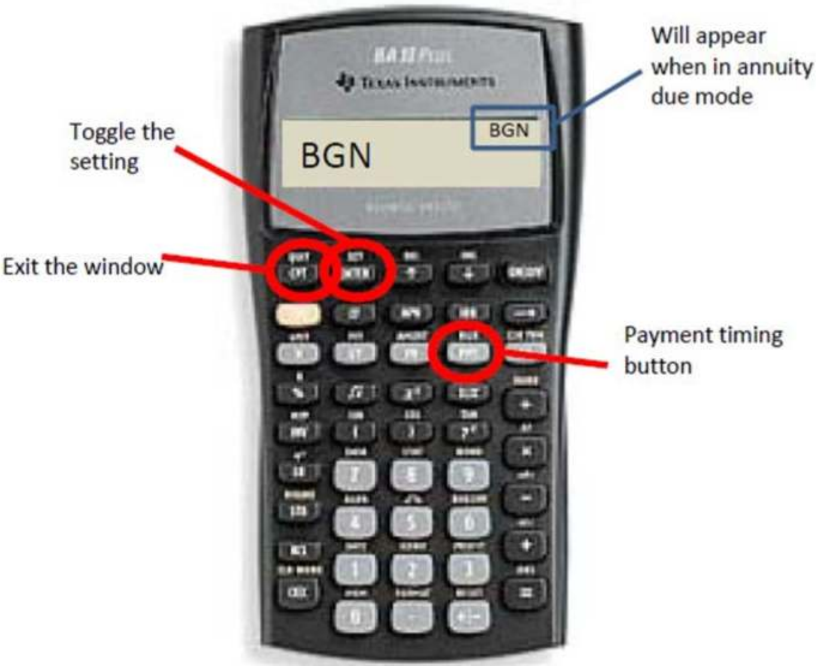
$$PV_{ORD}=PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}-1}}\right]$$

Formula 11.5 Annuity Due Present Value:

$$PV_{DUE}=PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right]\times(1+i)^{\frac{CY}{PY}}$$

Technology

Calculator


Will appear when in annuity due mode
BGN
BGN
Toggle the setting
Exit the window
Payment timing button

Annuity Type Settings

- The calculator default is for END mode, which is the ordinary annuity.
- The annuity type (payment timing) setting can be found on the second shelf above the PMT key. This function works as a toggle.
- To toggle the setting, complete the following sequence:
  - 2nd BGN (the current payment timing of END or BGN is displayed)
  - 2nd SET (it toggles to the other setting)
  - 2nd Quit (to get out of the window)
- When the calculator is in annuity due mode, a tiny BGN is displayed in the upper right of your calculator.

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11.CS: Case Study - Developing Product Payment Plans

The Situation

Lightning Wholesale works very closely with some of its smaller retailers, many of which are owned and operated by sole proprietors. These owners generally need assistance with their pricing and call on the staff at Lightning Wholesale to help develop promotional pricing plans and the amounts of the payments that can be advertised. Lightning Wholesale freely offers the help, since it means more sales and increased profits for the company.

Many of the requests are identical in nature, so the staff at Lightning Wholesale want to develop a payment plan chart for their ski product line. This chart is to illustrate both the three-month and six-month payment plan amounts that can be advertised at various interest rates.

The Data

- Lightning Wholesale has opted to only carry two ski brands in 2014: Ogasaka and Nordica.
- The list prices for Ogasaka and Nordica are $829.95 and $799.95, respectively.
- The typical monthly compounded interest rates charged by its retailers are 8%, 12%, 18%, and 28%.
- Lightning Wholesale recommends a 10% down payment on all finance plans for all of its retailers.

Important Information

- All retailers sell the skis at the list price.
- All ordinary payment plans are either three month or six month.
- Lightning Wholesale ignores sales taxes in its chart since every province has varying rates. The retailer can increase the payments in the payment chart by the appropriate sales tax.

Your Tasks

1. For both product lines and for each interest rate, develop both a three-month and a six-month payment plan amount chart that the retailers can advertise that incorporates the required down payment. For these advertised amounts, assume the final payment remains the same as all other payments (in application, though, the retailers will need to be cautioned that the final payment may be different and adjusted as needed, to which Lightning Wholesale can provide the necessary information as required).
2. Retailers ask you how to adjust the advertised payment plan chart amounts if they decide to sell the skis for some price other than the list price. What would you recommend? Provide calculations to support your answer.
3. Some retailers charge different interest rates and want to know if it is possible to just proportionally adjust the payment plan chart. For example, if a retailer charges 25.5% interest, this approach would then be to increase the 18% payment by 75% of the difference between the 18% and 28% level. Can retailers adjust your payment plan table in this way? Provide calculations using the provided numbers to support your answer.
4. Some retailers offer up to 12-month payment plans and ask if it is possible to just take the three-month payment numbers and divide by 4, or take the six-month payment numbers and divide by 2 to arrive at the 12-month payment plan numbers. Can retailers adjust your payment plan table in this way? Provide calculations to support your answer.

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11.E: Compound Interest- Annuities (Exercises)

11.1: Fundamentals of Annuities

Mechanics

For questions 1–4, use the information provided to determine whether an annuity exists.

1. A debt of four payments of $500 due in 6 months, 12 months, 18 months, and 24 months.
2. A debt of four quarterly payments in the amounts of $100, $200, $300, and $400.
3. Contributions to an RRSP of $200 every month for the first year followed by $200 every quarter for the second year.
4. Regular monthly deposits of $250 to an RRSP for five years, skipping one payment in the third year.

For questions 5–8, determine the annuity type.

Compounding Frequency	Payment Frequency	Payment Timing
5. Quarterly	Semi-annually	Beginning
6. Annually	Annually	End
7. Semi-annually	Semi-annually	Beginning
8. Monthly	Quarterly	End

For questions 9–10, draw an annuity timeline and determine the annuity type.

9. A $2,000 loan at 7% compounded quarterly is taken out today. Four quarterly payments of $522.07 are required. The first payment will be three months after the start of the loan.
10. A new RRSP is set up with monthly contributions of $300 for five years earning 9% compounded semi-annually. The RRSP will have $22,695.85 when complete. The first payment is today.

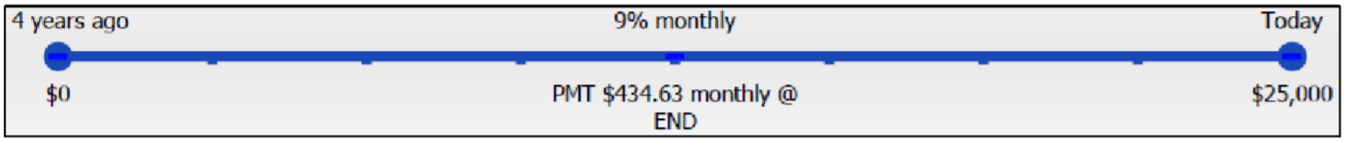
Applications

For questions 11–15, draw an annuity timeline and determine the annuity type. Calculate the value of N.

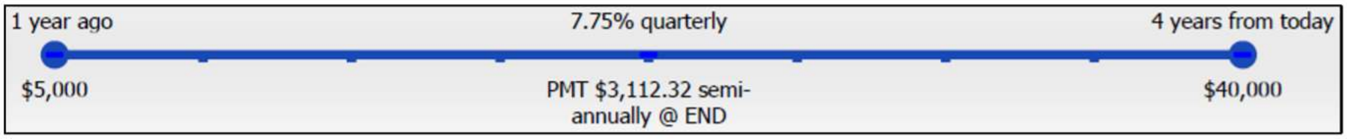
11. Marie has decided to start saving for a down payment on her home. If she puts $1,000 every quarter for five years into a GIC earning 6% compounded monthly she will have $20,979.12. She will make her first deposit three months from now.
12. Laroquette Holdings needs a new $10 million warehouse. Starting today the company will put aside $139,239.72 every month for five years into an annuity earning 7% compounded semi-annually.
13. Brenda will lease a $25,000 car at 3.9% compounded monthly with monthly payments of $473.15 starting immediately. After three years she will still owe $10,000 on the vehicle.
14. Steve takes out a two-year gym membership worth $500. The first of his monthly $22.41 payments is due at signing and includes interest at 8% compounded annually.
15. Each year, Buhler Industries saves up $1 million to distribute in Christmas bonuses to its employees. To do so, at the end of every month the company invests $81,253.45 into an account earning 5.5% compounded monthly.

For questions 16–20, assign the information in the timeline to the correct variables and determine the annuity type. Calculate the value of N.

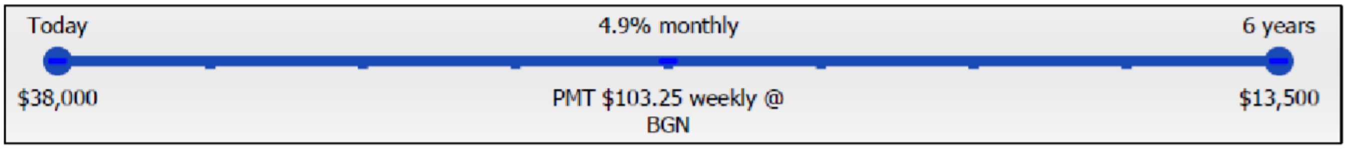
16.



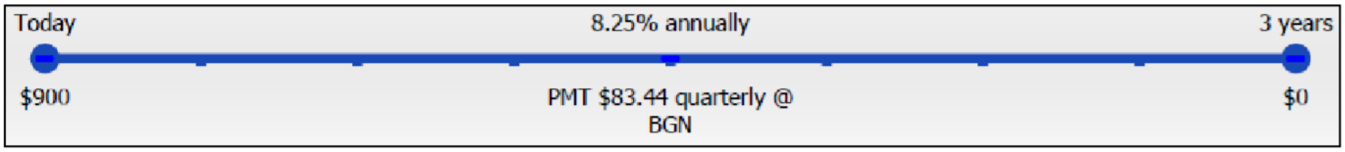
17.


1 year ago
7.75% quarterly
4 years from today
$5,000
PMT $3,112.32 semi-annually @ END
$40,000

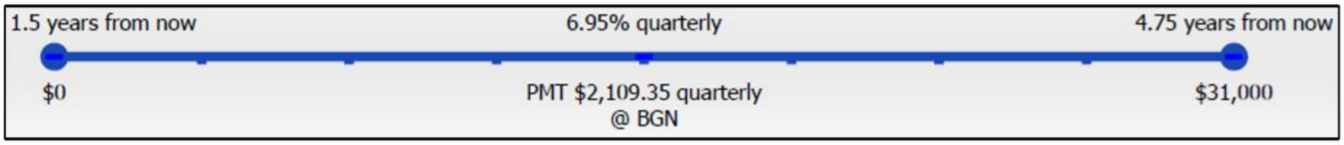
18.


Today
4.9% monthly
6 years
$38,000
PMT $103.25 weekly @ BGN
$13,500

19.


Today
8.25% annually
3 years
$900
PMT $83.44 quarterly @ BGN
$0

20.


1.5 years from now
6.95% quarterly
4.75 years from now
$0
PMT $2,109.35 quarterly @ BGN
$31,000

11.2: Future Value Of Annuities

Mechanics

For questions 1–4, calculate the future value.

Present Value	Interest Rate	Payments	Timing of Payment	Years
1. $0	7% quarterly	$2,000 quarterly	Beginning	10
2. $0	9% monthly	$375 monthly	End	20
3. $15,000	5.6% quarterly	$3,000 annually	End	30
4. $38,000	8% semi-annually	$1,500 monthly	Beginning	8

For questions 5–8, calculate the future value.

Present Value	Interest Rate	Payments	Timing of Payment
5. $0	6% annually for four years; then 7% semiannually for six years	$1,000 quarterly	End
6. $0	12% quarterly for six-and-a-half years; then 11% semi-annually for three-and-a-half years	$100 monthly	Beginning
7. $27,150	11% quarterly	$750 quarterly for five years; then $1,000 quarterly for 10 years	End

Present Value	Interest Rate	Payments	Timing of Payment
8. $50,025	8% annually for three years; then 10% annually for seven years	$5,000 annually for three years; then $4,000 annually for seven years	Beginning

Applications

9. Nikola is currently 47 years old and planning to retire at age 60. She has already saved $220,000 in her RRSP. If she continues to contribute $200 at the beginning of every month, how much money will be in her RRSP at retirement if it can earn 8.1% compounded monthly? No deposit is made the day she turns 60.
10. You are a financial adviser. Your client is thinking of investing $600 at the end of every six months for the next six years with the invested funds earning 6.4% compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?
11. The Saskatchewan Roughriders started a rainy day savings fund three-and-a-half years ago to help pay for stadium improvements. At the beginning of every quarter the team has deposited $20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today?
12. McDonald's major distribution partner, The Martin-Brower Company, needs at least $1 million to build a new warehouse in Medicine Hat two years from today. To date, it has invested $500,000. If it continues to invest $50,000 at the end of every quarter into a fund earning 6% quarterly, will it have enough money to build the warehouse two years from now? Show calculations to support your answer.
13. The human resource department helps employees save by taking preauthorized RRSP deductions from employee paycheques and putting them into an investment. For the first five years, Margaret has had $50 deducted at the beginning of every biweekly pay period. Then for the next five years, she increased the deduction to $75. The company has been able to average 8.85% compounded monthly for the first seven years, and then 7.35% compounded monthly for the last three years. What amount has Margaret accumulated in her RRSP after 10 years? Assume there are 26 biweekly periods in a year.
14. Joshua is opening up a Builder GIC that allows him to make regular contributions to his GIC throughout the term. He will initially deposit $10,000, then at the end of every month for the next five years he will make $100 contributions to his GIC. The annually compounded interest rates on the GIC in each year are 0.75%, 1.5%, 2.5%, 4.5%, and 7.25%. What is the maturity value of his GIC?
15. How much more money would an individual who makes $300 contributions to her RRSP at the beginning of the month have compared to another individual who makes $300 contributions to his RRSP at the end of the month? Assume a term of 30 years and that both RRSPs earn 9% annually.

Challenge, Critical Thinking, & Other Applications

16. When Shayla turned five years old, her mother opened up a Registered Education Savings Plan (RESP) and started making $600 end-of-quarter contributions. The RESP earned 7.46% semi-annually. At the end of each year, Human Resources and Skills Development Canada (HRSDC) made an additional deposit under the Canada Education Savings Grant (CESG) of 20% of her annual contributions into her RESP. Calculate the total maturity value available for Shayla's education when she turns 18.
17. Assume a 10-year ordinary annuity earning 10% compounded annually.
    a. If $5,000 is deposited annually, what is the maturity value?
    b. What is the maturity value if the deposits are doubled to $10,000? Compared to (a), what is the relationship between the size of the deposit and the maturity value, all other conditions being held equal?
    c. What is the maturity value if the $5,000 deposits are made semi-annually? Compared to (b), what is the relationship between the frequency of payments and the maturity value, all other conditions being held equal?
18. Carlyle plans to make month-end contributions of $400 to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn 9% compounded annually from age 20 to age 60, and then 5% compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?

19. To demonstrate the power of compound interest on an annuity, examine the principal and interest components of the maturity value in your RRSP after a certain time period. Suppose $200 is invested at the end of every month into an RRSP earning 8% compounded quarterly.
    a. Determine the maturity value, principal portion, and interest portion at 10, 20, 30, and 40 years. What do you observe?
    b. Change the interest rate to 9% quarterly and repeat (a). Comparing your answers to (a), what do you observe?
20. Compare the following maturity values on these annuities due earning 9% compounded semi-annually:
    a. Payments of $1,000 quarterly for 40 years.
    b. Payments of $1,600 quarterly for 25 years.
    c. Payments of $4,000 quarterly for 10 years.

Note in all three of these annuities that the same amount of principal is contributed. What can you learn about compound interest from these calculations?

11.3: Present Value Of Annuities

Mechanics

For questions 1–3, calculate the amount of money that must be invested today for an individual to receive the future payments indicated and have the remaining balance at the end of the term.

Future Value	Interest Rate	Payments	Timing of Payment	Years
1. $0	7% quarterly	$2,000 quarterly	Beginning	10
2. $250,000	5.6% quarterly	$3,000 annually	End	30
3. $380,000	8% semi-annually	$1,500 monthly	Beginning	8

For questions 4–6, calculate the amount of money that must be invested today for an individual to receive the future payments indicated and have the remaining balance at the end of the term.

Future Value	Interest Rate	Payments	Payment Timing
4. $0	7% semi-annually for six years; then 6% annually for four years	$1,000 quarterly	End
5. $327,150	11% quarterly	$1,000 quarterly for 10 years; then $750 quarterly for five years	End
6. $150,025	10% annually for seven years; then 8% annually for three years	$4,000 annually for seven years; then $5,000 annually for three years	Beginning

For questions 7–8, calculate the balance owing and total interest paid over the time period indicated (from the start) for the following ordinary loans.

Initial Loan Amount	Interest Rate	Payments	Balance Owing After
7. $35,000	8% quarterly	$853.59 monthly	Two years, six months
8. $48,000	9% monthly	$865.23 monthly	Four years, eight months

For questions 9–10, calculate the proceeds of the sale for the following sales of ordinary loan contracts.

Time Left on Loan on Date of Sale	Payments	Final Payment Amount	New Negotiated Interest Rate
9. Three-and-a-half years	$1,655.74 semi-annually	$1,655.69	14.85% monthly
10. Four-and-a-quarter years	$1,126.96 monthly	$1,127.21	12.9% quarterly

Applications

11. When Sinbad retires, he expects his RRSP to pay him $2,000 at the end of every month for 25 years. If his retirement annuity earns 3.8% compounded quarterly, how much money does he need to have in his RRSP when he retires?
12. Sandy's parents would like to have an annuity pay her $500 at the beginning of every month from September 1, 2012, to April 1, 2017, to help with her university tuition and living expenses. On May 1, 2017, they would like to give her a graduation gift of $5,000. If the annuity can earn 6.15% compounded monthly, how much money must be in the account on September 1, 2012?
13. The Workers' Compensation Board has determined that an injury in the workplace was your company's responsibility. As a result, your company has been ordered to pay the employee $3,000 at the end of every month for the next four years. Your human resource manager wants to set up an annuity to fund this obligation. If the proposed annuity can earn 5.7% compounded monthly for the first two-and-a-half years and then 6% compounded quarterly for the remaining one-and-a-half years, how much money should your company set aside today to meet its responsibilities?
14. Working in the accounting department, Jaycee needs to accurately record the debts of the company. Nine months ago, the company purchased new production equipment for $88,437.48 and financed it on a 12-month loan at 8.2% compounded quarterly. The payments at the end of every month have been $7,698.95. What amount should Jaycee record as the balance owing today? How much interest has been paid to date?
15. Sleep Country Canada completed a sale of an entire mattress and box spring set to a client for $2,250 to be paid in 12 equal month-end instalments with no interest. If it immediately sells this contract on the date of issue to CitiFinancial at 12% compounded annually, what are the proceeds of the sale?
16. Three years and two months ago, Mr. Magoo purchased a brand new Volkswagen Highline Jetta in Toronto for $32,854.75. He paid $5,000 as a down payment and financed the rest at 0.9% compounded monthly for six years. His payments have been $397.56 at the end of every month.
    a. What is the balance still owing on his vehicle today?
    b. If the dealership sells the loan contract today to a finance company at 9.9% compounded monthly, what are the proceeds of the sale? The last payment is for $397.85.

Challenge, Critical Thinking, & Other Applications

17. Lynne acquired a Sea Ray Sundancer boat and put $4,000 down. For the past two years, her end-of-month payments have been $1,049.01 including 9.32% compounded monthly. If she still owes $22,888.78 today, what was the purchase price of the boat?
18. Gerald has been granted power of attorney and is now responsible for setting up his aging parents in a seniors' home. The rent will be $2,490 at the beginning of every month for the first year, then increase by 5% the following year and 4% in the third year. Gerald wants to take money from his parents' estate and set up an annuity to pay their monthly rent. If he can get an annuity that earns 3.75% semi-annually, how much money from his parents' estate needs to be invested today to meet the rental payments over the next three years?
19. Compare the amount of money that needs to be invested today to provide the required payments from the investment fund annuities earning 9% compounded semi-annually:
    a. Payments of $1,000 quarterly for 40 years.
    b. Payments of $1,600 quarterly for 25 years.
    c. Payments of $4,000 quarterly for 10 years. Note that in all three of these annuities the same total payout occurs. Explain your results and comment on your findings.

20. HSBC Finance Canada is going to purchase the following ordinary loan contracts from the same company on the same date. In all cases, HSBC demands an interest rate of 18.9% compounded monthly on its purchases. What are the total proceeds of the sales?
    a. \$734.56 quarterly payments with four years remaining in the term, and the final payment is \$734.64.
    b. \$1,612.46 semi-annual payments with six-and-a-half years remaining, and the final payment is \$1,612.39.
    c. \$549.98 monthly payments with five years and two months remaining, and the final payment is \$550.28.

11.4: Annuity Payment Amounts

Mechanics

For questions 1–8, calculate the annuity payment amount.

Annuity Value	Single Payment Amount	Term of Annuity	Payment Frequency	Nominal Interest Rate	Compounding Frequency
1. $FV_{DUE} = \$25{,}000$	\$0	5 years	Monthly	8.25%	Monthly
2. $PV_{ORD} = \$500{,}000$	\$0	15 years	Quarterly	5.9%	Semi-annually
3. $PV_{DUE} = \$1{,}000{,}000$	\$0	25 years	Monthly	4.75%	Semi-annually
4. $FV_{ORD} = \$1{,}500{,}000$	\$0	35 years	Annually	9%	Annually
5. $PV_{ORD} = \$50{,}000$	$FV = \$5{,}000$	6½ years	Semi-annually	3.65%	Quarterly
6. $FV_{ORD} = \$5{,}000{,}000$	$PV = \$450{,}000$	4 years	Annually	4.35%	Monthly
7. $FV_{DUE} = \$1{,}000{,}000$	$PV = \$5{,}000$	40 years	Monthly	7.92%	Quarterly
8. $PV_{DUE} = \$50{,}000$	$FV = \$10{,}000$	5 years	Monthly	5.5%	Annually

Applications

9. To save approximately \$30,000 for a down payment on a home four years from today, what amount needs to be invested at the end of every month at 4.5% compounded monthly?
10. At age 60, Tiger has managed to save \$850,000 and decides to retire. He wants to receive equal payments at the beginning of each month for the next 25 years. The annuity can earn 5.4% compounded quarterly.
    a. If he plans on depleting the annuity, how much are his monthly payments?
    b. If he wants to have \$50,000 left over at the end of the annuity, how much are his monthly payments?
11. To purchase his new car, Scooby-Doo has obtained a six-year loan for \$40,000 at 8.8% compounded semi-annually.
    a. Determine the monthly payments required on the loan.
    b. Calculate the balance owing and the total interest paid after three years of making payments toward the loan.
    c. Instead of (b), Scooby-Doo is considering reducing the balance owing to \$15,000 after three years and paying off the loan in full at that time. What monthly payments are required?
12. Gold's Gym wants to offer its clients a monthly payment option on its annual membership dues of \$490. If the gym charges 7.75% compounded quarterly on its membership fee, what beginning-of-month payments should it advertise?
13. A 20-year marketable bond can be purchased today for \$13,402.90. It will make interest payments to the investor at the end of every six months, along with a \$10,000 lump-sum payment to the investor at the end of the term. If prevailing interest rates are 6.85% compounded semi-annually, how much is each interest payment?

14. Carling Industries needs to acquire some real estate to expand its operations. In negotiations with the Province of Nova Scotia, it will be allowed to purchase the $15 million parcel of land today and start making payments at the end of every six months for the next 10 years. If interest will be charged at 7.6% compounded semi-annually, what will be the required payments? (Round to the nearest dollar.)
15. Sinclair does not believe in debt and will only pay cash for all purchases. He has already saved up $140,000 toward the purchase of a new home with an estimated cost of $300,000. Suppose his investments earn 7.5% compounded monthly. How much does he need to contribute at the beginning of each quarter if he wants to purchase his home in five years?
16. A-One Courier Services needs to lease five vehicles for the next three years. Each vehicle retails for $23,750, and the interest rate on the lease is 5.85% compounded monthly. Under the lease terms, the company will make quarterly payments starting today such that the balance owing on each vehicle will be $10,300 at the end of the lease. Calculate the required lease payments.

Challenge, Critical Thinking, & Other Applications

17. A sales representative tells a production manager that if she purchases a new piece of machinery with a two-year life expectancy for $40,000 she will see a substantial reduction in operating costs. To purchase the machine, the production manager will need to obtain a two-year loan at 8% compounded quarterly. What is the least amount by which the monthly operating costs would need to be reduced for this purchase to make economic sense? Assume that operating costs are reduced at the beginning of each month.
18. The Kowalskis' only child is eight years old. They want to start saving into an RESP such that their son will be able to receive $5,000 at the end of every quarter for four years once he turns 18 and starts attending postsecondary school. When the annuity is paying out, it is forecast to earn 4% compounded monthly. While they make contributions at the end of every month to the RESP, it will earn 8% compounded semi-annually. Additionally, at the end of every year of contributions the government places a $500 grant into the RESP. What is the monthly contribution payment by the Kowalskis?
19. Santana wants his retirement money to pay him $3,000 at the beginning of every month for 20 years. He expects the annuity to earn 6.15% compounded monthly during this time. If his RRSP can earn 10.25% compounded annually and he contributes for the next 30 years, how much money does he need to invest into his RRSP at the end of every month? He has already saved $15,000 to date.
20. A lot of people fail to understand how interest rate changes affect their mortgages. Many think that if the interest rate on their mortgage rises from 5% to 6% their payments will rise by 1%. Assume a $100,000 mortgage with end-of-month payments for 25 years. Calculate the monthly mortgage payment at different semi-annually compounded interest rates of 4%, 5%, 6%, 7%, and 8%. What happens as the interest rate rises by 1% each time?

11.5: Number Of Annuity Payments

Mechanics

For questions 1–8, calculate the number of annuity payments required and express in a common date format.

Present Value	Future Value	Interest Rate	Annuity Payments	Payment Timing
1. $0	$100,000	6.35% Quarterly	$3,000 Quarterly	End
2. $0	$500,000	4.8% Monthly	$1,000 Monthly	Beginning
3. $0	$250,000	5.6% Semi-annually	$7,500 Annually	End
4. $0	$175,000	7% Annually	$2,500 Semi-annually	Beginning
5. $300,000	$0	4.55% Semi-annually	$18,000 Semi-annually	End
6. $100,000	$0	6.5% Annually	$10,000 Annually	Beginning
7. $50,000	$0	7.2% Quarterly	$500 Monthly	End

Present Value	Future Value	Interest Rate	Annuity Payments	Payment Timing
8. $1,000,000	$0	9% Monthly	$40,000 Quarterly	Beginning

Applications

9. An investment of $100,000 today will make advance quarterly payments of $4,000. If the annuity can earn 7.3% compounded semi-annually, how long will it take for the annuity to be depleted?
10. Amarjit wants to save up for a down payments on his first home. A typical starter home in his area sells for $250,000 and the bank requires a 10% down payment. If he starts making $300 month-end contributions to an investment earning 4.75% compounded monthly, how long will it take for Amarjit to have the necessary down payment?
11. The neighbourhood grocery store owned by Raoul needs $22,500 to upgrade its fixtures and coolers. If Raoul contributes $3,000 at the start of every quarter into a fund earning 5.4% compounded quarterly, how long will it take him to save up the needed funds for his store's upgrades?
12. Hi-Tec Electronics is selling a 52" LG HDTV during a special "no sales tax" event for $1,995 with monthly payments of $100 including interest at 15% compounded semi-annually. How long will it take a consumer to pay off her new television?
13. Andre has stopped smoking. If he takes the $80 he saves each month and invests it into a fund earning 6% compounded monthly, how long will it take for him to save $10,000?
14. How much longer will a $500,000 investment fund earning 4.9% compounded annually last if beginning-of-month payments are $3,500 instead of $4,000?
15. Consider a $150,000 loan with month-end payments of $1,000. How much longer does it take to pay off the loan if the interest rate is 6% compounded monthly instead of 5% compounded monthly?
16. In 1998, the Gillette Company launched the Mach 3 razor, having spent $750 million in research and development along with an additional $200 million in launching the product worldwide. Suppose the cost of borrowing was 10% compounded annually. If the forecast was to earn $80 million in profits at the end of every quarter, how long did Gillette forecast it would take to pay back its investment in the Mach 3?

Challenge, Critical Thinking, & Other Applications

17. You make $250 month-end contributions to your RRSP, which earns 9% compounded annually. a. How much less time will it take to reach $100,000 if you increase your payments by 10%? b. Which alternative requires less principal and by how much? (Assume all payments are equal.)
18. Most financial institutions tout the benefits of "topping up" your mortgage payments—that is, increasing from the required amount to any higher amount. Assume a 25-year mortgage for $200,000 at a fixed rate of 5% compounded semi-annually.
    a. How many fewer payments does it take to pay off your mortgage if you increased your monthly payments by 10%?
    b. How much money is saved by "topping up" the payments? Assume that all payments are equal amounts in your calculations.
19. For an ordinary $250,000 loan with monthly payments of $2,000, do the following calculations:
    a. How many payments are needed if the interest rate is 6% compounded annually? Semi-annually? Quarterly? Monthly?
    b. What is the total of the payments required under each alternative interest rate? The final payment amounts for each alternative are $193.19, $1,965.71, $1,950.13, and $1,312.84, respectively.
    c. What can you conclude from your various calculations in parts (a) and (b)?
20. For an ordinary $250,000 loan at 6% compounded monthly, do the following calculations:
    a. How many payments are needed if the payments are $1,500 monthly? $4,500 quarterly? $9,000 semi-annually? $18,000 annually? (Notice that all of these options nominally pay the same amount per year.)
    b. What is the total of the payments required under each alternative payment plan? The final payment amounts for each alternative are $371.24, $2,006.02, $420.61, and $8,300.30, respectively. c. What can you conclude from

your various calculations in parts (a) and (b)?

11.6: Annuity Interest Rates

Mechanics

For questions 1–8, solve for both the nominal interest rate indicated as well as the effective interest rate.

Present Value	Future Value	Compounding Frequency	Annuity Payment	Term	Payment Timing
1. $0	$150,000	Monthly	$12,000 Annually	7 years	End
2. $500,000	$0	Quarterly	$8,000 Quarterly	21 years, 6 months	End
3. $0	$750,000	Semi-annually	$300 Monthly	33 years, 6 months	Beginning
4. $100,000	$0	Monthly	$1,200 Monthly	10 years	Beginning
5. $25,000	$200,000	Semi-annually	$6,250 Semi-annually	8 years	End
6. $50,000	$25,000	Annually	$1,175 Monthly	2 years	End
7. $5,000	$35,000	Monthly	$2,650 Semi-annually	4 years, 6 months	Beginning
8. $300,000	$150,000	Quarterly	$4,975 Quarterly	13 years, 3 months	Beginning

Applications

9. Following his financial adviser's recommendations, Sanchez starts monthly contributions of $375 today to his RRSP. The plan is to have $240,000 in his RRSP after 20 years of monthly compounding. What nominal interest rate does the financial adviser think Sanchez's RRSP will be able to realize?
10. Helen's husband recently passed away. The life insurance company is offering her a lump-sum payout of $250,000 today, or month-end payments of $1,585 for 20 years.
    a. What monthly compounded and effective rate is the life insurance company using in its calculations?
    b. Helen thinks she can take the lump sum and invest it herself at 4.75% effectively. How much will her monthly payment increase?
11. Francisco just changed occupations. Unfortunately, he is not able to transfer his company pension with him to his new company. The administrators of the pension plan offer him the choice of a lump-sum payout of $103,075 today or beginning-of-month payments of $535 for the next 25 years. What semi-annually compounded rate of return are the pension administrators using in their calculations?
12. Under a wrongful dismissal suit, a court awarded a former employee $100,835.25 for end-of-month wages of $4,500 for the past 21 months. What effective interest rate is the court using in the judgment?
13. A life insurance company recommends that its younger clients convert their term life insurance to permanent life insurance. One of the agents tells a client that if he invests $10,000 today along with $1,435 at the beginning of every quarter for the next 10 years, he will own $100,000 of permanent life insurance. What semi-annually compounded rate of interest is being used?
14. Jake's Electronics wants to match a competitor's advertised payment plan on an identical stereo system. If the system retails for $1,011.35 including all sales taxes and Jake wants to advertise six equal end-of-month payments of $174, what effective rate of interest does he need to charge?
15. The marketing manager for Gold's Gym offers members a two-year membership for $650 in advance or beginning-of-month payments of $29. What monthly compounded interest is the marketing manager using in his pricing?
16. An investment today requires $1,125.51 to purchase. In return, the investment pays out $30 after every six months for the next 20 years, along with an additional final lump-sum payout of $1,000. What semi-annually compounded interest rate is being earned on the investment?

Challenge, Critical Thinking, & Other Applications

17. A retail store wants to offer its clients different two-year ordinary payment plans on their product purchases. The marketing manager understands the importance of odd-number pricing in these plans, where $499 is better than stating $500. On a typical $5,000 purchase, the marketing manager wants to offer payments of $229 monthly, $699 quarterly, or $1,399 semi-annually. The Competition Act of Canada requires full disclosure of the annual interest rate being charged on any plan. What interest rate must be published for each plan?
18. When you buy a car, a cash rebate is usually available if you finance the vehicle through your bank instead of the dealership; if you finance the vehicle through the dealership, you are not eligible for the cash rebate. Assume you can purchase a vehicle for $24,960 and finance it for four years with month-end payments at 0% through the dealership. Alternatively, you could get a loan from a bank and pay cash for your vehicle, which would entitle you to receive a $3,500 cash rebate. What monthly compounded interest rate would the bank have to charge to arrive at the same monthly payment as the dealership alternative? What decision rule can you create from this calculation?
19. On a $3,500 purchase, a company is thinking of offering a year-long month-end payment plan that requires payments of $299, $319, $334, or $349.
    a. If the goal of the plan is to offer a competitive interest rate comparable to a bank credit card that averages 18% effectively, which payment plan should be chosen?
    b. If the goal of the plan is to offer a competitive interest rate comparable to a department store credit card that averages 28% effectively, which payment plan should be chosen?
20. Margarite's goal is to save up $100,000 after 10 years of monthly contributions into an investment annuity starting today. Depending on the level of risk she chooses, her adviser tells her that under low-risk conditions she would need to contribute $645.19, under medium-risk conditions her contribution needs to be $523.32, and if she puts her money into high-risk investments she would need $401.14 per month. Based on the adviser's calculations, what are the effective interest rates on the low-, medium-, and high-risk investments?

Review Exercises

Mechanics

1. Sangarwe will deposit $300 every quarter into an investment annuity earning 4.5% compounded quarterly for seven years. What is the difference in the amount of money that she will have after seven years if payments are made at the beginning of the quarter instead of at the end?
2. Canseco wants to have enough money so that he could receive payments of $1,500 every month for the next nine-and-ahalf years. If the annuity can earn 6.1% compounded semi-annually, how much less money does he need if he takes his payments at the end of the month instead of at the beginning?
3. Kevin wants to save up $30,000 in an annuity earning 4.75% compounded annually so that he can pay cash for a new car that he will buy in three years' time. What is the difference in his monthly contributions if he starts today instead of one month from now?
4. Brianne has a $21,000 loan being charged 8.4% compounded monthly. What are the month-end payments on her loan if the debt will be extinguished in five years?
5. Consider an investment of $225,000 earning 5% annually. How long could it sustain annual withdrawals of $20,000 (including the smaller final payment) starting immediately?
6. The advertised month-end financing payments on a $28,757.72 car are $699 for a four-year term. What semi-annual and effective interest rate is being used in the calculation?

Applications

7. Kubb Bakery estimates it will need $198,000 at a future point to expand its production plant. At the end of each month, the profits of Kubb Bakery average $20,000, of which the owner will commit 70% toward the expansion. If the savings annuity can earn 7.3% compounded quarterly, how long will it take to raise the necessary funds?
8. An investment fund has $7,500 in it today and is receiving contributions of $795 at the beginning of every quarter. If the fund can earn 3.8% compounded semi-annually for the first one-and-a-half years, followed by 4.35% compounded monthly for another one-and-three-quarter years, what will be the maturity value of the fund?

9. A $17,475 Toyota Matrix is advertised with month-end payments of $264.73 for six years. What monthly compounded rate of return (rounded to one decimal) is being charged on the vehicle financing?
10. A variable rate loan has a balance remaining of $17,000 after two years of fixed end-of-month payments of $655. If the monthly compounded interest rate on the loan was 5.8% for the first 10 months followed by 6.05% for 14 months, what was the initial amount of the loan?
11. Hank has already saved $68,000 in his RRSP. Suppose he needs to have $220,000 saved by the end of 10 years. What are his monthly payments starting today if the RRSP can earn 8.1% compounded annually?
12. Many companies keep a “slush fund” available to cover unexpected expenses. Suppose that a $15,000 fund earning 6.4% compounded semi-annually continues to receive month-end contributions of $1,000 for the next five years, and that a withdrawal of $12,000 is made two-and-a-half years from today along with a second withdrawal of $23,000 four years from today. What is the maturity value of the fund?
13. Many consumers carry a balance each month on their credit cards and make minimal payments toward their debt. If a consumer owes $5,000 on a credit card being charged 18.3% compounded daily interest, how long will it take him to pay off his debt with month-end payments of $100?
14. You have a loan for $20,000 on which you are charged 6% compounded quarterly. What payment amount at the end of every six months would reduce the loan to $15,000 after two years? What is the interest portion of the total payments made?

Challenge, Critical Thinking, & Other Applications

15. Stan and Kendra's children are currently four and two years old. When their older child turns 18, they want to have saved up enough money so that at the beginning of each year they can withdraw $20,000 for the first two years, $40,000 for the next two years, and $20,000 for a final two years to subsidize their children's cost of postsecondary education. The annuity earns 4.75% compounded semi-annually when paying out and 6.5% compounded monthly when they are contributing toward it. Starting today, what beginning-of-quarter payments must they deposit until their oldest reaches 18 years of age in order to accumulate the needed funds?
16. Karen is saving $1,500 at the end of every six months into an investment that earns 9.4% compounded monthly for the next 20 years. The maturity value will then be rolled into an investment earning 5.85% compounded annually, from which she plans on withdrawing $23,800 at the beginning of each year. How long will the annuity sustain the withdrawals (including the smaller final payment)?
17. In an effort to clear out last year's vehicle inventory, Northside Ford advertises a vehicle at $46,500 with 0% financing for five years of end-of-month payments. Alternatively, consumers can pay cash and receive a $6,000 rebate. What is the maximum monthly compounded interest rate that a bank could charge that would result in equal or lower monthly payments?
18. Delaney is 18 years old and wants to sustain an annual income of $30,000 in today's dollars for 17 years at the end of every year when she retires at age 65 (the amount will remain fixed once set at age 65). If the annually compounded annuity can earn 4.65% in retirement and 9.5% during contributions, how much does she need to invest at the end of every month? Assume the annual rate of inflation is 2.7%.
19. A mortgage can take up to 25 years to pay off. Taking a $250,000 home, calculate the month-end payment for 15-, 20-, and 25-year periods using semi-annually compounded interest rates of 4%, 5.5%, and 7% for each period. What do you observe from your calculations?
20. Being able to start an RRSP with a lump-sum investment can reduce your end-of-month contributions. For any 35-year term RRSP earning 8.7% compounded annually, calculate the monthly contribution necessary to have a maturity value of $1,000,000 if the starting lump sums are $5,000, $10,000, $15,000, and $20,000. What do you observe from your calculations?

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CHAPTER OVERVIEW

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12.0: Introduction

It has been a hectic day at work. This morning your newly leased computer equipment arrived and your manager reminded you that as soon as it is set up, the debt obligation associated with the computer lease must be recorded in the accounting books. As well, a few of your corporate partners had contributed toward a new scholarship program for your employees and you must determine exactly how much the scholarship can afford to pay out from its interest earnings without touching the principal. If that were not enough, your company's pension plan is under review. Your pension earners' monthly income payments need to be increased based on the rate of inflation. On a related note, some of your employees want to participate in a special component of your RRSP benefits program in which they invest single payments today and receive future benefit payments down the road. You must help them figure out how much money they must deposit today to participate in this program.

What does this example illustrate? A majority of businesses lease computers and equipment for tax benefit purposes along with the ability to regularly upgrade equipment to meet rapid technological changes. Whether you are the business selling the leased products or acquiring them, you must understand how leases work. Company and government pension plans such as the Canada Pension Plan and Old Age Security regularly increase their payments to match inflation. Many companies, organizations, and even not-for-profit enterprises beyond just the educational sector set up scholarship programs for employees and their families. These scholarships are designed to provide benefits today and endlessly into the future.

On the personal side, you must arrange for the retirement income that you will eventually withdraw from your RRSP savings to increase regularly to match inflation so that your purchasing power remains the same throughout all stages of retirement. On a similar note, most contributions to RRSPs are regularly increased as annual income rises throughout a career. In addition, as a Canadian you have access to Canada Pension Plan and Old Age Security benefits, which are constantly increasing annuity payments available to you in your retirement years. You also save for a lot of events and big purchases in your lives; many consumers defer the purchase of these items until cash is available.

In this chapter, you will explore four specific situations relating to annuities: deferred annuities, constant growth annuities, perpetuities, and leases. Then you will practise with some personally relevant applications in the form of procedures for purchasing your vehicle at the lowest cost and for basic RRSP planning.

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12.1: Deferred Annuities

The power of compound interest, as you have already seen, is amazing. Your investment realizes exponential growth, and over long periods of time the results are spectacular.

Taking advantage of this principle, many parents (and grandparents too) invest large amounts of money when their children are young to have enough money to pay for their college or university education. A simple investment of $10,000 at birth with no further contributions could sustain approximately four years’ worth of $1,200 monthly payments to the child starting at age 18. That is over $55,000 toward the child's education!

Many students now stay at home longer with their parents while they pursue postsecondary education. Some accumulate sums of wealth at an early age through part-time earnings when they have little or no expenditure. Assume a 21 year old student had accumulated $50,000 and instead of using that money on a new car or backpacking across Europe, she invested it into her RRSP. By age 65 and without contributing another dime, she could have a retirement fund that sustains 20 years of $15,000 monthly income, or approximately $3.6 million!

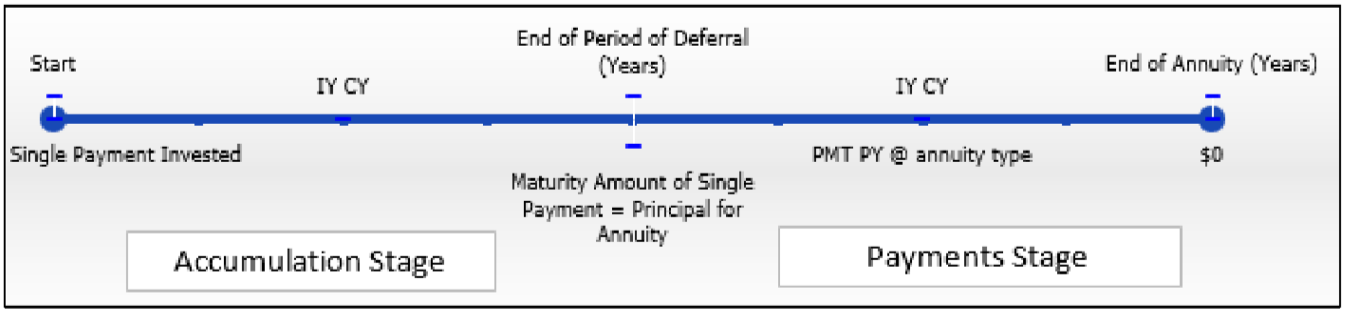
Compare this to someone who makes $300 monthly payments every month from age 21 to age 65, therefore contributing 528 payments totalling $158,400 out-of-pocket over the years. Under equal conditions, this person will receive about $12,000 monthly for 20 years, or approximately $2.9 million. See the difference? This person not only put $108,400 more into the RRSP but receives $0.7 million less in income during retirement.

Although investing the $50,000 into an RRSP instead of buying a new car is not quite as sexy, can you imagine never having to contribute to your RRSP again and being secure in your retirement before you even start your career? The financial freedom you will experience for the rest of your life would be enviable.

This section explores the concept of investing single payments today with the goal of using the maturity value to sustain an annuity afterwards. This is more commonly referred to as a deferred annuity.

What Is a Deferred Annuity?

A **deferred annuity** is a financial transaction where annuity payments are delayed until a certain period of time has elapsed. Usually the annuity has two stages, as depicted in this figure.



1. **Accumulation Stage**. A single payment is allowed to earn interest for a specified duration. There are no annuity payments during this period of time, which is commonly referred to as the **period of deferral**.
2. **Payments Stage**. The annuity takes the form of any of the four annuity types and starts at the beginning of this stage as per the financial contract. Note that the maturity value of the accumulation stage is the same as the principal for the payments stage.

The interest rate on deferred annuities can be either variable or fixed. However, since deferred annuities are commonly used to meet a specific need, fixed interest rates are more prevalent since they allow for certainty in the calculations.

The Formula

For a deferred annuity, you apply a combination of formulas that you have already used throughout this book. The accumulation stage is not an annuity, so it uses the various single payment compound interest formulas from Chapter 9. The payments stage is an annuity, so it uses the various annuity formulas from Chapter 11.

How It Works

For deferred annuities, the most common unknown variables are either the present value, the length of the period of deferral, the annuity payment amount, or the number of annuity payments that are sustainable for a fixed income payment. Follow this sequence of steps for each of these variables:

<table>
<tr><th>Solving for the Present Value</th><th>Solving for the Period of Deferral</th><th>Solving for the Annuity Payment Amount</th><th>Solving for the Number of Annuity Payments</th></tr>
<tr><td colspan="4">Step 1: Draw a timeline and identify the variables that you know, along with the annuity type.</td></tr>
<tr><td colspan="2">Step 2: Starting at the end of your timeline, calculate the present value of the annuity using the steps from Section 11.3 (Formulas 11.4 or 11.5). Round your answer to two decimals.</td><td colspan="2">Step 2: Starting at the beginning of the timeline, calculate the future value of the single payment using the steps from Section 9.2 (Formula 9.3). Round your answer to two decimals.</td></tr>
<tr><td>Step 3: Take the principal of the annuity, and using the steps from Section 9.3 (Formula 9.3) calculate the present value for the single amount</td><td>Step 3: Solve for the number of compounding periods using the applicable steps from Section 9.7 (Formula 9.3). The single payment investment is the present value, and the principal of the annuity is the future value.</td><td>Step 3: Calculate the annuity payment amount using steps from Section 11.4 (Formula 11.4 or 11.5).</td><td>Step 3: Calculate the number of annuity payments using steps from Section 11.5 (Formula 11.4 or 11.5).</td></tr>
</table>

Important Notes

Rounding. The maturity value of the single payment or the present value of the annuity is always rounded to two decimals. Since an accumulation fund is different from a payment annuity, logistically the money is transferred between different bank accounts, which means that only two decimals are carried either forwards or backwards through this step of the required calculations.

Things To Watch Out For

Avoid these three common sources of error when you work with deferred annuities:

1. **Combining the Deferral Period and the Annuity Term**. It is an error to treat the period of deferral and the term of the annuity as simultaneous time periods. For example, if a deferred annuity has a three-year period of deferral and a 10-year annuity term, this is sometimes interpreted, mistakenly, as an annuity ending 10 years from today. These time segments are separate and consecutive on the timeline! The correct interpretation is that the annuity term ends 13 years from today, since the 10-year term does not start until the three-year deferral terminates.
2. **Incorrect Timing between Stages**. A common mistake is to determine incorrectly when the period of deferral ends and the annuity starts. This error usually results from forgetting that the payments on ordinary annuities start one payment interval after the annuity starts, whereas annuity due payments start immediately. Thus, if the first quarterly payment on an ordinary annuity is to be paid 6¾ years from today, then the period of deferral is 6½ years. If the deferral is for an annuity due, then the period of deferral is 6¾ years.
3. **Confusing** N. A deferred annuity requires different calculations of N using either Formula 9.2 or Formula 11.1. In the accumulation stage, recall that N must represent the number of compound periods calculated by Formula 9.2. In the payments stage, the N must represent the number of annuity payments calculated by Formula 11.1.

Exercise 12.1.1: Give It Some Thought

a. If a deferred annuity has a four-year period of deferral and a seven-year annuity term, how many years from today will the term of the annuity end?
b. If an ordinary deferred annuity makes its first monthly payment 25 months from now, how long is the period of deferral? What if it were a deferred annuity due?

Answer

a. 11 years
b. 24 months; 25 months

Example $12.1.1$: Investing an Inheritance for your Retirement

Frasier is 33 years old and just received an inheritance from his parents' estate. He wants to invest an amount of money today such that he can receive $5,000 at the end of every month for 15 years when he retires at age 65. If he can earn 9% compounded annually until age 65 and then 5% compounded annually when the fund is paying out, how much money must he invest today?

Solution

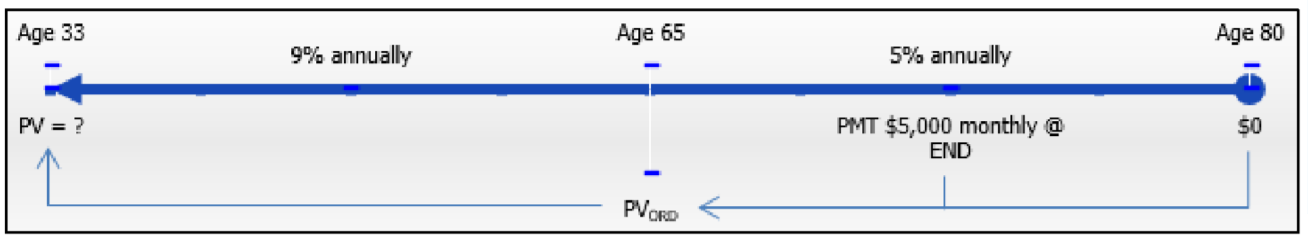
Calculate the single payment that must be invested today. This is the present value (PV) of the deferred annuity.

What You Already Know

Step 1:

The deferred annuity has monthly payments at the end with an annual interest rate. Therefore, this is an ordinary general annuity.

The timeline for the deferred annuity appears below.



Ordinary General Annuity: FV = $0, IY = 5%, CY = 1, PMT = $5,000, PY = 12, Years = 15

Period of Deferral: $FV = PV_{ORD}$, IY = 9%, CY = 1, Years = 32

How You Will Get There

Step 2:

Calculate the periodic interest rate (i, Formula 9.1), number of annuity payments (N, Formula 11.1), and present value of the ordinary general annuity (PV_{ORD}, Formula 11.4).

Step 3:

Discount the principal of the annuity back to today. Calculate the periodic interest rate (i, Formula 9.1), number of single payment compound periods (N, Formula 9.2), and present value of a single payment (PV, Formula 9.3 rearranged).

Perform

Step 2:

$$i = 5\%/1 = 5\%; N = 12 \times 15 = 180 \text{ payments}$$

$$PV_{ORD} = \$5,000\left[\frac{1-\left[\frac{1}{(1+0.05)^{\frac{1}{12}}}\right]^{180}}{(1+0.05)^{\frac{1}{12}}-1}\right] = \$636,925.79$$

Step 3:

$$i = 9\%/1 = 9\%; N = 1 \times 32 = 32 \text{ compounds}$$

$$\begin{aligned}\$636,925.79 &= PV(1+0.09)^{32}\\ PV &= \$636,925.79 \div (1.09)^{32}\\ &= \$40,405.54\end{aligned}$$

Calculator Instructions

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	END	180	5	Answer: -636,925.79	5000	0	12	1
Accumulation	√	32	9	Answer: 40,405.53861	0	636925.79	1	1

If Frasier invests $40,405.54 today, he will have enough money to sustain 180 withdrawals of $5,000 in retirement.

✓ Example 12.1.2: Planning the Deferral Period

Bashir wants an annuity earning 4.3% compounded semi-annually to pay him $2,500 at the beginning of every month for 10 years. To achieve his goal, how far in advance of the start of the annuity does Bashir need to invest $50,000 at 8.25% compounded quarterly? Assume 91 days in a quarter.

Solution

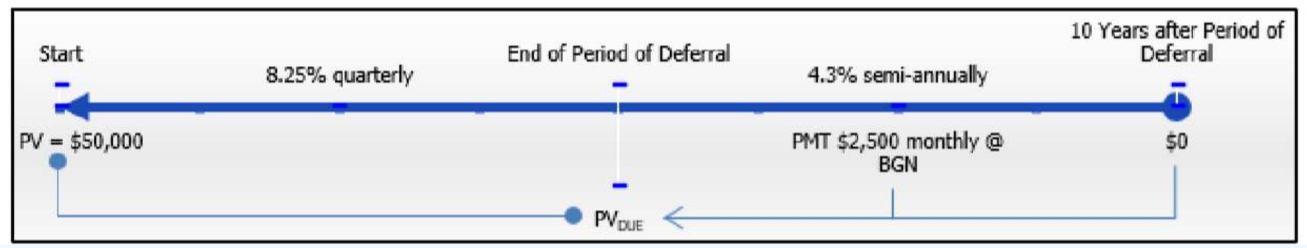
Calculate the amount of time between today and the start of the annuity. This is the period of deferral, or N.

What You Already Know

Step 1:

The deferred annuity has monthly payments at the beginning with a semi-annual interest rate. Therefore, this is a general annuity due.

The timeline for the deferred annuity appears below.



General Annuity Due: FV = $0, IY = 4.3%, CY = 2, PMT = $2,500, PY = 12, Years = 10

Period of Deferral: PV = $50,000, $FV = PV_{DUE}$, IY = 8.25%, CY = 4

How You Will Get There

Step 2:

Calculate the periodic interest rate (i, Formula 9.1), number of annuity payments (N, Formula 11.1), and present value of the ordinary general annuity (PV_{ORD}, Formula 11.4).

Step 3:

Determine the number of compounds during the accumulation stage. Calculate the periodic interest rate (i, Formula 9.1) followed by the number of single payment compound periods (N, Formula 9.3 rearranged).

Perform

Step 2:

$$i = 4.3\%/2 = 2.15\%; N = 12 \times 10 = 120 \text{ payments}$$

$$PV_{DUE} = \$2,500\left[\frac{1-\left[\frac{1}{(1+0.0215)^{\frac{2}{12}}}\right]}{(1+0.0215)^{\frac{2}{12}}-1}\right] \times (1+0.0215)^{\frac{2}{12}} = \$244,780.93$$

Step 3:

$$\begin{gathered}
i=8.25\%/4=2.0625\% \\
\$244,780.93=\$50,000(1+0.020625)^{N} \\
4.895618=1.020625^{N} \\
\ln(4.895618)=N\times\ln(1.020625) \\
\ln(4.895618)=N\times\ln(1.020625) \\
N=\frac{1.588340}{0.020415}=77.801923\text{ quarterly compounds} \\
\text{Years }=77\div4=19.25=19\text{ years, }3\text{ months} \\
0.801923\times91\text{ days }=72.975105=73\text{ days}
\end{gathered}$$

Calculator Instructions

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	BGN	120	4.3	Answer: -244,780.9336	2500	0	12	2
Accumulation	END	Answer: 77.801924	8.25	-5000	0	244780.93	4	4

To achieve his goal, Bashir needs to invest the $50,000 19 years, 3 months and 73 days before the annuity starts.

✓ Example 12.1.3: How Much Income Will It Provide?

On the day of their granddaughter's birth, Henri and Frances deposited $3,000 into a trust fund for her future education. The fund earns 6% compounded monthly. When she turns 18, they then want it to make payments at the end of every quarter for five years. If the income annuity can earn 4.5% compounded quarterly, what is the amount of each annuity payment to the granddaughter?

Solution

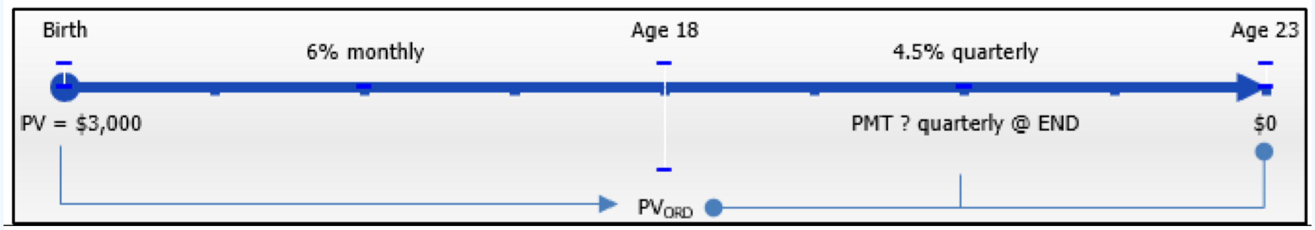
Calculate the amount of the annuity payment (PMT) during the income payments stage of the deferred annuity.

What You Already Know

Step 1:

The deferred annuity has quarterly payments at the end with a quarterly interest rate. Therefore, this is an ordinary simple annuity.

The timeline for the deferred annuity appears below.



Period of Deferral: PV = $3,000, IY = 6%, CY = 12, Years = 18

Ordinary Simple Annuity: PV_{ORD} = FV after deferral, FV = $0, IY = 4.5%, CY = 4, PY = 4, Years = 5

How You Will Get There

Step 2:

Calculate the future value of the single payment investment. Calculate the periodic interest rate (i, Formula 9.1), number of single payment compound periods (N, Formula 9.2), and future value of a single payment amount (FV, Formula 9.3).

Step 3:

Work with the ordinary simple annuity. First, calculate the periodic interest rate (i, Formula 9.1), number of annuity payments (N, Formula 11.1), and finally the annuity payment amount (PMT, Formula 11.4).

Perform

Step 2:

$i = 6\%/12 = 0.5\%; N = 12 \times 18 = 216$ compounds

$$FV = \$3,000(1+0.005)^{216} = \$8,810.30$$

Step 3:

$i = 4.5\%/4 = 1.125\%; N = 4 \times 5 = 20$ payments

$$\$8,810.30 = PMT\left[\frac{1-\left[\frac{1}{(1+0.01125)^{\frac{4}{4}}}\right]^{N}}{(1+0.01125)^{\frac{4}{4}}-1}\right]$$

$$PMT = \frac{\$8,810.30}{\left[\frac{1-\left[\frac{1}{(1+0.01125)^{\frac{4}{4}}}\right]^{N}}{(1+0.01125)^{\frac{4}{4}}-1}\right]} = \frac{\$8,810.30}{\frac{0.200480}{0.01125}} = \$494.39$$

Calculator Instructions

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	END	216	6	-3000	0	Answer: 8,810.297916	12	12
Accumulation	√	20	4.5	-8810.30	Answer: 494.392721	0	4	4

The granddaughter will receive $494.39 at the end of every quarter for five years starting when she turns 18. Since the payment is rounded, the very last payment is a slightly different amount, which could be determined exactly using techniques discussed in Chapter 13.

✓ Example 12.1.4: How Long Can the Annuity Be Sustained?

Emile received a $25,000 one-time bonus from his employer today, and he immediately invested it at 8% compounded annually. Fourteen years from now, he plans to withdraw $2,300 at the beginning of every month to use as his retirement income. If the income annuity can earn 3.25% compounded semi-annually, what is the term of the annuity before it is depleted (including the smaller final payment)?

Solution

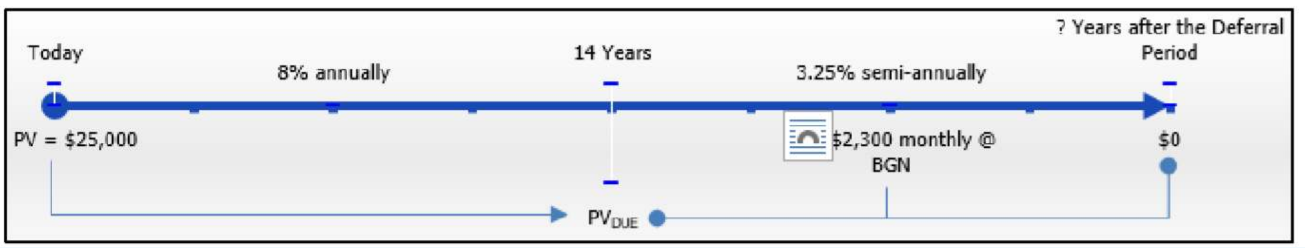
Figure out how long the income annuity is able to sustain the income payments. This requires you to calculate the number of annuity payments, or N.

What You Already Know

Step 1:

The deferred annuity has monthly payments at the beginning with a semi-annual interest rate. Therefore, this is a general annuity due.

The timeline for the deferred annuity appears below.



Period of Deferral: PV = \$25,000, IY = 8%, CY = 1, Years = 14

General Annuity Due: $PV_{DUE} = FV$ after deferral, FV = \$0, IY = 3.25%, CY = 2, PMT = \$2,300, PY = 12

How You Will Get There

Step 2:

Calculate the future value of the single deposit. Calculate the periodic interest rate (i, Formula 9.1), number of single payment compound periods (N, Formula 9.2), and future value of a single payment amount (FV, Formula 9.3).

Step 3:

Work with the general annuity due. Calculate the periodic interest rate (i, Formula 9.1) and the number of annuity payments (N, Formula 11.5 rearranged for N). Finally substitute into the annuity payments Formula 11.1 to solve for Years.

Perform

Step 2:

$$i = 8\%/1 = 8\%; N = 1 \times 14 = 14 \text{ compounds}$$

$$FV = \$25,000(1+0.08)^{14} = \$73,429.84$$

Step 3:

$$i = 3.25\%/2 = 1.625\%$$

$$\$73,429.84 = \$2,300\left[\frac{1-\left[\frac{1}{(1+0.01625)^{\frac{2}{12}}}\right]^{N}}{(1+0.01625)^{\frac{2}{12}}-1}\right] \times (1+0.01625)^{\frac{2}{12}}$$

$$\begin{aligned}
&31.840361 = \frac{1-0.997317^{N}}{0.002690} \\
&0.085656 = 1-0.99730 \\
&\ln(0.997317) = N \times \ln(0.914343) \\
&N = 33.332019 = 34 \text{ payments} \\
&34 = 12 \times \text{ Years}
\end{aligned}$$

$$\text{Years } = 2.8\overline{3} = 2 \text{ years, } 10 \text{ months}$$

Calculator Instructions

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	END	14	8	-25000	0	Answer: 73,429.84061	1	1

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Accumulation	BGN	Answer: 33.332019	3.25	-73429.84	2300	0	12	2

The annuity will last 2 years and 10 months before being depleted.

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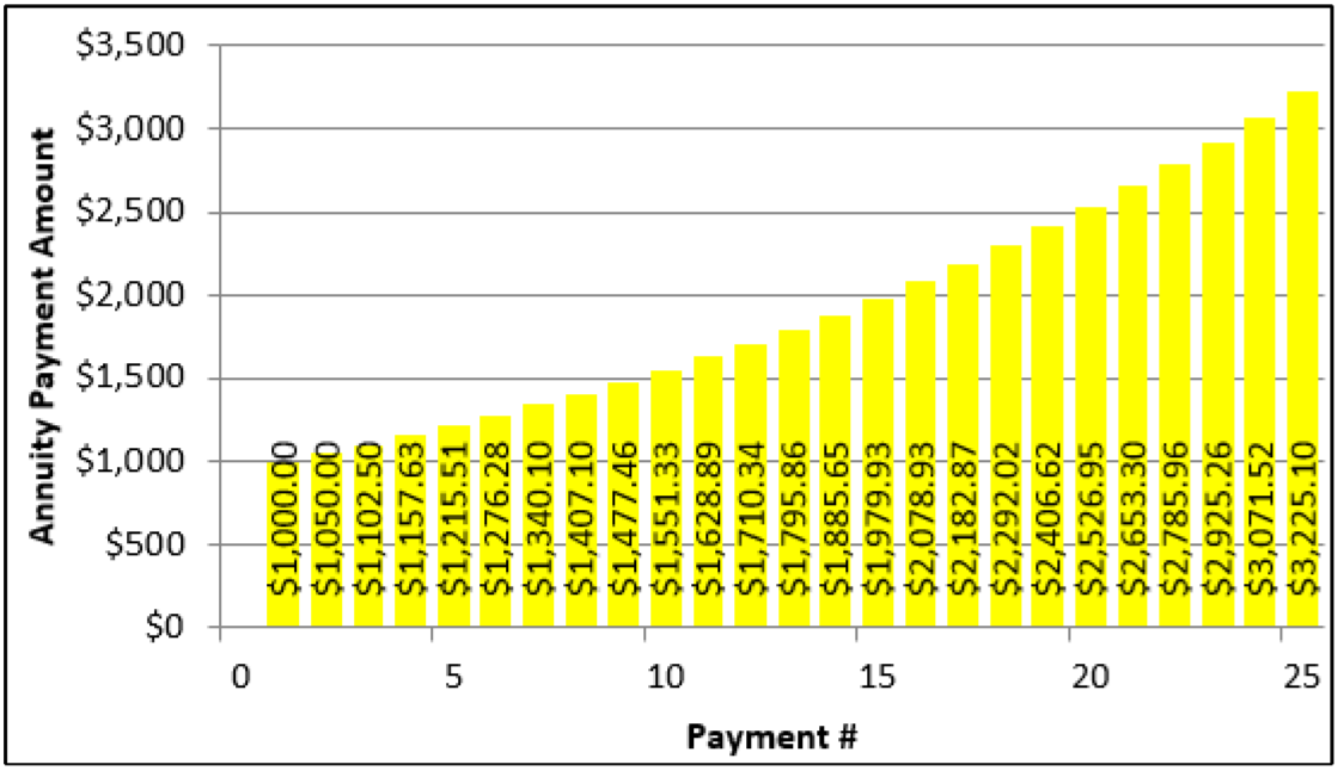
12.2: Constant-Growth Annuities

Many experts recommend that you should save around 10% of your annual income toward your RRSP contributions. Like most people, when you graduate college or university, your employer will offer you a starting salary that is usually at the lower end of their pay scale. In most companies, you are then eligible to receive annual raises in accordance with a predetermined pay structure or through performance reviews. This represents an annual income that always rises. As a result, your RRSP contributions should always rise annually too.

All annuity calculations so far have permitted fixed contributions only. More realistically, in many financial situations, such as your RRSP, the annuity payments should constantly increase on a regular basis. For this situation you need to study constant growth annuities.

The Concept of Constant Growth

A **constant growth annuity** is an annuity in which each annuity payment is increased by a fixed percentage. The figure here illustrates a $1,000 initial payment growing by 5% with each subsequent payment.



In addition to your RRSP contributions, annuity payments are regularly increased in many situations:

- When you draw from your retirement savings, you must increase your payments each year to keep up with the cost of living. This increasing demand on your retirement fund must be factored into a proper RRSP savings plan.
- Payments by the Canada Pension Plan and Old Age Security, along with most private company pension plans, increase annually to match the rate of inflation.

The change in each payment is a fixed percentage and therefore represents a percent change in the annuity payment. This can be represented by the percent change symbol, $\Delta\%$, used in Section 3.1. This variable represents the periodic percent change, where the period is consistent with the payment interval. Thus, if the payments are monthly then the percent change variable needs to be expressed as the percent change per month.

The Impact of Constant Growth on the Annuity Payment

To understand how constant growth affects the annuity payment variable, take a look at how the first four payments in any constant growth annuity are represented. Recall in annuities that N represents the number of payments, therefore:

$N=1$: First payment $= PMT = PMT(1+\Delta\%)^{N-1}$ (since $N=1$, this produces a zero exponent on the percent change)

$N=2$: Second payment $= PMT(1+\Delta\%) = PMT(1+\Delta\%)^{N-1}$

$N=3$: Third payment $= PMT(1+\Delta\%)(1+\Delta\%) = PMT(1+\Delta\%)^2 = PMT(1+\Delta\%)^{N-1}$

$N=4$: Fourth payment $= PMT(1+\Delta\%)(1+\Delta\%)(1+\Delta\%) = PMT(1+\Delta\%)^3 = PMT(1+\Delta\%)^{N-1}$

Notice that every annuity payment ultimately is represented by $PMT(1+\Delta\%)^{N-1}$ regardless of the number of annuity payments made.

The Impact of Constant Growth on the Annuity Interest Rate

You must adjust the annuity periodic interest rate to isolate the growth in the annuity from interest, since the growth in the annuity payments is already reflected in the \(PMT(1 + Δ\%)^{N − 1}\) above. Over the course of any single period, interest compounds at a rate calculated as $1+i$. On the other hand, contributions over the same time period grow by $1+\Delta\%$. Therefore, the interest in the account grows more than the growth in the contributions by a rate that is calculated by:

$$\frac{(1+i)}{(1+\Delta\%)}-1$$

For illustrative purposes, assume an annuity with a periodic interest rate of 10% and a periodic growth rate of 5%. Apply the above calculation:

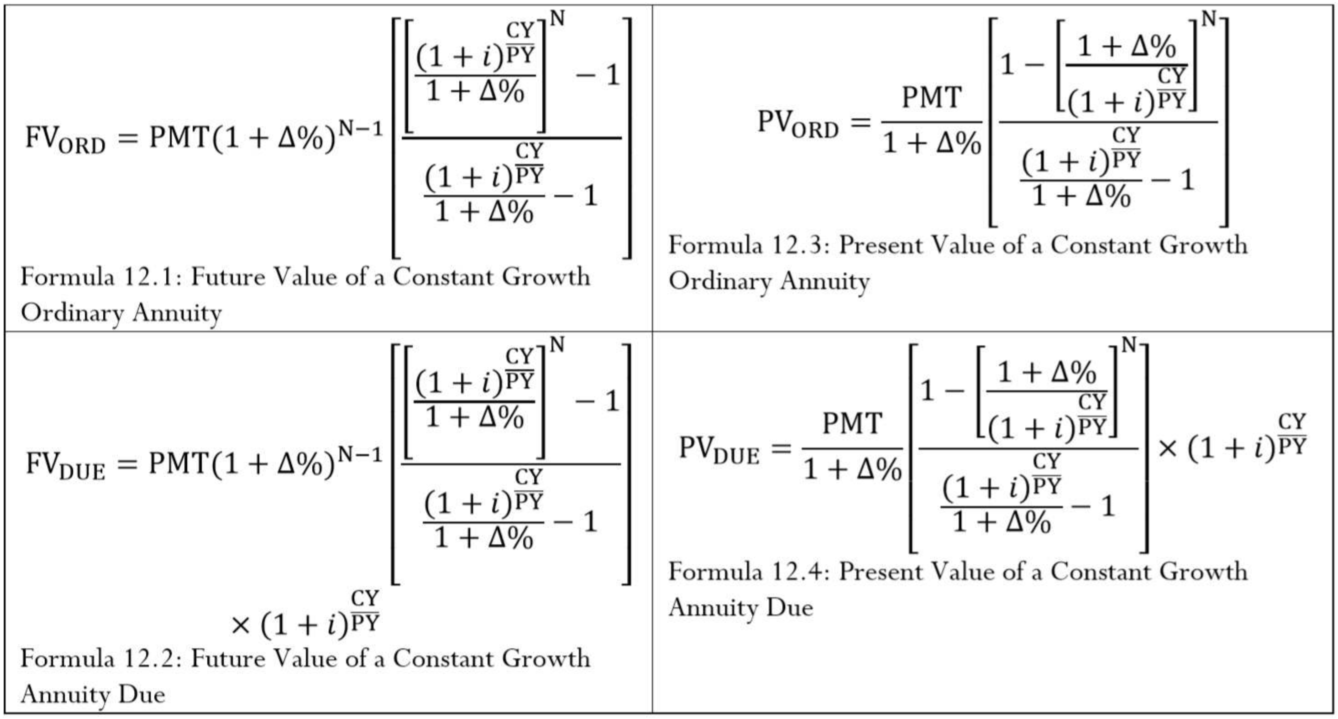
$$\frac{1+0.1}{1+0.05}-1=\frac{1.1}{1.05}-1=0.047619$$

Therefore, each period the net growth attributable solely to interest represents a periodic compounding of 4.7619% and not 10%. This interest rate, sometimes called the net rate, must replace the periodic interest rate you use in all annuity formulas.

The Formula

The four formulas for the future value and present value of both ordinary and annuity dues are shown below, incorporating the concept of constant growth. Notice in every formula that the periodic interest rate is changed to net rate. The percent change formula in the denominator does not require the $\frac{CY}{PY}$ exponent since the variable is already expressed in the same periodic terms as the payments.

1. **Future Value Formulas**. The annuity payment is modified to incorporate the growth in the payments from PMT to $PMT(1+\Delta\%)^{N-1}$ as previously illustrated. The first payment has zero growth, which results in an exponent having one period of growth less than the number of payments made.
2. **Present Value Formulas**. When bringing the annuity back to its beginning, this represents the 0th payment since a first payment has not occurred yet. Applying the $PMT(1+\Delta\%)^{N-1}$ formula results in $PMT(1+\Delta\%)^{0-1} = PMT(1+\Delta\%)^{-1} = \frac{PMT}{1+\Delta\%}$, which then substitutes for PMT in the formula.



$\mathrm{FV_{ORD}} = \mathrm{PMT}(1+\Delta\%)^{\mathrm{N}-1}\left[\frac{\left[\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}\right]^{\mathrm{N}}-1}{\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right]$ Formula 12.1: Future Value of a Constant Growth Ordinary Annuity	$\mathrm{PV_{ORD}} = \frac{\mathrm{PMT}}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right]$ Formula 12.3: Present Value of a Constant Growth Ordinary Annuity
$\mathrm{FV_{DUE}} = \mathrm{PMT}(1+\Delta\%)^{\mathrm{N}-1}\left[\frac{\left[\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}\right]^{\mathrm{N}}-1}{\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right] \times (1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}$ Formula 12.2: Future Value of a Constant Growth Annuity Due	$\mathrm{PV_{DUE}} = \frac{\mathrm{PMT}}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right] \times (1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}$ Formula 12.4: Present Value of a Constant Growth Annuity Due

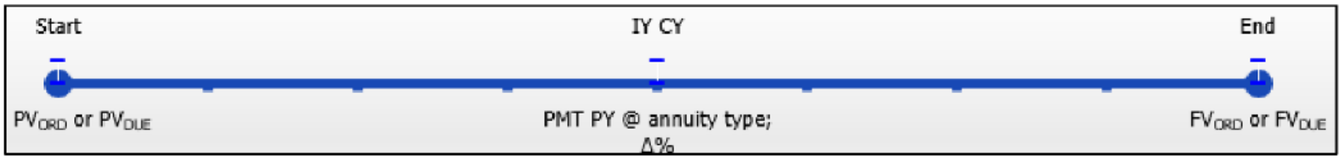
Formula for Future Value and Growth Value - 12.1, 12.2, 12.3, 12.4

Make the following notes about the variables:

1. PMT. Every payment is constantly increasing in a constant growth annuity. Therefore, the first payment is the clearly identifiable value. To ascertain the value of any other payment, use the formula $PMT(1+\Delta\%)^{N-1}$, as illustrated previously. For example, if a $1,000 payment is growing at 5% and the value of the 10th payment needs to be known, it is calculated as $\$1,000(1+0.05)^{10-1} = \$1,551.33$.
2. i. The rate of interest resulting from Formula 9.1, converted to match the payment interval if necessary. The periodic interest rate is adjusted to reflect the net rate by dividing it by $1+\Delta\%$.
3. $\Delta\%$. The periodic percent change matching the payment interval between each successive payment in the annuity. If payments are made quarterly, then the percent change per quarter is required. The value of $\Delta\%$ in these formulas is restricted to a value less than the equivalent periodic interest rate i (where i is expressed as the interest rate per payment interval). Situations where $\Delta\% \geq (1+i)^{\frac{CY}{PY}} - 1$ are beyond the scope of this textbook. If $\Delta\%$ is assigned a value of 0% then these formulas revert back to Formulas 11.2 to 11.5, respectively. Hence, the formulas from Chapter 11 are sometimes referred to as **zero growth annuity formulas** and represent simplified versions of these complete annuity formulas.

How It Works

Follow the same steps discussed for future value in Section 11.2 and the steps for present value in Section 11.3. The only notable difference is that you must identify the periodic growth rate for the annuity payment and, of course, use the new Formulas 12.1 to 12.4. The figure below illustrates a typical timeline when constant growth is involved. Note the $\Delta\%$ after the annuity type and that only one of FV_{ORD}, FV_{DUE}, PV_{ORD}, or PV_{DUE} ever appears on the timeline.



The most common applications involving constant growth annuities require you to calculate the future value, present value, or the first annuity payment. In the first two cases, you use Formulas 12.1 to 12.4 to solve for FV and PV. If the first annuity payment is the unknown variable, you can rearrange any of Formulas 12.1 to 12.4 algebraically to isolate PMT as needed.

Important Notes

Most financial calculators, including the Texas Instruments BAII Plus, are not pre-programmed with constant growth annuities. They are designed to handle fixed payment annuities only.

That said, on the web or in your readings you may come across methods that show you how to adapt your calculator input to "trick" the calculator. It is important to note that these methods are generally complex, requiring you to memorize a lot of adaptations and conditional modifications. Outputs are not necessarily correct unless further adapted. Ultimately, these tricks are not recommended. This textbook solves constant growth annuities only using formulas and through the Excel template.

Paths To Success

Although all of the discussion has been about growth, you can also use these formulas in situations involving constant reduction since the requirement of $\Delta\% < (1+i)^{\frac{CY}{PY}} - 1$ remains true. Use a negative value for the growth rate in all calculations.

Additionally, if you wish to know the total value of the annuity payments in a constant growth annuity, you can use Formula 11.2 with a few adaptations:

- Treat the first payment as the annuity payment amount or PMT.
- Substitute the periodic growth rate for the equivalent periodic interest rate while setting CY and PY both to 1. This eliminates the $\dfrac{CY}{PY}$ exponent.
- Retain using N as the number of payments.

Thus, substituting and simplifying produces the following calculation:

$$\text{Sum of constant growth payments } = PMT\left[\frac{[(1+\Delta\%)]^{\mathrm{N}} - 1}{\Delta\%}\right]$$

Exercise 12.2.1: Give It Some Thought

If an annuity payment is increased by \$5 each time from \$100 to \$105 to \$110 to \$115, does this represent a situation of constant growth?

Answer

No. A constant growth is represented by a fixed percentage growth, not a fixed dollar amount growth.

Example 12.2.1: Constant Growth RRSP Contributions

Bradford is 27 years old and starts investing \$3,000 per year into his RRSP earning 9% compounded annually. He decides that each year he will increase his contributions by 2.5%. Calculate the maturity value at age 65 for both an ordinary annuity and an annuity due.

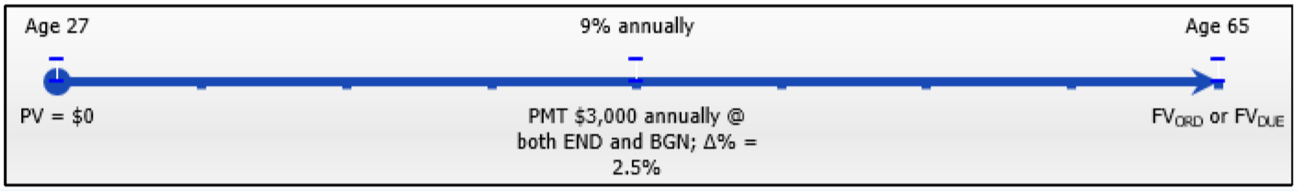
Solution

If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate two maturity values, one for the ordinary annuity, or FV_{ORD}, and one for the annuity due, or FV_{DUE}.

What You Already Know

Step 1:

There are annual contributions with an annual compound interest rate. The beginning payment creates a simple constant growth annuity due, while the end payments create an ordinary simple constant growth annuity. The timeline is below.


Age 27
9% annually
Age 65
PV = $0
PMT $3,000 annually @ both END and BGN; Δ% = 2.5%
FV_{ORD} or FV_{DUE}

Step 2:

Both annuities: PV = \$0, IY = 9%, CY = 1, PMT = \$3,000, PY = 1, Years = 38, Δ% = 2.5%

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

With PV = \$0, skip this step.

Step 5:

Apply Formula 11.1 and Formulas 12.1 and 12.2.

Perform

Step 3:

$$i = 9\%/1 = 9\%$$

Step 5:

$$N = 1 \times 38 = 38 \text{ payments}$$

$$FV_{ORD} = \$3,000(1+0.025)^{37}\left[\frac{\left[\frac{(1+0.09)^{\frac{1}{1}}}{(1+0.025)^{\frac{1}{1}}}\right]^{38}-1}{\frac{(1+0.09)^{\frac{1}{1}}}{(1+0.025)^{\frac{1}{1}}}-1}\right] = \$1,102,199.91$$

$$FV_{DUE} = \$1,102,199.91 \times (1+0.09)^{\frac{1}{1}} = \$1,201,397.90$$

The 38 contributions amount to a total principal contribution of \$186,681.89. The maturity value of the ordinary annuity is \$1,102,199.91 and the annuity due is one compound higher at \$1,201,397.90.

Example 12.2.2: Investment Necessary to Support a Constant Growth Annuity

How much money is required today to fund a 10-year annuity earning 5.6% compounded quarterly where the first monthly payment will be \$2,000 and each payment will grow by 0.2%? Calculate as both an ordinary annuity and annuity due.

Solution

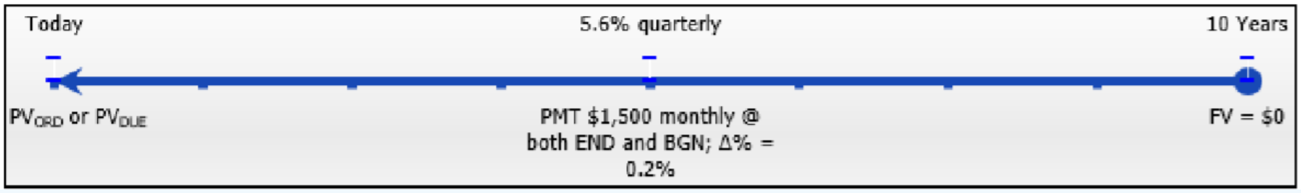
If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate two principal amounts, one for the ordinary annuity, or PV_{ORD}, and one for the annuity due, or PV_{DUE}.

What You Already Know

Step 1:

There are monthly payments with a quarterly compound interest rate. The beginning payments create a general constant growth annuity due, while the end payments create an ordinary general constant growth annuity.

The timeline appears below.



Step 2:

Both annuities: FV = $0, IY = 5.6%, CY = 4, PMT = $2,000, PY = 12, Years = 10, Δ% = 0.2%

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

With FV = $0, skip this step.

Step 5:

Apply Formula 11.1 and Formulas 12.3 and 12.4.

Perform

Step 3:

$$i = 5.6\%/4 = 1.4\%$$

Step 5:

$$N = 12 \times 10 = 120 \text{ payments}$$

$$PV_{ORD} = \frac{\$2,000}{1+0.002}\left[\frac{1-\left[\frac{(1+0.002)}{(1+0.014)^{\frac{4}{12}}}\right]^{120}}{\frac{(1+0.014)^{\frac{4}{12}}}{(1+0.002)}-1}\right] = \$205,061.74$$

$$PV_{DUE} = \$205,061.7409 \times (1+0.014)^{\frac{4}{12}} = \$205,061.7409 \times 1.004645 = \$206,014.26$$

The 120 payments each increased by 0.2% amounts to a total payout of $270,944.49. To fund these payouts, an ordinary constant growth annuity requires $205,061.74 today, whereas a constant growth annuity due requires a little more, equaling $206,014.26 because the first payment is withdrawn immediately.

✓ Example $12.2.3$: Where to Start?

After a discussion with her financial adviser on February 28, 2012, Jennifer has determined that she needs $2 million in her RRSP when she retires on February 28, 2051. She has decided to start making annual contributions to her RRSP starting February 28, 2013, and grow those contributions by 3% every year. If her RRSP can earn 8.5% compounded annually, in what amount should she make her first contribution?

Solution

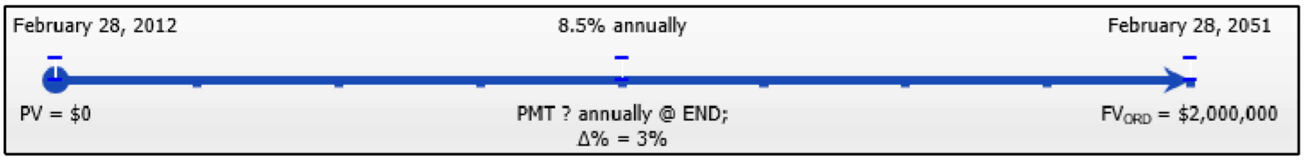
If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate the amount of her first annuity payment (PMT).

What You Already Know

Step 1:

There are annual contributions at the end of the interval with an annual compound interest rate. Therefore, this is an ordinary simple constant growth annuity.

The timeline appears below.



Step 2:

PV = $0, FV_{ORD} = $2,000,000, IY = 8.5%, CY = 1, PY = 1, Years = 39, Δ% = 3%

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

With PV = $0, skip this step.

Step 5:

Apply Formula 11.1 and Formula 12.1. Once you have substituted and simplified this formula, rearrange it for PMT.

Step 3:

$$i = 8.5\%/1 = 8.5\%$$

Step 5:

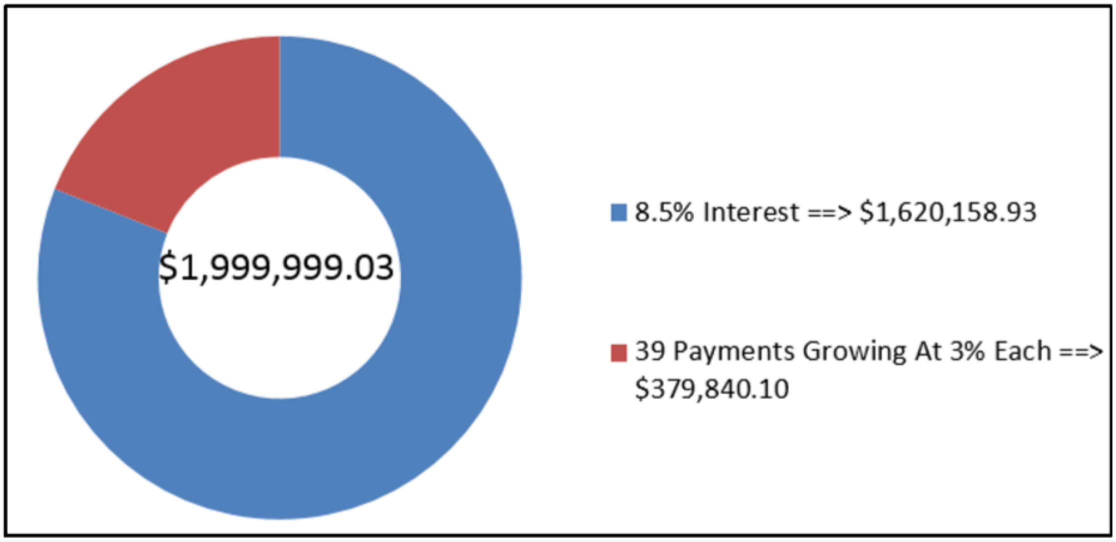
$$N = 1 \times 39 = 39 \text{ payments}$$

$$\$2,000,000 = PMT(1+0.03)^{38}\left[\frac{\left[\frac{(1+0.085)^{\frac{1}{3}}}{(1+0.05)^{\frac{1}{1}}}\right]^{39} - 1}{\frac{(1+0.055)^{\frac{1}{1}}}{(1+0.03)^{\frac{1}{1}}} - 1}\right]$$

$$\$2,000,000 = PMT(3.074783)\left[\frac{6.605154}{0.053398}\right]$$

$$\begin{aligned}\$2,000,000 &= PMT(380.340030)\\ \$5,258.45 &= PMT\end{aligned}$$

Jennifer's first payment is $5,258.45 and then increased by 3% each year for her 39 payments to provide enough principal to achieve her $2,000,000 financial goal. Note that because of the rounding of the first annuity payment, she in fact ends up marginally short of her goal in the amount of $0.97.



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12.3: Perpetuities

Some of the scholarships offered to students at your college were created decades ago, yet these scholarships continue to pay out money to students each and every year. Where does all of the money come from? Did somebody bestow a tremendously large endowment fund all those many years ago to sustain the scholarship over decades? Or does an individual or supporting organization simply pay it out of pocket each year? How is it possible for these scholarships to pay out through all the past years and all of the foreseeable years?

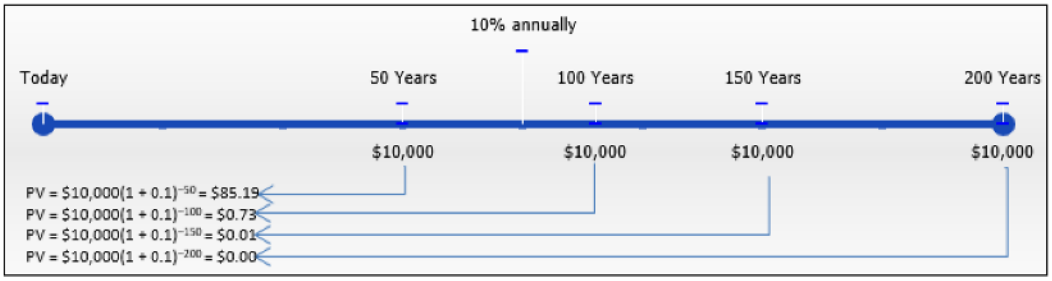
What would you do if you win a large lottery prize such as Lotto 6/49 or Lotto Max? Today's headlines continually showcase lottery winners who win "the big one," becoming overnight millionaires. With their new-found earnings, they rush out to buy million dollar homes and luxury vehicles and indulge in the pleasures of life. Sometime shortly thereafter, stories are told about how these millionaires file for bankruptcy. What if someone had told these people that instead of spending the money all at once, they could invest it and use the interest to live on forever; whole generations of their family could benefit from the winnings. Just imagine: if you invest $5 million at 5% interest, you would earn $250,000 of interest each and every year that you could withdraw endlessly.

These scenarios highlight the importance of perpetuities, which are annuities that have an infinite term. At your individual level, any sum of money you invest as a perpetuity can be used to generate income forever. At the professional level, many companies and not-for-profit organizations such as sports clubs establish scholarship programs and bursaries for their employees, members, or clients. In some contracts, payments such as royalties continue forever. Some divided stocks have their price determined by indefinite future dividend amounts.

This section explores the concept of perpetuities. You will calculate the investment required to sustain a perpetuity along with the payment amount.

What Are Perpetuities?

A **perpetuity** is a special type of annuity that has fixed, regular payments continuing indefinitely. If the principal of the investment is never withdrawn, then the interest earned each period can be withdrawn without affecting the future interest earnings of the investment. Therefore, the annuity continues to earn the same amount of interest each and every future interval and can pay out interest forever. How can a single deposit of principal sustain an infinite amount of payments? Mathematically, money that occurs in a far distant future becomes worthless when it is brought back to today, where it has no value.

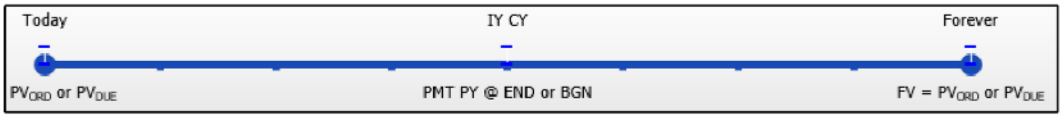

10% annually
Today
50 Years
100 Years
150 Years
200 Years
$10,000
$10,000
$10,000
$10,000
PV = $10,000(1 + 0.1)^{-50} = $85.19
PV = $10,000(1 + 0.1)^{-100} = $0.73
PV = $10,000(1 + 0.1)^{-150} = $0.01
PV = $10,000(1 + 0.1)^{-200} = $0.00

To illustrate the point of money being worthless today, the figure here calculates some present values of $10,000 payments that occur far into the future. Through the various calculations, notice that when the payments occur farther into the future the present value diminishes to practically nothing. In fact, after about 160 years the $10,000 becomes worth almost $0 today. Any payments made after this point in time result in such minuscule additions to the present value that in essence they really have no further impact. Under this premise, it is then possible to determine a value today that is equivalent to the infinite future annuity payment stream.

Ordinary Perpetuities and Perpetuities Due

A perpetuity is a special type of annuity. It comes in both ordinary and annuity due types. As well, the payment frequency and compounding frequency create either a simple or general annuity structure. Perpetuities can even occur after a deferral period, and hence deferred perpetuities are possible.

A typical timeline for a perpetuity appears here. It is identical to any other annuity timeline except for two distinguishing characteristics:



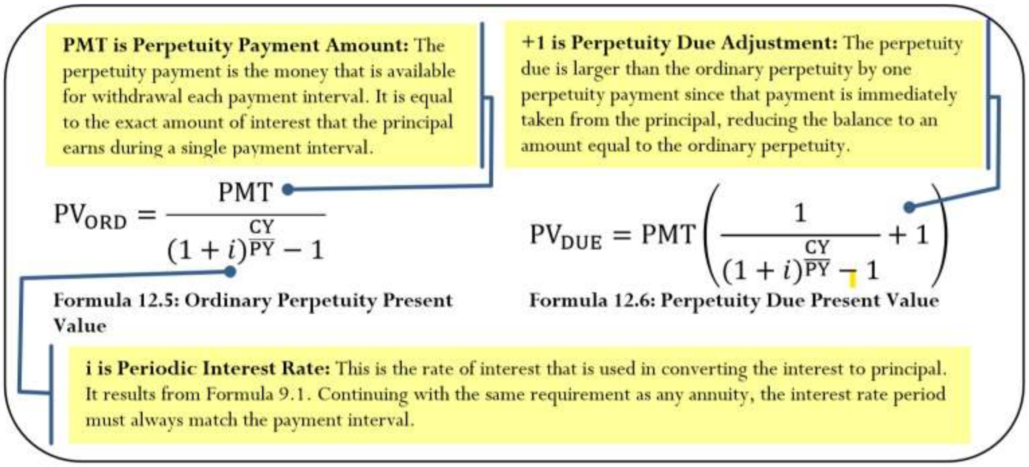
1. Future Time Period. Instead of a specific time period being indicated on the right-hand side of the timelines, the word "Forever" appears, to represent the infinite nature of the perpetuity.
2. Equivalency of FV and PV. The future value of the perpetuity is the same as the present value since only the interest is ever paid out and the principal is never touched.

The Formula

Like the annuity formulas, the perpetuity formulas are designed to accommodate both simple and general annuities through the $\frac{CY}{PY}$ exponent, which ensures that the compounding interval matches the payment interval. These new formulas represent simplified versions of Formulas 11.4 and 11.5. To illustrate, recall Formula 11.4:

$$PV_{ORD} = PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}-1}}\right]$$

Looking at the numerator, as N becomes increasingly larger the $\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}$ approaches a value of zero, which effectively removes it from the equation. This leaves a numerator of 1 over the denominator.



PMT is Perpetuity Payment Amount: The perpetuity payment is the money that is available for withdrawal each payment interval. It is equal to the exact amount of interest that the principal earns during a single payment interval.

+1 is Perpetuity Due Adjustment: The perpetuity due is larger than the ordinary perpetuity by one perpetuity payment since that payment is immediately taken from the principal, reducing the balance to an amount equal to the ordinary perpetuity.

$$PV_{ORD} = \frac{PMT}{(1+i)^{\frac{CY}{PY}}-1}$$

Formula 12.5: Ordinary Perpetuity Present Value

$$PV_{DUE} = PMT\left(\frac{1}{(1+i)^{\frac{CY}{PY}}-1}+1\right)$$

Formula 12.6: Perpetuity Due Present Value

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal. It results from Formula 9.1. Continuing with the same requirement as any annuity, the interest rate period must always match the payment interval.

Formulas 12.5, 12.6

How It Works

The steps required to solve for the present value of a perpetuity remain essentially unchanged from solving for the present value of any annuity. These steps, introduced in Section 11.3, are reviewed for the case of perpetuities:

Step 1: Identify the perpetuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that may be known, including \(IY, CY, PMT, PY\), and either PV_{ORD} or PV_{DUE}. If a PV_{ORD} or PV_{DUE} is known, then set the FV to the same amount. If no present value is known, then set FV to \$0, since money in the distant future is worthless. Unlike in regular annuities, you do not need to identify Years because of the absence of N from all perpetuity calculations.

Step 3: Use Formula 9.1 to calculate i.

Step 4: You never require this step since the FV is located so far into the future that you can automatically ascertain that its present value is equal to zero.

Step 5: To calculate the present value, apply either Formula 12.5 or Formula 12.6, depending on the type of perpetuity.

As with regular annuities, if you are dealing with a deferred perpetuity then you must modify these procedures as you would to calculate deferred annuities (refer to Section 12.1). Most perpetuity applications require you to calculate either the investment needed (PV_{ORD} or PV_{DUE}) or the payment (PMT). Use algebraic substitution and rearrangement when other variables are required.

Important Notes

The BAII Plus calculator is set up for fixed term annuities only. Therefore, it has no specific built-in function or manner in which to enter a perpetuity. However, since a perpetuity is a specialized version of a regular annuity, a few minor adaptations to the annuity inputs allow you to calculate perpetuities. Input all variables as usual except for the changes shown in the table below.

Variable	Solving for PV_{ORD} or PV_{DUE}	Solving for PMT
FV	Assign FV = 0 since money in the distant future is worthless today.	Assign FV equal to either PVORD or PVDUE, whichever is known. Recall that the principal is never touched.
N	Use a large value such as 10,000.*	Use a large value such as 2,500.*

* This will place the FV far into the distant future. A larger value could be used, but this must be done with caution. Depending on the quantities appearing in the perpetuity calculation, the power resulting from the exponent of a very large N may exceed the computational abilities of the technology and produce an error. If you experience an error, try lowering the value of N by a little bit.

Paths To Success

Formulas 12.5 and 12.6 represent simplified versions of Formulas 11.4 and 11.5. If you find these new formulas confusing, or if you just want to remember a single formula for present values, you can solve any perpetuity question with the Chapter 11 formulas. If you use these formulas, you must substitute an extremely large value of N into the formula using the technology substitutions discussed in the table above.

? Exercise 12.3.1: Give It Some Thought

a. If a single payment is placed today into an ordinary perpetuity, will its future value be higher, lower, or the same?
b. If a lump sum is placed today into a perpetuity due, will its future value be higher, lower, or the same?

Answer

a. It is the same after every withdrawal of interest and higher between withdrawals.
b. It is the same immediately before any withdrawal of interest and always lower between withdrawals. Recall that the first payment is withdrawn immediately, resulting in a principal that remains smaller than the original amount

✓ Example 12.3.1: Implementing a New Employee Benefit Program

Kendra is the human resources director and wants to offer a new employee perk through a corporate scholarship program. Each year, 10 $1,000 scholarships are to be made available to the children of the company's employees. If the first scholarships are to be offered annually starting one year from now, what amount must be invested today at 7.5% compounded annually to fund the scholarship program in perpetuity?

Solution

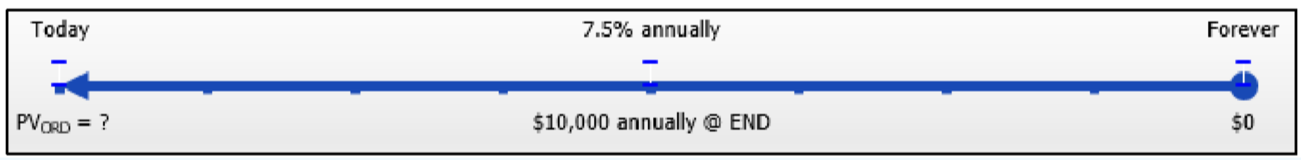
Step 1:

Calculate an amount today that must be invested such that it can pay out the needed funds at the end of every year in perpetuity. Payment and compounding intervals are the same, therefore this is an ordinary simple perpetuity. Determine the present value (PV_{ORD}).

What You Already Know

Step 1 (continued):

The timeline for the perpetuity appears below.



Step 2:

IY = 7.5%, CY = 1, PMT = 10 × $1,000 = $10,000, PY = 1

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Not needed for perpetuities.

Step 5:

Apply Formula 12.5.

Perform

Step 3:

$i = 7.5\%/1 = 7.5\%$

$$PV_{ORD} = \frac{\$10,000}{(1+0.075)^{\frac{1}{1}}-1} = \$133,333.33$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	10000	7.5	Answer: -133,333.3333	10000	0	1	1

If Kendra has her company place $133,333.33 into the perpetuity account, it will earn $10,000 interest at 7.5% compounded annually each year, which will fund the scholarship program in perpetuity.

✓ Example 12.3.2: How Much Can Be Paid Out?

In memory of his late father, who was a highly successful marketing manager, Brian would like to set up a marketing scholarship program with his university's school of business. He is able to donate $100,000 from his father's estate to set up the fund. If the perpetuity fund can earn 6.3% compounded semi-annually and the first scholarship is to be provided immediately, what annual scholarship amount can the fund offer?

Solution

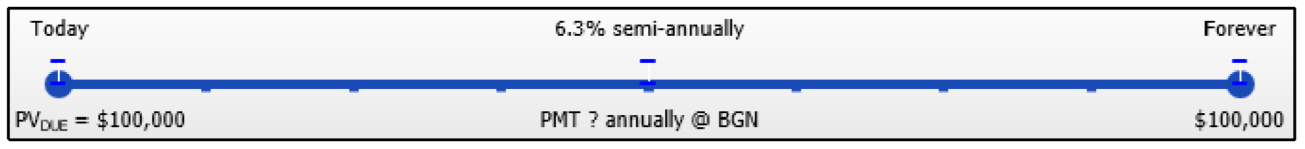
Step 1:

Calculate the amount of the perpetuity payment that the contribution can sustain. Payments and compounding intervals are different, while payments start immediately. Therefore, this is a general perpetuity due. Determine the perpetuity payment amount (PMT).

What You Already Know

Step 1 (continued):

The timeline for the scholarship appears below.



Step 2:

IY = 6.3%, CY = 2, PV_{DUE} = $100,000, PY = 1

How You Will Get There

Step 3:

Apply Formula 9.1.

Step 4:

Not needed in perpetuities.

Step 5:

Substitute into Formula 12.6, rearranging for PMT.

Perform

Step 3:

$i = 6.3\%/2 = 3.15\%$

Step 5:

$$\$100,000 = PMT\left(\frac{1}{(1+0.0315)^{\frac{2}{1}}-1}+1\right)$$

$$\$100,000 = PMT(16.626892)$$

$$\$6,014.35 = PMT$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	2500	6.3	-100000	Answer: 6,014.353018	100000	1	2

Brian's donation will allow an immediate scholarship amount of $6,014.35, which will leave $100,000 − $6,014.35 = $93,985.65 principal in the account. This principal amount will be able to earn $6,014.35 in perpetuity each year.

✓ Example 12.3.3: Valuation of Common Shares

One of the methods for valuing common shares is to determine the present value of its future dividends. If IBM Corporation is forecast to have year-end dividends of $1.50 per share for the next three years followed by $1.80 per share in perpetuity, how much should an investor be willing to pay if he requires a 12% compounded quarterly rate of return?

Solution

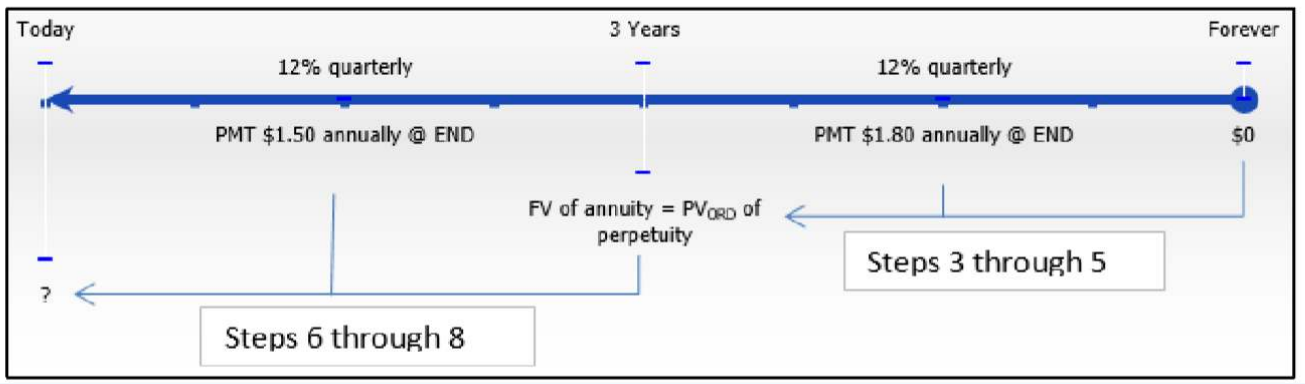
Step 1:

There are different compounding and payments, and payments are made at the end of the interval. Therefore, this is a combination of an ordinary general annuity for the first three years followed by an ordinary general perpetuity. Calculate the present value of the common shares (PV_{ORD}).

What You Already Know

Step 1 (continued):

The timeline for the common shares appears below.


Today
3 Years
Forever
12% quarterly
12% quarterly
PMT $1.50 annually @ END
PMT $1.80 annually @ END
$0
FV of annuity = PV$_{ORD}$ of perpetuity
Steps 3 through 5
?
Steps 6 through 8

Step 2:

Ordinary General Perpetuity: IY = 12%, CY = 4, PMT = $1.80, PY = 1

How You Will Get There

Starting from the right side of the timeline, calculate the present value of the perpetuity first.

Step 3:

Apply Formula 9.1.

Step 4:

Not needed in perpetuities.

Step 5:

Apply Formula 12.5. Then calculate the present value of the three-year annuity.

Step 6:

Apply Formulas 9.2 and 9.5 (rearranging for PV) to find the future value single payment (which is the PV_{ORD} of the perpetuity).

Step 7:

Apply Formula 11.1 and Formula 11.4 to the annuity.

Step 8:

Add the results of step 6 and step 7 to get the share value today.

Perform

Step 3:

$i = 12\%/4 = 3\%$

Step 5:

$$PV_{ORD} = \frac{\$1.80}{(1+0.03)^{\frac{4}{1}}-1} = \$14.341622$$

Step 6:

$N = 4 \times 3 = 12 \text{ compounds}$

$$\$14.341622 = PV(1+0.03)^{12}$$

$$PV = \$14.341622 \div (1+0.03)^{12} = \$10.058925$$

Step 7:

$N = 1 \times 3 = 3 \text{ payments}$

$$PV_{ORD} = \$1.50\left[\frac{1-\left[\frac{1}{(1+0.03)^{\frac{4}{1}}}\right]^{3}}{(1+0.03)^{\frac{4}{1}}-1}\right] = \$3.568914$$

Step 8:

$$\$10.058925 + \$3.568914 = \$13.63$$

Calculator Instructions

Steps	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
3 - 5	END	10000	12	Answer: -14.341622	1.8	0	1	4
6 - 8	√	3	√	Answer: -13.627839	1.5	14.341622	√	√

If an investor desires a 12% rate of return on his investment, he would be willing to pay $13.63 per share to receive the dividends in perpetuity.

Contributors and Attributions

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12.4: Leases

Average vehicle costs in North America have risen from $10,668 in 1982 to $25,683 in 2009,[1] a whopping 141% increase over 27 years. With vehicle purchase prices running well ahead of inflation, leasing now represents an attractive option for many Canadians with many added conveniences. On the upside, lease payments are substantially lower than purchase payments. As a bonus you are able to return your vehicle after a certain number of years and upgrade to the most recent models available. And while you are in possession of the vehicle, it usually remains under the manufacturer warranty, which minimizes your maintenance and repair expenses. On the downside, you never actually own the vehicle and always make car payments, while various research has also calculated that leasing is actually much more expensive than vehicle ownership in the long run.

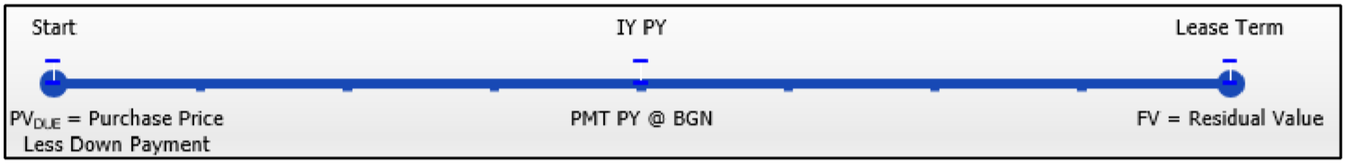
Businesses take advantage of leasing not only for vehicles, but for real estate and office space along with production equipment and office equipment. Businesses constantly need to change to avoid obsolescence, particularly in their fleet of computers. As a business grows, it must expand or even change its office space. Leasing allows organizations to do all of this. Leasing also offers significant tax benefits to most companies as well.

This section explains the basic principles of a lease and then illustrates leasing calculations such as purchase prices, residual values, lease payments, and leasing interest rates.

How Leases Work

A **lease** is a contract by which the owner of an asset gives another party an exclusive right to possess and use the asset under specified conditions for a specific period of time in return for agreed-upon payments. In other words, you are able to borrow something for a period of time and make payments on it. At the end of the lease, the asset is usually returned to the owner, or it can be purchased outright for a depreciated price. The owner of the asset is known as the **lessor**, while the borrowing party is known as the **lessee**.

A typical leasing timeline is illustrated below. Important lease components include the present value, annuity timing, term, interest rate, and future value.



1. Present Value of the Lease. The purchase price of an asset is known by many names, such as its selling price, list price, or even the manufacturer's suggested retail price (MSRP). A down payment, which is a portion of the purchase price required up front, may or may not be required depending on the asset being leased. If it is required, it is deducted from the purchase price to determine the amount of money being financed.
2. Annuity Timing. All leases take the form of annuities due since you must first pay for the item before you are allowed to borrow it. Car dealerships will not let you drive that new car home until you demonstrate the financial capability to pay for it!
3. Lease Term. A lease term is no different from an annuity term. It is the specified period of time for which the lessor leases the asset.
4. Lease Interest Rate. The nominal interest rate and compounding frequency are determined by the owner of the asset. Most vehicle leases default to rates that are compounded monthly; however, advertisements typically present effective rates.
5. Future Value of the Lease. The future value is more commonly known as the residual value, which is the projected value of an asset at the end of its lease term. A residual value does not necessarily occur in every lease calculation, as some assets, like computer equipment, may only have an insubstantial value upon lease expiration. Vehicle leases always have a residual value. Thus, the lessee has a choice to give the car dealership either the keys to the car or a cheque in the amount of the residual value, thereby purchasing the car.

In accounting, under certain circumstances, when an officer of the company signs a lease for the acquisition of a capital good (i.e., a lease of an asset, not a rental of property like a building), this creates a debt. Most leases are closed and cannot be canceled without severe financial penalties; therefore, a liability is recorded on the balance sheet.

The amount recorded is the present value of the remaining lease payments, and it is calculated using a discount rate equivalent to the interest rate the business would have had to pay if the business had purchased the asset instead. The residual value, if any, is included in the amount of this debt, which is known as a **capitalized lease liability**.2

For example, assume a company signed a two-year lease with quarterly payments of $1,000 and no residual value. Alternatively, it could have purchased the asset and financed it through a bank loan for 8% quarterly. Therefore, applying Formula 11.5, the present value of the eight $1,000 payments discounted at 8% quarterly is $7,471.99. This is the amount that appears on the balance sheet.

The Formula

A lease is an annuity due involving some special characteristics. To solve a leasing situation you can use any of the Chapter 11 annuity due formulas.

How It Works

The typical unknown variable in a leasing situation is the present value, future value, lease payment amount, or lease interest rate. To solve a lease for those variables you follow the exact same steps as found in Section 11.2, Section 11.3, Section 11.4, and Section 11.6.

Things To Watch Out For

Here are some of the most common mistakes made when working with leases:

1. **Cash Flow Violation**. This mistake occurs when the cash flow sign convention on the calculator or in Excel is violated. From the perspective of the lessee, the present value of the lease is a positive number since the lessee is receiving the asset. Payments on the lease are therefore negative. The future value or residual value of the lease (if other than zero) is also a negative number, since it represents the amount still owing on the asset if the lessee wishes to purchase it.
2. **Confusing the Present Value and the Purchase Price**. When calculating the present value of the lease, be careful to answer the question that is asked. If you need to know how much money was financed for the lease, this is your present value, whereas if you need to know the purchase price of the asset, you must add any down payment to the present value to arrive at the right answer.

? Exercise $12.4.1$: Give It Some Thought

Assume two cars are financed at the same amount at the same interest rate for the same period of time, but one is leased and the other is purchased. Are the lease payments larger than, smaller than, or equal to the purchase payments?

Answer

Smaller. The lease has a residual value at the end, meaning the lease payments do not have to pay as much principal as would be the case under the purchase payments.

✓ Example $12.4.1$: What Will the Lease Payments Be?

Mazda is advertising a new Mazda3 Sport with an MSRP inclusive of all taxes, deliveries, and fees for $39,091.41. It offers a 48-month lease with a lease rate of 3.9% effectively. The residual value of the vehicle is $13,354.00. Determine the monthly lease payments if a $2,500 down payment is made.

Solution

Step 1:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with annual compounding create a general annuity due. Calculate the annuity payment amount (PMT).

What You Already Know

Step 1 (continued):

The timeline for the vehicle lease appears below.

Step 2:

PV_{DUE} =\$39,091.41−\$2,500.00 = \$36,591.41, IY = 3.9%, CY = 1, PY = 12, Years = 4, FV = \$13,354.00

How You Will Get There

Step 3:

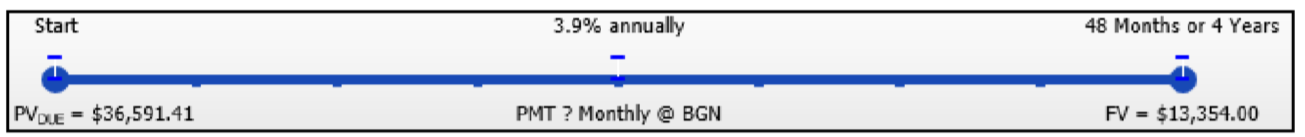
Apply Formula 9.1.

Step 4:

The future value needs to be moved to the start, the same date as PV_{DUE}. Apply Formulas 9.2 and 9.3 (rearranged for PV). The calculated PV is money the lease is not required to pay off, so it is subtracted from PV_{DUE} to arrive at the amount required by the annuity.

Step 5:

Apply Formulas 11.1 and 11.5 (rearranged for PMT) using the calculated step 4 PV_{DUE} amount.



Perform

Step 3:

$i = 3.9\%/1 = 3.9\%$

Step 4:

$N = 1 \times 4 = 4 \text{ compounds}, \$13,354 = PV(1+0.039)^4$

$\text{new } PV_{DUE} = \$36,591.41 - \$11,459.06497 = \$25,132.34503$

Step 5:

$N = 12 \times 4 = 48 \text{ payments}$

$$\$25,132.34503 = PMT\left[\frac{1-\left[\frac{1}{(1+0.039)^{\frac{1}{12}}}\right]^{48}}{(1+0.039)^{\frac{1}{12}}-1}\right] \times (1+0.039)^{\frac{1}{12}}$$

$$\$25,132.34503 = PMT\left[\frac{0.141900}{0.003193}\right] \times 1.003193$$

$$PMT = \frac{\$25,132.34503}{44.578554} = \$563.78$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	48	3.9	36591.41	Answer: -563.776579	-13354	12	1

With \$2,500 down on this vehicle, a four-year lease has beginning-of-month payments of \$563.78.

Example 12.4.2: Recording the Lease Liability

Bradford Enterprises just updated its computer mainframe systems and leased its new equipment under a two-year contract requiring fixed quarterly payments of \$17,300. Alternatively, it could have borrowed the money through a financial institution at 6.5% compounded semi-annually. What liability amount (the capitalized lease liability) should be recorded on the company's balance sheet

a. on the day the contract was signed?
b. after five payments have been made?

Solution

Step 1:

This is a leasing question, so payments are made at the beginning of the period. Quarterly payments with semi-annual compounding create a general annuity due. Calculate the present value (PV_{DUE}) at two time points— today and after five payments.

What You Already Know

Step 1 (continued):

The timeline for the computer equipment lease appears below.

Step 2:

FV = \$0, IY = 6.5%, CY =2, PMT = \$17,300, PY = 4, Years = 2

How You Will Get There

Step 3:

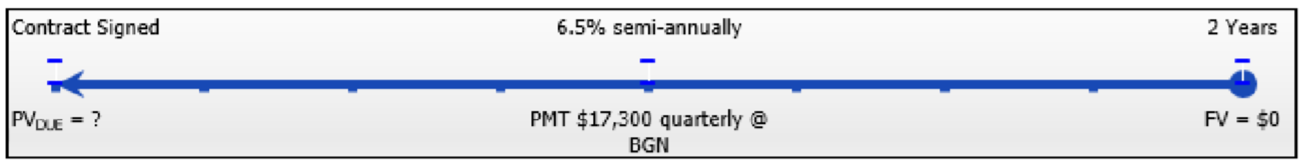
Apply Formula 9.1.

Step 4:

Since FV = \$0, skip this step.

Step 5:

Apply Formulas 11.1 and 11.5 twice (for both today and after five payments).



Perform

Step 3:

$i = 6.5\%/2 = 3.25\%$

Step 5:

a. For today, $N = 4 \times 2 = 8$ payments

$$PV_{DUE} = \$17,300 \left[\frac{1 - \left[\frac{1}{(1+0.0325)^{\frac{2}{4}}} \right]^{8}}{(1+0.0325)^{\frac{2}{4}} - 1} \right] \times (1+0.0325)^{\frac{2}{4}} = \$130,954.37$$

b. After five payments, $N = 4 \times 3/4 = 3$ payments left

$$PV_{DUE} = \$17,300 \left[1 - \left[\frac{1}{(1+0.0325)^{\frac{2}{4}}} \right]^{3} \right] \times (1+0.0325)^{\frac{2}{4}} = \$51,080.99$$

$$FV_{DUE} = \$17,300 \left[\frac{}{(1+0.0325)^{\frac{2}{4}}-1} \right] \times (1+0.0325)^{4} = \$51,080.99$$

Calculator Instructions

Calculation	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Today	BGN	8	6.5	Answer: 130,954.37	-17300	0	4	2
After 5 payments	√	3	√	Answer: 51,080.99	√	√	√	√

On the day that the computer equipment is leased, the accountants will record a capital lease liability of $130,954.37. This amount will reduce with each payment, such that after five payments the liability would be $51,080.99.

✓ Example 12.4.3: What Is It Worth at the End?

A Nissan 370Z Roadster A7 advertised in the local paper offers monthly payments of $728.70 for 48 months at 3.83% compounded monthly. The lease requires a $5,000 down payment, and the vehicle has an MSRP (including all fees and taxes) of $53,614.83. What residual value is used in the calculation?

Solution

Step 1:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate its residual value, which is the same thing as the future value (FV_{DUE}).

What You Already Know

Step 1 (continued):

The timeline for the vehicle lease appears below.

Step 2:

PV = $53,614.83 − $5,000 = $48,614.83, IY = 3.83%, CY =12, PMT = $728.70, PY = 12, Years = 4

How You Will Get There

Step 3:

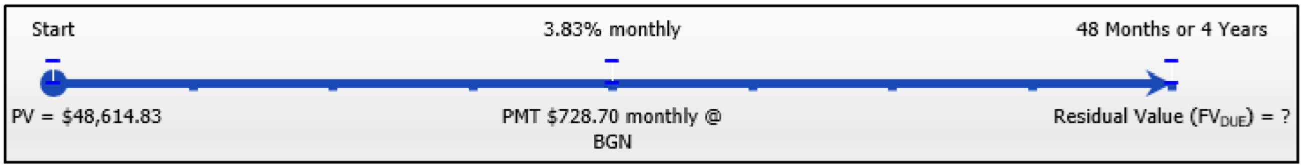
Apply Formula 9.1.

Step 4:

For the PV, apply Formulas 9.2 and 9.3.

Step 5:

Apply Formulas 11.1 and 11.2. The residual value is the difference between the answers to step 4 (what the lessee owes) and step 5 (what the lessee paid).


Start
3.83% monthly
48 Months or 4 Years
PV = $48,614.83
PMT $728.70 monthly @ BGN
Residual Value (FV_{DUE}) = ?

Perform

Step 3:

$i = 3.83\%/12 = 0.3191\overline{6}\%$

Step 4:

$N = 12 \times 4 = 48$ compounds; $FV = \$48,614.83(1+0.003191)^{48} = \$56,649.58522$

Step 5:

$N = 12 \times 4 = 48$ payments

$$\begin{aligned} FV_{DUE} = \$728.70 & \left[\frac{\left[(1+0.003191)^{\frac{12}{12}}\right]^{48}-1}{(1+0.003191)^{\frac{12}{12}}-1}\right] \times (1+0.003191)^{\frac{12}{12}} \\ & = \$37,854.63271 \end{aligned}$$

$$\text{Residual Value } = \$56,649.58522 - \$37,854.63721 = \$18,794.95$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	48	3.83	48614.83	-728.7	Answer: -18,794.95252	12	12

In determining the lease payments, Nissan expects its Roadster to have a residual value of $18,794.95 after the four year lease term expires. The owner will either have to pay this amount or give the keys back.

✓ Example 12.4.4: What Rate Are They Using?

An advertisement during The Bachelorette tells you that you can lease a Dodge Challenger SE for $161.00 biweekly for five years. The MSRP of the vehicle is $26,480, and a $1,500 down payment is required. If the residual value of the vehicle is $8,146.16, what annually compounded interest rate does the calculation use?

Solution

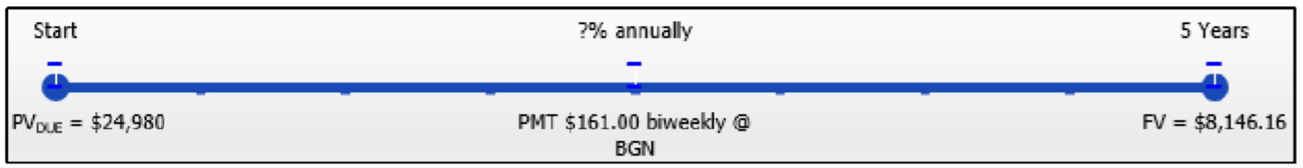
Step 1:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate the annually compounded interest rate (IY) with CY = 1.

What You Already Know

Step 1 (continued):

The timeline for the vehicle lease appears below.



Step 2:

PV_{DUE} = $26,480 − $1,500 = $24,980, CY = 1, PMT = $161, PY = 26, Years = 5, FV = $8,146.16

How You Will Get There

Step 3:

Apply Formulas 11.1 and 11.5. Enter this information into the calculator to solve for IY (which automatically applies Formula 9.1 to convert the i to an IY).

Perform

Step 3:

$N = 26 \times 5 = 130$ payments

$$\$24,980 = \$161 \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{1}{26}}} \right]^{130}}{(1+i)^{\frac{1}{26}} - 1} \right] \times (1+i)^{\frac{1}{26}}$$

See calculator instructions for the solution. IY = 4.99%

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	130	Answer: 4.989999	24980	-161	-8146.16	26	1

To arrive at payments of $161 biweekly on a Dodge Challenger with $1,500 down and a residual value of $8,146.16, Dodge is using a 4.99% compounded annually interest rate.

References

1. Statistics Canada, New Motor Vehicle Sales, June 2011 (Catalogue no. 63-007-X, p. 15), publications.gc.ca/collection...llection_2011/statcan/63-007-X/63-007-x2011006-eng.pdf (accessed August 18, 2013).
2. For illustrative purposes this textbook makes some simplifying assumptions:
    i. Accounting rules require the asset to be recorded at the present value of the minimum lease obligations or its fair market value, whichever is less. The examples in this book assume that the present value is the lesser number.
    ii. Accounting rules require at least one of four specific conditions to be met for a lease to qualify as a capital lease. The examples assume that this criterion is met.
    iii. Issues involving capital leases such as depreciation or executory costs are ignored.

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12.5: Application - How To Purchase A Vehicle

What is the cheapest way to acquire a vehicle? With 20 million cars on the road for 12 million Canadian households, that is an average of 1.7 cars per household in Canada. In fact, on average Canadians purchase about 10 to 12 cars in their lifetime. With average prices in the $25,000 to $30,000 range per vehicle, and factoring in rising inflation over your lifetime, you will spend approximately $550,000 on acquiring vehicles! Yet despite this huge amount of money being spent, shockingly few consumers take the time to analyze their financing options critically. Would you spend $550,000 without doing some research? I thought not.

This section guides you through the practical task of finding the cheapest of the alternatives for obtaining a vehicle. It lays out the four primary means of vehicle ownership and develops a procedure through which you can make a smart purchase decision.

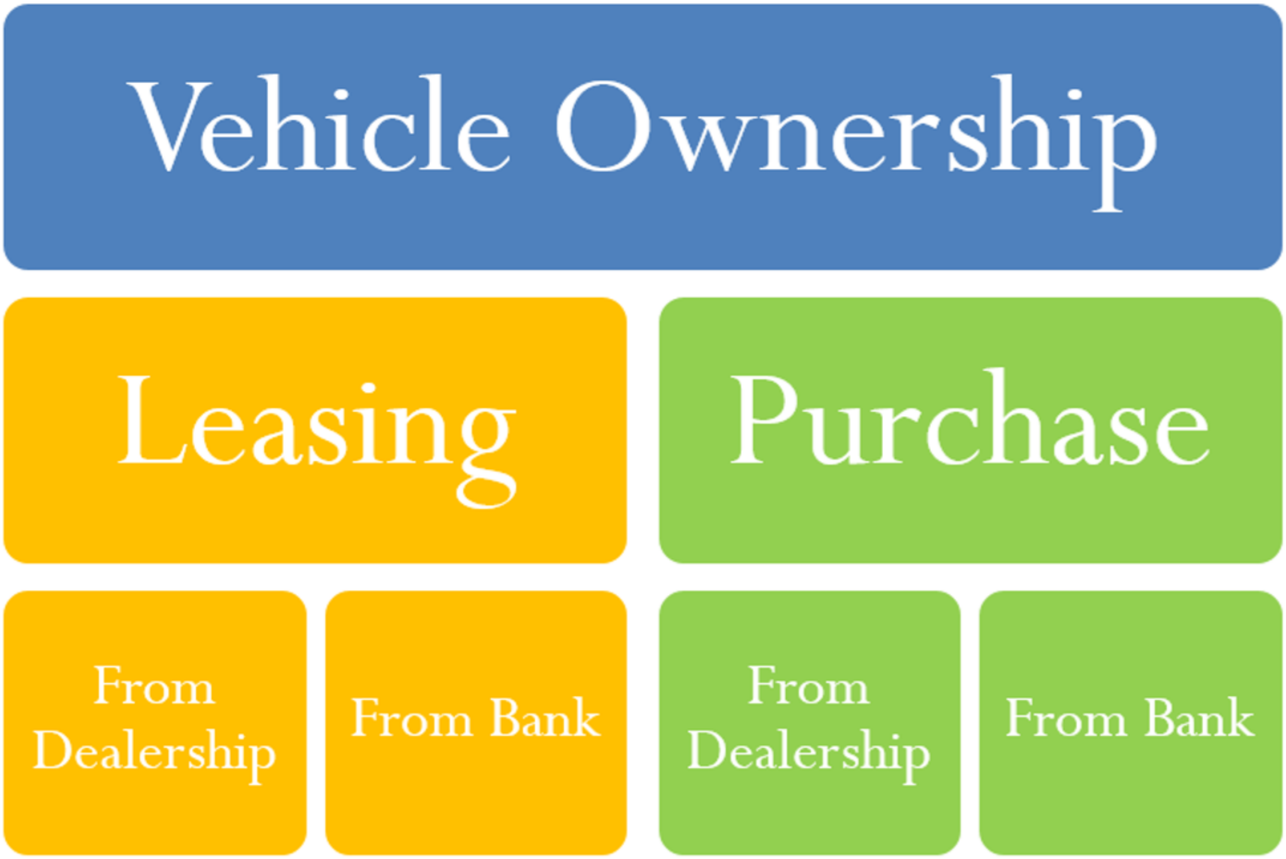
What Are the Choices?

A few matters need to be cleared up from the outset. This section sticks to examining vehicle leases aimed at eventual ownership. Studies of vehicle leases have revealed that leasing a vehicle without intent to own is much more costly than ownership, even after factoring in maintenance and repairs. Give it some thought: Leasing a vehicle with no intent ever to own it means that you will be making monthly car payments for the rest of your driving life. That is called the "forever-in-debt" club. That does not sound like much fun, does it?

Smart vehicle ownership means exploring the various financial paths to vehicle ownership and ultimately choosing the lowest cost option that results both in 100% ownership of the vehicle for as long as you decide to retain the vehicle and in eventual termination of your car payments. That definitely sounds much more financially appealing.

One of the ways to purchase the vehicle is to pay cash for it. While this is technically possible, very few Canadians have $30,000 lying around that they can just write a cheque against. Therefore, this option is not pursued as a normal alternative. If you cannot pay cash, then you must finance the purchase by borrowing the money from an organization.

The figure below illustrates the four paths available for a typical vehicle acquisition.



The two basic options are to lease the vehicle or purchase it. You can borrow the money from either the dealership or a financial institution of your choice (from here on just called a bank for simplicity), such as RBC or a credit union. Note that if you lease the vehicle, you intend to pay the residual value upon the termination of the lease term.

Through the Internet and publications such as newspapers and flyers, you can usually obtain a lot of information about each of these four options without having to visit the dealership or a financial institution. Sometimes you may have to look in the fine print. The goal is to explore your monthly payment under equal conditions for each of the four options and then select the lowest payment.

The Formula

Borrowing funds for a car requires either a loan or a lease. Both options represent annuities and have previously been discussed in this textbook. Therefore, to figure out your lowest cost alternative to vehicle ownership you do not need any new formulas, just a new process.

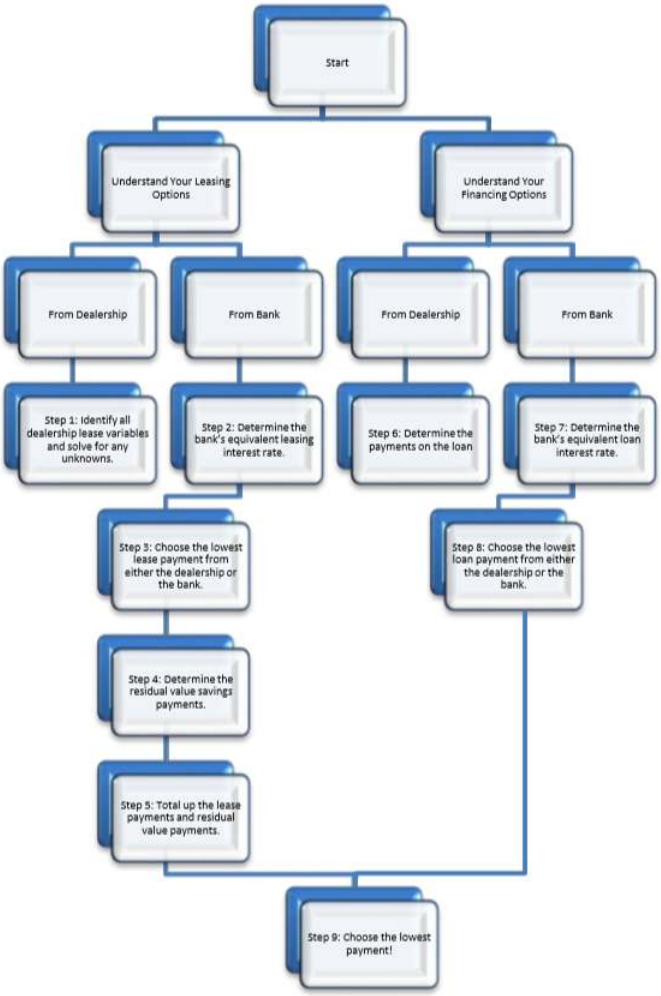
How It Works

The secret to choosing well is that most vehicles actually have two prices at the dealership. One is the financing price and the other is a cash price, which results from a cash rebate that lowers the price. To be eligible to receive this cash rebate, the dealership must receive payment in full for the vehicle on the date of purchase. If you elect to have a bank be your source of funds, then the bank will provide a cheque to the dealership for the vehicle on your behalf, and therefore you will have paid cash! Thus, you need to borrow less money from the bank to acquire the vehicle since the cash price is lower. If you elect for the dealership to be your source of funds, then the dealership is not paid in full right away, and you are not eligible for the cash rebate. Thus, you pay the financing price, which is higher.

Of course, the dealerships want you to borrow from them since you pay the higher price and they receive an additional revenue stream from the interest they charge you. Dealerships put very tempting offers in front of consumers. Because most consumers do not understand the time value of money, they fail to compare these offers with other alternatives.

Do you pay more for the car but at a lower interest rate from the dealership, or do you pay less for the car but at a higher interest rate from the bank? Only a time value of money calculation can determine that. The consumer’s best decision is based on which alternative has the lowest payment under equal conditions.

The steps for figuring out the lowest cost automotive option are illustrated in the figure below.


Start
Understand Your Leasing Options
Understand Your Financing Options
From Dealership
From Bank
From Dealership
From Bank
Step 1: Identify all dealership lease variables and solve for any unknowns.
Step 2: Determine the bank's equivalent leasing interest rate.
Step 6: Determine the payments on the loan
Step 7: Determine the bank's equivalent loan interest rate.
Step 3: Choose the lowest lease payment from either the dealership or the bank.
Step 8: Choose the lowest loan payment from either the dealership or the bank.
Step 4: Determine the residual value savings payments.
Step 5: Total up the lease payments and residual value payments.
Step 9: Choose the lowest payment!

To understand the process illustrated, it is much easier to obtain information on vehicles from dealership advertisements or websites. Most commonly, the price of the vehicle is the same regardless of which dealership you might visit. Therefore, acquiring the vehicle from a dealership commonly represents a singular path with known information. On the other hand, approximately 75 banks and 485 credit unions are available to consumers in Canada. This represents up to 560 different choices, from which it would be extremely time consuming to gather all required information. The process establishes a threshold against which the banks are compared. You need contact only those financial institutions that pass the initial screening.

Step 1: Understand your leasing terms from the dealership. You must identify your leasing variables, including the lease interest rate, lease term, leasing payment structure, any down payment required, the purchase price, and the residual value. If any of these variables is unknown, identify its value by applying the appropriate annuity due formula.

Step 2: Determine the equivalent leasing interest rate from your bank. Banks advertise interest rates, not payments. Thus, you calculate the interest rate a bank must charge to be equivalent to the dealership using the identical leasing structure using the same lease term, lease payments, compounding, and residual value. The present value used in this calculation is the dealer's purchase price from step 1 less the cash rebate. Solve for the bank's equivalent nominal interest rate (IY).

Step 3: Make your leasing decision. After conducting some research, if you find a bank with a lower interest rate than the calculated IY from step 2, then choose the bank lease (you should contact the bank to confirm the rate). Recalculate the leasing payments at the bank's interest rate. If you cannot find a lower interest rate from a bank, then select the dealership lease payments from step 1.

Step 4: A lease requires you to save up for the residual value. In choosing a leasing option, you also must save enough funds to pay off the residual value at the lease end. Set up an annuity due savings account using the same term and payment frequency. Use an appropriate interest rate from your savings institution. Calculate the annuity payment required to meet your residual value savings goal.

Step 5: Total the lease payment. Your lease payment from step 3 plus the residual value annuity payment from step 4 is the total out-of-pocket commitment required per payment interval to own the vehicle through leasing.

That completes the analysis of the leasing alternative. The remainder of the steps now involve examining your financing options.

Step 6: Figure out the loan payment from the dealership. Using the same term and payment frequency as the lease, calculate the payments to purchase the vehicle. Remember that a vehicle loan is an ordinary annuity. Use the financing price of the vehicle and the financing interest rate (not the lease rate) offered by the dealership.

Step 7: Figure out the equivalent loan interest rate from your bank. Using the same process as in step 2, aim to determine the equivalent interest rate from a bank loan that matches the payments with the dealership. With an identical structure and using the cash price, calculate the bank's equivalent nominal interest rate (IY).

Step 8: Make the loan decision. If a bank offers an interest rate lower than the calculated IY from step 7, then select the bank loan (and you should probably contact the bank to confirm the rate). Recalculate the loan payments at the bank's interest rate. If you are unable to find a bank with a lower interest rate, then select the dealership loan payments from step 6.

Step 9: The final decision. In step 5, you calculated the total lease payment. From step 8, you have the lowest loan payment. The obvious choice is the lowest payment.

Important Notes

1. At any given point in time, numerous vehicle rebates that adjust the purchase price of the vehicle are available (subject to various conditions). If such rebates apply to your purchase situation, modify the vehicle's purchase price accordingly in your calculations.
2. While leases are generally non-negotiable, the purchase price of a vehicle may be negotiable depending on the dealership and automotive manufacturer's policies. If haggling is allowed, estimate the amount by which you could reduce the purchase price and use the reduced amount for the calculations in steps 6 and 7.
3. If you have a trade-in for your vehicle acquisition, the value of the trade-in is treated in the same manner as a down payment toward the vehicle. It reduces the purchase price, so you modify your calculations accordingly.
4. The fundamental concept of time value of money says that to decide between the different leasing and financing timelines, all money must be moved to the same focal date. Note that in the process explained above you are making the final decision by comparing the payment amounts on the annuity due against the payment amounts on the ordinary annuity. A clear discrepancy in the timing of the payments is evident. In technically correct financial theory, to obey the fundamental concept of time value of money these payment amounts should then be present valued to the start of the transaction (the focal date) and the lower present value should be chosen. However, the reality is that most consumers base their purchase decision on the payment amount and what they can afford rather than on the total amount. Therefore, it is more logical to base the car acquisition decision strictly on the physical out-of-pocket annuity payment amounts.

Example 12.5.1: How Should You Acquire This Vehicle?

Haber Suzuki advertises a Suzuki SX4 Hatchback for sale. The following information is known:

- Leasing information: Seven-year term, 0.9% compounded annually lease rate, \$90.75 biweekly payments, \$8,250 residual value, no down payment required.
- Purchase information: Cash rebate of \$3,500 available, purchase financing at 2.99% compounded annually.
- Bank options: The lowest vehicle lease rate from a bank is 4.4% compounded annually, while the lowest loan rate is 8.25% compounded annually.
- Savings accounts: The best rate available is 2% compounded annually. What is the lowest cost method to purchase this vehicle?

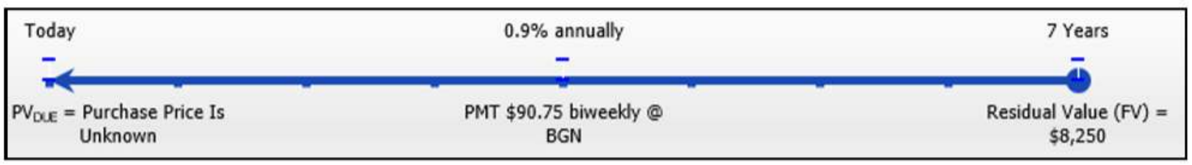
Solution

Examine the four paths to vehicle ownership and choose the path with the lowest biweekly payment.

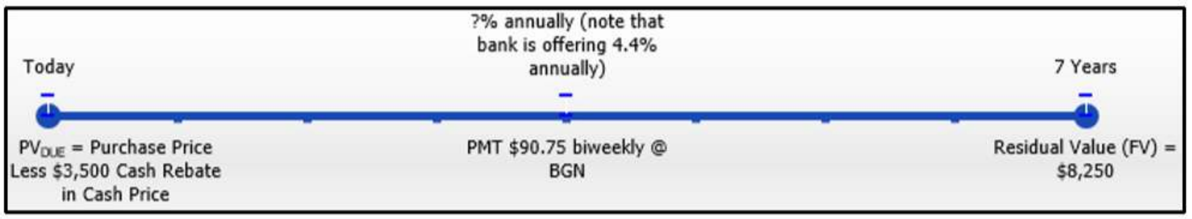
What You Already Know

There are five timelines, one for each of the options (two leases, one savings account, two purchases):

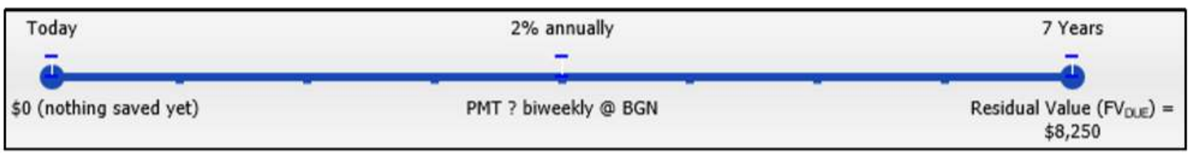
Dealership Lease: IY = 0.9%, CY = 1, PMT = \$90.75, PY = 26, FV = \$8,250, Years = 7


Today
0.9% annually
7 Years
PV$_{DUE}$ = Purchase Price Is Unknown
PMT \$90.75 biweekly @ BGN
Residual Value (FV) = \$8,250

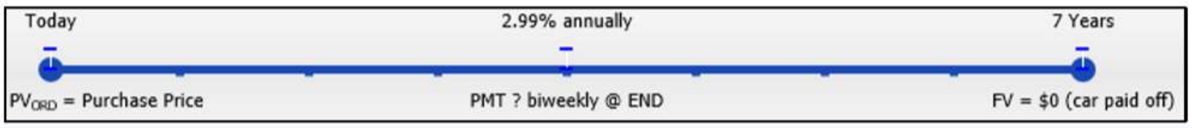
Bank Lease: PV_{DUE} = Purchase Price − \$3,500, CY = 1, PMT = \$90.75, PY = 26, FV = \$8,250, Years = 7


?% annually (note that bank is offering 4.4% annually)
Today
7 Years
PV$_{DUE}$ = Purchase Price Less \$3,500 Cash Rebate in Cash Price
PMT \$90.75 biweekly @ BGN
Residual Value (FV) = \$8,250

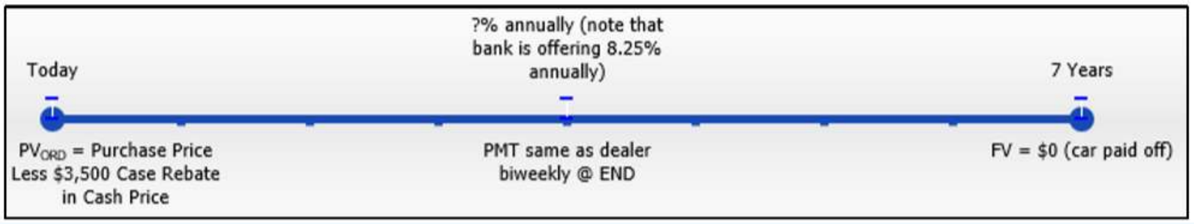
Residual Savings Plan: PV = \$0, IY = 2%, CY = 1, PY = 26, FV = \$8,250, Years = 7


Today
2% annually
7 Years
\$0 (nothing saved yet)
PMT ? biweekly @ BGN
Residual Value (FV$_{DUE}$) = \$8,250

Dealership Loan: PV_{ORD} = Purchase Price, IY = 2.99%, CY = 1, PY = 26, FV = \$0, Years = 7


Today
2.99% annually
7 Years
PV$_{ORD}$ = Purchase Price
PMT ? biweekly @ END
FV = \$0 (car paid off)

Bank Loan: PV_{ORD} = Purchase Price − \$3,500, CY = 1, PMT = Dealer loan payment, PY = 26, FV = \$0, Years =


?% annually (note that bank is offering 8.25% annually)
Today
7 Years
PV$_{ORD}$ = Purchase Price Less \$3,500 Case Rebate in Cash Price
PMT same as dealer biweekly @ END
FV = \$0 (car paid off)

How You Will Get There

Step 1:

In the dealership lease, the purchase price of the vehicle for the lease is missing. Find the present value of the residual value by applying Formulas 9.1, 9.2, and 9.3 (rearranged for PV). Then apply Formulas 11.1 and 11.5 to find the present value of the lease payments. Add the PV and PV_{DUE} to arrive at the purchase price.

Step 2:

In the bank lease, calculate the equivalent bank leasing interest rate. Adjust the purchase price to reflect the cash price and use your calculator to solve Formula 11.5 for IY.

Step 3:

If the bank rate is higher, choose the payment from step 1. If the bank rate is lower, then recalculate the lease payment using Formulas 9.1, 9.2, and 9.3 (rearranged for PV) to adjust the PV_{DUE} and apply Formula 11.5 (rearranged for PMT).

Step 4:

In the residual savings plan, save to pay off the residual value using the savings account interest rate. Apply Formulas 9.1 and 11.3 (rearranged for PMT).

Step 5:

Add the payments from steps 3 and 4. This is the total lease payment.

Step 6:

In the dealership loan, calculate the loan payments from the dealership by applying Formulas 9.1 and 11.4 (rearranged for PMT).

Step 7:

In the bank loan, calculate the equivalent bank loan interest rate. Adjust the purchase price and use your calculator to solve Formula 11.4 for IY.

Step 8:

If the bank rate is higher, choose the payment from step 6. If the bank rate is lower, recalculate the loan payment, first using Formulas 9.1 and 9.3 (rearranged for PV) to adjust the PV_{ORD} and then using Formula 11.4 (rearranged for PMT).

Step 9:

Choose the lowest payment from step 5 or step 8.

Perform

Step 1:

Residual Value:

$i = 0.9\%/1 = 0.9\%; N = 1 \times 7 = 7 \text{ compounds };$

$\$8,250 = PV(1+0.009)^7;$

$$PV = \$8,250 \div (1+0.009)^7 = \$7,748.466949$$

Lease Payments:

$N = 26 \times 7 = 182 \text{ payments}$

$$PV_{DUE} = \$90.75\left[\frac{1-\left[\frac{1}{(1+0.009)^{\frac{1}{26}}}\right]^{182}}{(1+0.009)^{\frac{1}{26}}-1}\right] \times (1+0.009)^{\frac{1}{26}} = \$16,011.97657$$

$$\text{Total } PV = \$7,748.466949 + \$16,011.97657 = \$23,760.44 = \text{ purchase price of the vehicle}$$

Step 2:

$\text{Cash price of vehicle } = \$23,760.44 - \$3,500.00 = \$20,260.44$

Solve Formula 11.5 (adjusting for residual value):

$$\$20,260.44 - \frac{\$8,250}{\left((1+i)^{\frac{1}{26}}\right)^{182}} = \$90.75\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{26}}}\right]^{182}}{(1+i)^{\frac{1}{26}}-1}\right] \times (1+i)^{\frac{1}{26}}$$

From calculator, IY = 4.5331% compounded annually.

Step 3:

The bank lease rate of 4.4% is lower than the equivalent lease rate of 4.5331%. Choose the lease from the bank and recalculate the payment.

$i = 4.4\%/1 = 4.4\%; N = 1 \times 7 = 7 \text{ compounds}$

$$\begin{aligned} \$8,250 &= PV(1+0.044)^7 \\ PV &= \$8,250 \div (1+0.044)^7 \\ &= \$6,103.099605 \end{aligned}$$

$$\text{new } PV_{DUE} = \$20,260.44 - \$6,103.099605 = \$14,157.34039$$

$N = 182$ payments (same as step 1)

$$\$14,157.34039 = PMT\left[\frac{1-\left[\frac{1}{(1+0.044)^{\frac{1}{26}}}\right]^{182}}{(1+0.044)^{\frac{1}{26}}-1}\right] \times (1+0.044)^{\frac{1}{26}}$$

$$\$14,157.34039 = PMT\left[\frac{0.260230}{0.001657}\right] \times 1.001657$$

$$PMT = \frac{\$14,157.34039}{157.261348} = \$90.02$$

Step 4:

$i = 2\%/1 = 2\%; N = 182 \text{ payments}$ (same as step 1)

$$\$8,250 = PMT\left[\frac{\left[(1+0.02)^{\frac{1}{26}}\right]^{182}-1}{(1+0.02)^{\frac{1}{26}}-1}\right] \times (1+0.02)^{\frac{1}{26}}$$

$$\$8,250 = PMT\left[\frac{0.148685}{0.000761}\right] \times 1.000761$$

$$PMT = \frac{\$8,250}{195.292254} = \$42.24$$

Step 5:

The total lease payment is \$90.02 + \$42.24 = \$132.26 biweekly.

Step 6:

$\text{Purchase price } = \$23,760.44$ (from step 1) ; $i = 2.99\%/1 = 2.99\%; N = 182 \text{ payments}$ (same as in step 1)

$$\$23,760.44 = PMT\left[\frac{1-\left[\frac{1}{(1+0.0299)^{\frac{1}{26}}}\right]^{182}}{(1+0.0299)^{\frac{1}{26}}-1}\right.$$

$$\$23,760.44 = PMT\left[\frac{0.186355}{0.001133}\right]$$

Step 7:

Cash price of vehicle =$20,260.44 (from step 2).

$$\$20,260.44 = \$144.56\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{26}}}\right]^{182}}{(1+i)^{\frac{1}{26}}-1}\right]$$

From calculator, IY = 8.0841% compounded annually.

Step 8:

Since the bank loan rate of 8.25% is higher than the equivalent loan rate of 8.0841%, remain with the dealership loan and make payments of $144.56.

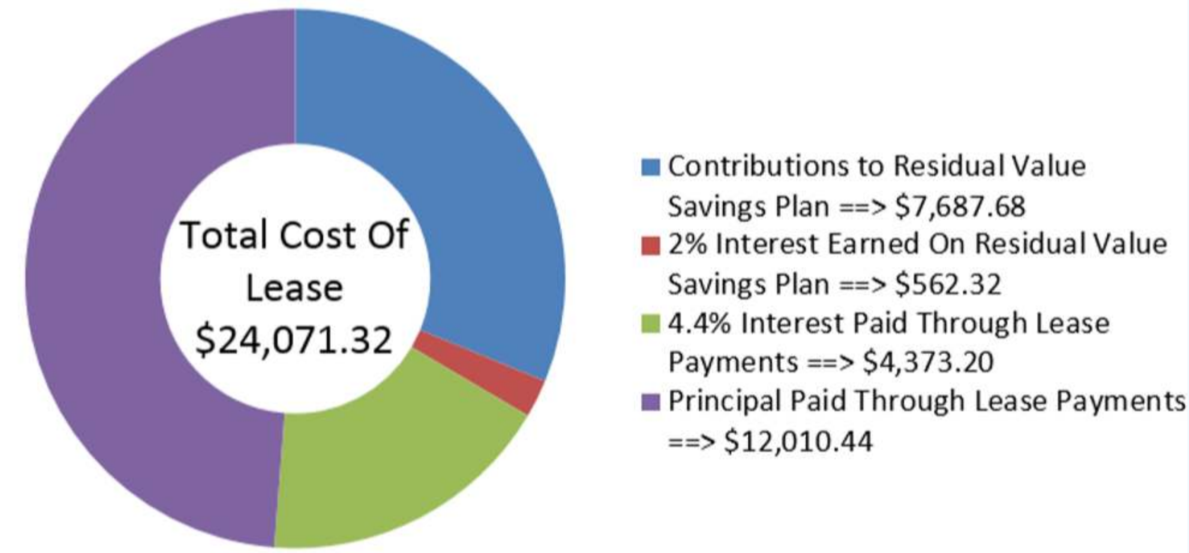
Step 9:

The lowest payment is from step 5, which is $132.26.

Calculator Instructions

Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	182	0.9	Answer: 23,760.44352	-90.75	-8250	26	1
2	√	√	Answer: 4.533113	20260.44	√	√	√	√
3	√	√	4.4	√	Answer: -90.024284	√	√	√
4	√	√	2	0	Answer: -42.244378	8250	√	√
6	END	√	2.99	23760.44	Answer: -144.558127	0	√	√
7	√	√	Answer: 8.084126	20260.44	-144.56	√	√	√

The best option to acquire this vehicle is to lease it from the bank. This requires biweekly lease payments of $90.02 and residual savings payments of $42.24 for a total of $132.26 per period for seven years. At the end of the lease, the accumulated savings is used to pay the $8,250 residual value.



Contributors and Attributions

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12.6: Application - Planning Your RRSP

How and when should you start planning your RRSP? When you are in your late teens or early twenties, you are probably not thinking much about your retirement income. You are at a stage in life when, besides paying for education or paying off student loans, you move out on your own and need to acquire many possessions ranging from appliances to furniture to transportation. Your income is probably at its lowest point in your career. You find yourself on a limited budget with high demands.

Ironically, though, this is the best time to start your RRSP. Throughout the last few chapters you have already witnessed the incredible power of compound interest. Through discussion, examples, and even some of the exercises, it should be clear that the earlier you invest your money, the less you have to pay out of your pocket to reach your retirement income goals.

But why should you go to the trouble and expense of contributing to an RRSP? For those counting on the government to provide retirement income, it was pointed out that the typical Canadian in retirement earns $529.09 per month from the Canada Pension Plan (CPP) and $514.74 from Old Age Security (OAS). Both of these amounts are pre-tax, which means your net income from these sources will be even less. This is not much to live off of.

Are companies not supposed to provide pension plans for their employees? While some companies do offer pension plans, these public and private-sector companies are few and far between. The Pension Investment Association of Canada (PIAC) manages approximately 130 pension funds representing approximately 80% of Canada's pension industry by asset size. In a 2007 study conducted by the PIAC, it found that approximately 20.5% of Canadian membership companies are either closing or considering closing their defined benefit pension funds. In Ontario, the percentage is much higher at 36.2%. These closures are in the private sector. Additionally approximately 20% of Canadian organizations are reducing future benefits.1

The bottom line is that you should not count on somebody else to create your retirement income for you. Also, do not bank on being lucky enough to work for a company that has a good pension plan. You need to take care of yourself.

This section introduces a simplified model of RRSP planning that reasonably approximates the financial commitment that will provide a satisfactory retirement income. Ultimately, you should always consult a financial adviser as you plan your RRSP.

Income Planning

Income planning requires you to project future earnings, determine income needs in retirement, and then develop a savings plan toward that goal. Look at these details:

1. Projected Retirement Income. The first step is to determine your annual retirement income at the age of retirement. The general consensus is that a retiree needs approximately 70% of pre-retirement gross income to live comfortably and maintain the same lifestyle. It is difficult to know what that amount will be 30 or 40 years from now. However, today's typical retirement income is known. The average Canadian needs about $40,000 of gross income. You can adjust this number higher if necessary. You can then project today's amount forward to the age of retirement using an estimated rate of inflation based on historical data.
2. Principal Required at Age of Retirement. The second step requires you to determine the principal needed to fund the annual income requirement at the age of retirement. You must estimate the number of years for which withdrawals are to be made from the RRSP balance. In essence, how long are you going to live? While no one knows the answer, the average Canadian male lives to age 78 while the average Canadian female lives to age 82. These are good starting points. It would not make sense to lower these numbers since you do not want to be caught short, but you could raise these numbers if necessary. A look at your parents, grandparents, and other ancestors combined with your own lifestyle choices might indicate the typical life span you may expect. While you are not guaranteed to live that long, at least you can base a reasonable financial decision upon it. You must also factor in inflation during the retirement years. Each year, your income must rise to keep up with the cost of living. Finally, you should use a

conservative, low-risk interest rate in these calculations because you certainly cannot afford to lose your savings to another "Black Monday" (which refers to October 19, 1987, when stock markets around the world crashed).

3. Annuity Savings Payments. The final step is to determine the payments necessary to reach the principal required for retirement. You should consider that your income generally rises over your lifetime, which means as you get older you could afford to contribute a larger amount to your RRSP. Thus, contributions start small when you are young and increase as you grow older. The interest rate you use must reflect the market and your risk tolerance. Since 1980, the Toronto Stock Exchange (TSX) has averaged approximately 6% annual growth, which is a good starting point for determining what interest rate to use in your calculations.

The Formula

RRSP planning does not involve any new formulas. Instead, you must combine previously studied single payment concepts from Chapter 9 and annuity concepts from both Chapter 11 and Chapter 12.

How It Works

Your end goal is to calculate the regular contributions to your RRSP necessary to achieve a balance in your RRSP from which you can regularly withdraw in retirement to form your income. The calculations presented ignore other sources of retirement income such as the CPP or OAS, as if you were solely responsible for your own financial well-being. You can then treat any other income as bonus income.

Follow the steps:

Step 1: Calculate the annual retirement income you will need. Choose a value of annual income in today's dollars along with an annual rate of inflation to use. Then, by applying Formula 9.1 (Periodic Interest Rate), Formula 9.2 (Number of Compound Periods for Single Payments), and Formula 9.3 (Compound Interest for Single Payments) you can move that income to your required age of retirement.

Step 2: Calculate the present value of the retirement income. Most people receive their retirement income monthly, so you divide the result from step 1 by 12. Retirement income usually starts one month after retirement, thus forming an ordinary annuity. Using a reasonable annual rate of inflation, you also divide the inflation rate by 12 to approximate the growth rate to be used in calculating a constant growth annuity required during retirement. Estimate the number of years the retirement fund must sustain and select a low-risk conservative interest rate. Then apply Formula 9.1 (Periodic Interest Rate), Formula 11.1 (Number of Annuity Payments), and Formula 12.3 (Present Value of a Constant Growth Ordinary Annuity) to arrive at the principal required at the age of retirement.

Step 3: Calculate the annuity payment required to achieve your goal. If a single payment is already invested today, deduct its future value at the age of retirement (using Formula 9.1, Formula 9.2, and Formula 9.3) from the amount of money determined in step two. The principal at the age of retirement (from step 2) now becomes the future value for the ordinary constant growth annuity. Use an interest rate that reflects market rates and your risk level, along with an appropriate growth rate you select for the annuity payments you will make. This growth rate may or may not match the growth rate determined in step 2. Thus, arriving at the annuity payment amount requires Formula 9.1 (Periodic Interest Rate), Formula 11.1 (Number of Annuity Payments), and Formula 12.1 (Future Value of a Constant Growth Ordinary Annuity) rearranged for PMT.

Important Notes

This procedure approximates the required regular RRSP contribution. However, recognize that the procedure is somewhat simplified and certainly does not factor in all variables that may apply in an individual situation. It is always best to consult a certified financial planner to ensure that your RRSP plan works under current market conditions and under any applicable restrictions.

Also, the steps listed above also assume an ordinary annuity structure. In the event of an annuity due, substitute the appropriate annuity due formula as required.

Paths To Success

One of the hardest tasks in planning an RRSP is to guess interest rates and rates of inflation. If you are unsure of what numbers to use, some safe values are as follows:

1. A rate of inflation or growth rate of 3% compounded annually, equaling 0.25% per month
2. During retirement, an interest rate of 4% compounded annually
3. While you contribute toward your RRSP, an interest rate of 6% compounded annually, based on the approximate historical average over the past 31 years (1980–2011) for the TSX.

? Exercise 12.6.1: Give It Some Thought

Assume that two people—Person A and Person B—are saving for the same retirement amount under equal conditions except as noted. In each of the following cases, determine whose RRSP contributions are higher:

a. Person A starts contributing at age 18 while Person B starts contributing at age 25.
b. Person A earns 7% compounded annually while Person B earns 6% compounded annually.
c. Person A contributes monthly while Person B contributes semi-monthly.
d. Person A contributes through an ordinary annuity while Person B uses an annuity due.
e. Person A earns 4.5% interest during retirement while Person B earns 4% interest during retirement.
f. Person A has quarterly compounded earnings while Person B has monthly compounded earnings.

Answer

a. Person B, because he has a shorter period to accumulate the same amount as Person A.
b. Person B, because a lower interest rate requires more principal, hence larger contributions.
c. Person A, because the principal does not increase as often as Person B’s does.
d. Person A, because there is one fewer compounds than for Person B.
e. Person B earns less interest, so a larger principal is required at retirement than Person A requires.
f. Person A, because the interest is not converted to principal as often as it is for Person B.

✓ Example 12.6.1: An 18-Year Old with No Savings Planning for Retirement

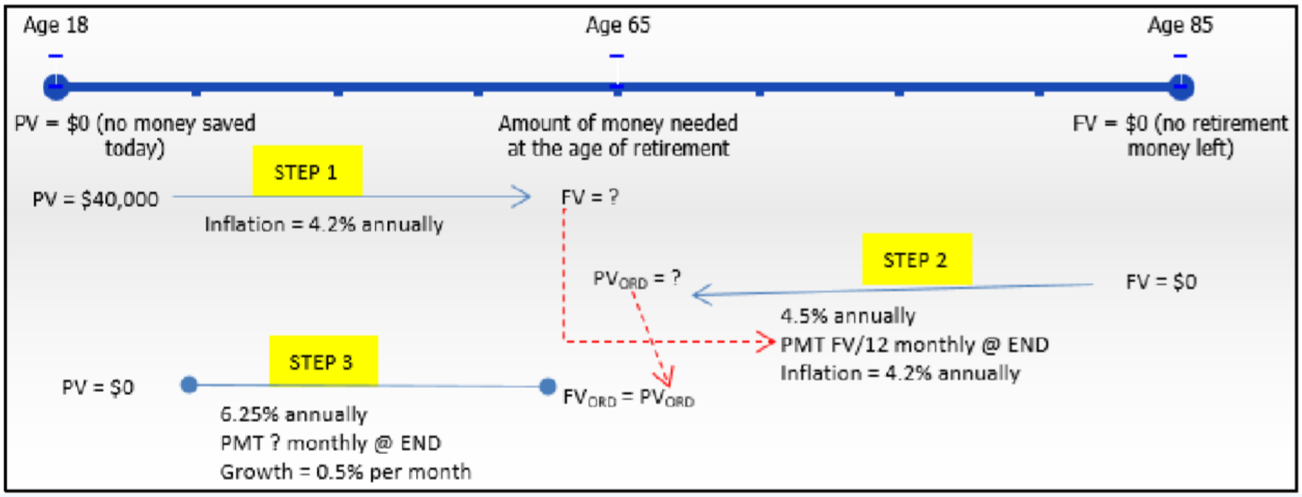
Jesse just turned 18 and plans on retiring when he turns 65. In today's funds, he wants to earn $40,000 annually in retirement. Based on the past 50 years, he estimates inflation at 4.2% compounded annually. He would like to receive 20 years of monthly payments from his retirement money, which is forecasted to earn 4.5% compounded annually. He believes his RRSP can earn 6.25% compounded annually during his contributions, he has no money saved to date, and he would like to increase each payment by 0.5%. One month from now, what is the amount of his first monthly RRSP contribution?

Solution

Jesse wants to make regularly increasing contributions to his RRSP at the end of the monthly payment interval with annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate his first payment (PMT).

What You Already Know

Jesse's RRSP plan appears in the timeline.


Age 18
Age 65
Age 85
PV = $0 (no money saved today)
Amount of money needed at the age of retirement
FV = $0 (no retirement money left)
STEP 1
PV = $40,000
FV = ?
Inflation = 4.2% annually
STEP 2
PV_ORD = ?
FV = $0
4.5% annually
PMT FV/12 monthly @ END
Inflation = 4.2% annually
STEP 3
PV = $0
FV_ORD = PV_ORD
6.25% annually
PMT ? monthly @ END
Growth = 0.5% per month

Step 1:

PV = $40,000, IY = 4.2%, CY = 1, Years = 47

Step 2:

FV = $0, IY = 4.5%, CY = 1, PY = 12, Years = 20, Δ% = 4.2%/12 = 0.35% per payment

Step 3:

PV = $0, FV_{ORD} = Step 2 answer, IY = 6.25%, CY = 1, PY = 12, Years = 47, Δ% = 0.5%

How You Will Get There

Step 1:

Move Jesse's desired annual income from today to the age of retirement by applying Formula 9.1, Formula 9.2, and Formula 9.3.

Step 2:

Calculate the amount of money needed at the age of retirement by applying Formula 9.1, Formula 11.1, and Formula 12.3.

Step 3:

Calculate Jesse's regular monthly payment by applying Formula 9.1, Formula 11.1, and Formula 12.1.

Perform

Step 1:

$$i = 4.2\%/1 = 4.2\%; N = 1 \times 47 = 47 \text{ compounds}; \; FV = \$40,000(1+0.042)^{47} = \$276,594.02$$

Step 2:

$$i = 4.5\%/1 = 4.5\%; N = 12 \times 20 = 240 \text{ payments}$$

Note that the monthly payment is $276,594.02 ÷ 12 = $23,049.50

$$PV_{ORD} = \frac{\$23,049.50}{1+0.0035}\left[\frac{1-\left[\frac{1+0.0035}{(1+0.045)^{\frac{1}{12}}}\right]^{240}}{\frac{(1+0.045)^{\frac{1}{12}}}{1+0.0035}-1}\right] = \$5,398,479.88$$

Step 3:

$$i = 6.25\%/1 = 6.25\%; N = 12 \times 47 = 564 \text{ payments}$$

$$\left[\left[\frac{(1+0.0625)^{\frac{1}{12}}}{1+0.005}\right]^{564}-1\right]$$

$$\$5,398,479.88 = PMT(1+0.005)^{564-1}\left[\frac{}{\frac{(1+0.0625)^{\frac{1}{12}}}{1+0.005}-1}\right]$$

$$\begin{aligned}\$5,398,479.88 &= PMT(16.576497)[574.367284]\\ \$5,398,479.88 &= PMT(9,520.997774)\\ \$567.01 &= PMT\end{aligned}$$

Calculator Instructions

Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	END	47	4.2	-40000	0	Answer: 276,594.0176	1	1

On the basis that the rate of inflation is relatively accurate, Jesse’s first RRSP contribution one month from now is $567.01. Each subsequent payment increases by 0.5% resulting in a balance of $5,398,479.88 at retirement, from which he can make his first withdrawal of $23,049.50 increasing at 0.35% per month for 20 years.

✓ Example $12.6.2$: A 30-Year Old with Savings Planning for Retirement

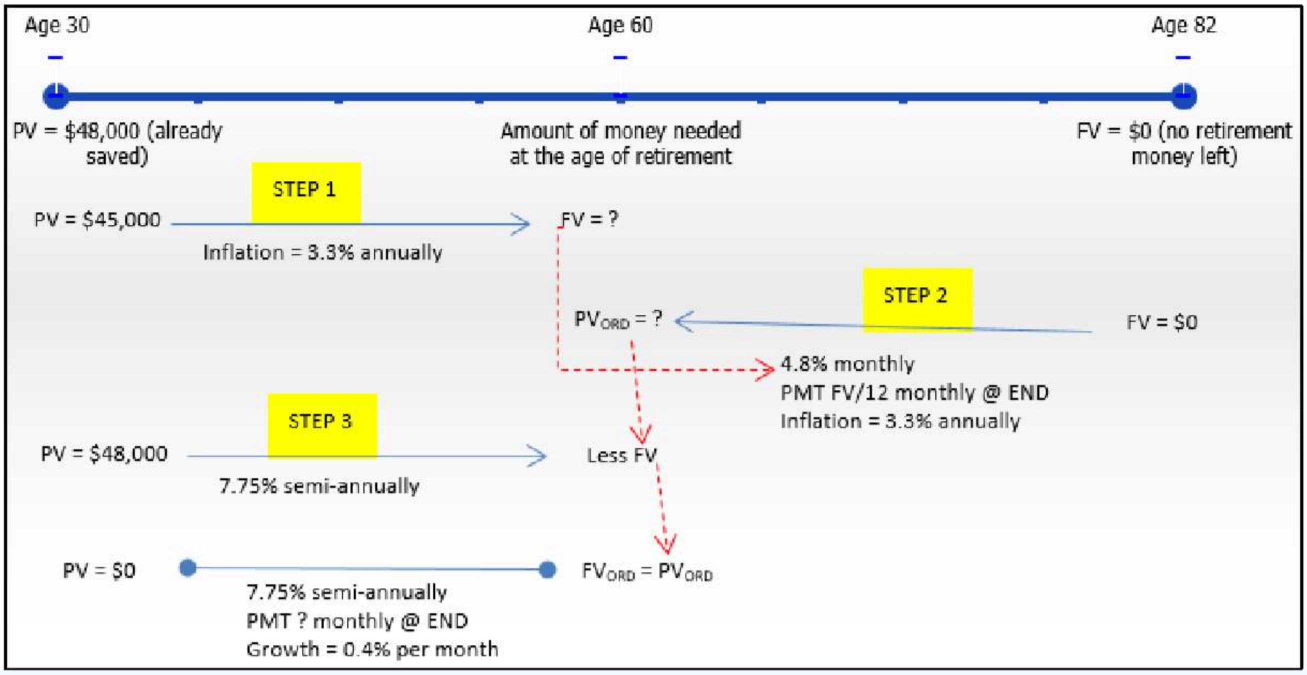
Marilyn just turned 30 and plans on retiring when she turns 60. In today's funds, she wants to earn $45,000 in retirement. Based on the past 30 years, she estimates inflation at 3.3% compounded annually. She would like to receive 22 years of month-end payments from her retirement fund, which is forecasted to earn 4.8% interest compounded monthly. Based on the results to date, she believes her RRSP can earn 7.75% compounded semiannually during her contributions. She has already saved $48,000, and she will increase each contribution by 0.4%. One month from now, what is the amount of her first monthly RRSP contribution?

Solution

She will make regularly increasing contributions to her RRSP at the end of the monthly payment interval with semi-annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate her first payment (PMT).

What You Already Know

Marilyn’s RRSP plan appears in the timeline.


Age 30
Age 60
Age 82
PV = $48,000 (already saved)
Amount of money needed at the age of retirement
FV = $0 (no retirement money left)
STEP 1
PV = $45,000
FV = ?
Inflation = 3.3% annually
STEP 2
PV$_{ORD}$ = ?
FV = $0
4.8% monthly
PMT FV/12 monthly @ END
Inflation = 3.3% annually
STEP 3
PV = $48,000
Less FV
7.75% semi-annually
PV = $0
FV$_{ORD}$ = PV$_{ORD}$
7.75% semi-annually
PMT ? monthly @ END
Growth = 0.4% per month

Step 1:

PV = $45,000, IY = 3.3%, CY = 1, Years = 30

Step 2:

FV = $0, IY = 4.8%, CY = 12, PY = 12, Years = 22, Δ% = 3.3%/12 = 0.275% per payment

Step 3:

PV = $48,000, FV_{ORD} = Step 2 answer, IY = 7.75%, CY = 2, PY = 12, Years = 30, Δ% = 0.4%

How You Will Get There

Step 1:

Move her desired annual income from today to the age of retirement by applying Formula 9.1, Formula 9.2, and Formula 9.3.

Step 2:

Calculate the amount of money needed at the age of retirement by applying Formula 9.1, Formula 11.1, and Formula 12.3.

Step 3:

Deduct her current savings from her required future contributions by calculating the future value of her present savings and deducting it from the step 2 answer by applying Formula 9.1, Formula 9.2, and Formula 9.3. Calculate her regular monthly payment by applying Formula 9.1, Formula 11.1, and Formula 12.1.

Perform

Step 1:

$$i = 3.3\%/1 = 3.3\%; N = 1 \times 30 = 30 \text{ compounds; } FV = \$45,000(1+0.033)^{30} = \$119,185.15$$

Step 2:

$$i = 4.8\%/12 = 0.4\%; N = 12 \times 22 = 264 \text{ payments}$$

Note that the monthly payment is $119,185.15/12 = $9,932.10

$$PV_{ORD} = \frac{\$9,932.10}{1+0.00275}\left[\frac{1-\left[\frac{1+0.00275}{(1+0.004)^{\frac{12}{12}}}\right]^{264}}{\frac{(1+0.004)^{\frac{12}{12}}}{1+0.00275}-1}\right] = \$2,226,998.08$$

$$1 + 0.00275$$

Step 3:

Current Savings:

$$i = 7.75\%/2 = 3.875\%; N = 2 \times 30 = 60 \text{ compounds }; FV = \$48,000(1+0.03875)^{60} = \$469,789.65$$

$$\text{new } FV_{ORD} = \$2,226,998.08 - \$469,789.65 = \$1,757,208.43$$

RRSP Payments:

$$i = 7.75\%/2 = 3.875\%; N = 12 \times 30 = 360 \text{ payments}$$

$$\$1,757,208.43 = PMT(1+0.004)^{360-1}\left[\frac{\left[\frac{(1+0.03875)^{\frac{2}{12}}}{1+0.004}\right]^{360} - 1}{\frac{(1+0.03875)^{\frac{2}{12}}}{1+0.004} - 1}\right]$$

$$\begin{aligned}\$1,757,208.43 &= PMT(4.191822)[564.766990] \\ \$1,757,208.43 &= PMT(2,367.403054) \\ \$742.25 &= PMT\end{aligned}$$

Calculator Instructions

Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	END	30	3.3	-45000	0	Answer: 119,185.1533	1	1
3(Savings)	√	60	7.75	-48000	√	Answer: 469,789.6504	2	2

On the basis that the rate of inflation is relatively accurate, Marilyn's first RRSP contribution one month from now is $742.25. Each subsequent payment increases by 0.4% resulting in a balance of $2,226,998.08 (including her current savings) at retirement, from which she makes her first withdrawal of $9,932.10 increasing at 0.275% per month for 22 years.

References

1. Pension Investment Association of Canada, Pension Plan Funding Challenges: 2007 PIAC Survey, December 2007, www.piacweb.org/files/Pension%20Plan%20Funding%20Challenges%202007%20PIAC%20Survey%20Dec%2007.pdf (accessed September 26, 2010).

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12.S: Summary

Key Concepts

12.1: Deferred Annuities

- The stages of deferred annuities
- The four common unknown variables and how to solve for them

12.2: Constant-Growth Annuities

- The concept of constant growth and the modifications required to annuity formulas
- The four new annuity formulas
- How to solve constant growth scenarios

12.3: Perpetuities

- An explanation of perpetuities
- Ordinary and due types of perpetuities
- The two perpetuity formulas
- How to solve perpetuity scenarios

12.4: Leases

- An explanation of lease characteristics and how leases operate
- Capitalized lease liabilities explained
- How to solve leasing scenarios

12.5: Application - How To Purchase A Vehicle

- Four financial choices available to make payments on a vehicle
- The basis and procedure upon which the selection of the vehicle ownership choice is made
- Key considerations to keep in mind when purchasing a vehicle

12.6: Application - Planning Your RRSP

- The three components of retirement income planning
- The procedure for retirement income planning

The Language of Business Mathematics

capitalized lease liability

The present value of the remaining lease payments and residual value on a capital good using a discount rate equivalent to the interest rate the business would have had to pay if it had purchased the asset instead.

constant growth annuity

An annuity in which each annuity payment is increased by a fixed percentage.

deferred annuity

A financial transaction where annuity payments are delayed until a certain period of time has elapsed.

down payment

A portion of the purchase price required up front.

lease

A contract by which the owner of an asset gives another party an exclusive right to possess and use the asset under specified conditions for a specific period of time in return for agreed-upon payments.

lessee

The borrower of a leased asset.

lessor

The owner of a leased asset.

net rate

The growth in a constant growth annuity that is attributable solely to the interest rate and not to the growth in the annuity payment.

period of deferral

The time segment of a deferred annuity where the single payment earns interest and no contributions are made to the investment.

perpetuity

A special type of annuity that has fixed, regular payments continuing indefinitely.

residual value

The projected value of an asset at the end of its lease term.

zero growth annuity formula

Any annuity formula where the growth rate is 0%. All formulas presented in Chapter 11 incorporate zero growth

The Formulas You Need to Know

Symbols Used

$\Delta\%$ = the constant growth rate per payment interval (percent change)

CY = compounding frequency

FV_{DUE} = future value of an annuity due

FV_{ORD} = future value of an ordinary annuity

i = periodic interest rate

N = number of annuity payments

PMT = annuity payment amount

PV_{DUE} = present value of an annuity due

PV_{ORD} = present value of an ordinary annuity

PY = payment frequency

Formulas Introduced

Formula 12.1 Future Value of a Constant Growth Ordinary Annuity:

$$FV_{ORD} = PMT(1+\Delta\%)^{N-1}\left[\frac{\left[\frac{(1+i)^{\frac{CY}{PY}}}{1+\Delta\%}\right]^{N}-1}{\frac{(1+i)^{\frac{CY}{PY}}}{1+\Delta\%}-1}\right]$$

Formula 12.2 Future Value of a Constant Growth Annuity Due:

$$FV_{DUE} = PMT(1+\Delta\%)^{N-1}\left[\frac{\left[\frac{(1+i)^{\frac{CY}{PY}}}{1+\Delta\%}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}}\right]\times(1+i)^{\frac{CY}{PY}}$$

$$\left. \frac{}{1+\Delta\%} - 1 \right]$$

Formula 12.3 Present Value of a Constant Growth Ordinary Annuity:

$$PV_{ORD} = \frac{PMT}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+i)^{\frac{CY}{PY}}}\right]^N}{\frac{(1+i)^{\frac{CY}{PY}}}{1+\Delta\%}-1}\right] \times (1+i)^{\frac{CY}{PY}}$$

Formula 12.4 Present Value of a Constant Growth Annuity Due:

$$PV_{DUE} = \frac{PMT}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+i)^{\frac{CY}{PY}}}\right]^N}{\frac{(1+i)^{\frac{CY}{PY}}}{1+\Delta\%}-1}\right] \times (1+i)^{\frac{CY}{PY}}$$

Formula 12.5 Ordinary Perpetuity Present Value:

$$PV_{ORD} = \frac{PMT}{(1+i)^{\frac{CY}{PY}-1}}$$

Formula 12.6 Perpetuity Due Present Value:

$$PV_{DUE} = PMT\left(\frac{1}{(1+i)^{\frac{CY}{PY}}-1}+1\right)$$

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12.CS: Case Study - Should You Go On Strike?

The Situation

It is time to negotiate a new contract with some of Lightning Wholesale's unionized employees. The company believes in dealing fairly with its employees. Based on the current economic environment, cost of living increases, and the financial health of the organization, management feels that the best it can offer is a 3% wage increase. From its own analysis, the union believes that the company is holding out and that a 5.5% wage increase is more than possible. Unfortunately, negotiations have broken down, and the union has turned to its employee group seeking strike action. The union is certain of achieving its wage increase through the strike action, though it advises the employees that they may need to go on strike for three months to achieve the goal. The employees are trying to figure out their best course of action—should they vote to go on strike or not?

The Data

- The typical employee in the unionized group currently earns $48,000 per year, which is paid out at the end of every month equally.
- Six employees have five years until retirement.
- Eight employees have 10 years until retirement.
- Nine employees have 15 years until retirement.
- Seven employees have 20 years until retirement.

Important Information

- During the three months that employees would be on on strike, employees receive no wages from Lightning Wholesale.
- The time value of money is unknown, but employees have four annually compounded estimates of 6%, 5%, 4%, and 3%.
- Assume employees make their strike vote according to their best financial outcome.
- The decision to go on strike is determined by the majority vote.
- No future wage increases are important when making this decision to strike or not.

Your Tasks

The employees are uncertain of the time value of money, so they need to run a few scenarios. Perform steps 1 through 3 below using EACH of the time value of money estimates as a different scenario. Determine the outcome of the strike vote (go on strike or not go on strike).

1. Calculate the present value of the company offer for each of the employee groups.
2. Calculate the present value of the union increase for each of the employee groups.
3. Cast the votes according to your results and determine the strike vote outcome under each time value of money possibility.
4. Management is trying to figure out the most likely outcome of the strike vote so that they can adjust their bargaining strategy if necessary. Based on the completed scenario analysis, what outcome should management plan on?
5. From your scenario analysis, what are some key decision-making variables that the employees need to consider before casting a vote to go on strike?

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12.E: Compound Interest- Special Applications Of Annuities (Exercises)

12.1: Deferred Annuities

Mechanics

Accumulation Stage Payments			Payments Stage			
Present Value	**Nominal Interest Rate**	**Period of Deferral**	**Annuity Payments**	**Annuity Payment Timing**	**Annuity Term**	**Nominal Interest Rate**
1. ?	7.7% semiannually	15½ years	$4,200 quarterly	End	12 years	5.9% semiannually
2. $40,000	6.1% quarterly	?	$908.56 monthly	Beginning	10 years	4.2% monthly
3. $100,000	8.5% annually	20 years	$20,000 semiannually	Beginning	?	3.9% quarterly
4. $17,500	7.5% monthly	14 years, 8 months	? annually	End	5 years	5% annually
5. $28,900	8.2% quarterly	?	$17,018.98 semiannually	End	7 years, 6 months	2.8% annually
6. $55,000	10.8% monthly	13 years, 10 months	$9,850.80 quarterly	End	?	6.6% quarterly
7. ?	8.9% annually	11 years	$1,500 monthly	Beginning	20 years	3% monthly
8. $15,000	5.5% semiannually	18 years	? monthly	Beginning	4 years	4.1% semiannually

Applications

9. What is the present value of a deferred annuity with a deferral period of 17 years at 6.7% compounded semi-annually followed by a 10-year annuity due paying $1,250 every month at 4.78% compounded semi-annually?
10. Your objective is an annuity due paying $5,000 semi-annually for 5½ years at 4% compounded quarterly. How far in advance of this would you need to invest $20,000 at 6.82% compounded monthly? Assume 30 days in a month.
11. If $38,000 is invested for 15 years at 9.4% compounded quarterly and then pays out $10,000 at the beginning of each year while earning 2.4% compounded annually, how far from today would the last payment occur?
12. Jeff and Sarah want an ordinary annuity to pay their daughter $1,000 monthly for three years and nine months for the duration of her educational studies starting August 1, 2024. What lump-sum amount do they need to invest on August 1, 2014, if the deferred annuity can earn 6.6% compounded monthly during the accumulation stage and 4% compounded quarterly during the income payments stage?
13. On July 13, 2011, Harriet invested $24,500 at 8.35% compounded semi-annually. Ten years later, she plans on withdrawing $5,000 at the end of every three months. If the income annuity can earn 4.5% compounded monthly, on what date will Harriet receive her last payment?
14. At the age of 44, Parker just finished all the arrangements on his parents' estate. He is going to invest his $80,000 inheritance at 5.5% compounded quarterly until he retires at age 55, and then wants to receive month-end payments for the next 25 years. The income annuity is expected to earn 3.85% compounded annually. What are his monthly annuity payments during his retirement?
15. Helga invested $34,000 on September 24, 2010, at 7.75% compounded monthly. She plans on using the money to fund an annuity due starting January 24, 2028, with the last payment being made on December 24, 2045. If the annuity due is expected to earn 5.8% compounded monthly, how much will she receive each month?

Challenge, Critical Thinking, & Other Applications

16. ING Direct just purchased a loan contract on its date of sale from a small retailer for $3,350.64 at a negotiated rate of 18% compounded monthly. If the contract calls for 24 month-end payments of $200 after a period of no payments, how long is the deferral period?
17. Consider the following two investors saving into their RRSP earning 9% compounded annually throughout:
    - Scully invests $275 at the end of every month from age 18 to age 35, then stops contributing until age 65 retirement.
    - Mulder starts his RRSP later and invests $275 at the end of every month from age 35 to age 65.

    a. How much more money does the person with the higher balance have at age 65?
    b. How much money did each investor nominally put into the RRSP?
    c. What time value of money concept is being illustrated?
18. Once Jason graduated college at age 22, he invested $350 into his RRSP at the beginning of every month until age 40. He then stopped his contributions and let the amount earn interest until today, when at age 62 he decided to retire. He wants his retirement money to last until age 85. If his account can earn 10.4% compounded quarterly before age 62 and 4.8% compounded annually after that, how much money can he expect to receive at the end of every quarter?
19. Amber would like her RRSP earning 5.1% compounded semi-annually to pay her $2,500 at the end of every month for 20 years once she retires at age 65. What lump-sum amount at age 25 would Amber need to invest? Suppose she can get rates of 11.2% compounded annually for the first 35 years followed by 5.9% compounded annually until she needs the money at retirement.
20. Compute the following scenarios using different interest rates of 6%, 8%, and 10% compounded annually throughout.
    a. What is the present value of a deferred annuity with a 10-year deferral period followed by a 10-year ordinary annuity with annual $10,000 payments?
    b. What is the annual annuity payment if a lump sum of $50,000 is invested for 10 years followed by a 10-year ordinary annuity?
    c. What is the term of the annuity if a lump sum of $50,000 is invested for 10 years followed by an ordinary annuity paying $20,000 annually?
    d. Discuss your observations from all of the above scenarios.

12.2: Constant-Growth Annuities

Mechanics

Solve each of the following constant growth annuities for

a. **The future value**
b. **The present value**

Rate of Growth	Interest Rate	First Annuity Payment	Payment Timing	Term
1. 0.75%	6% quarterly	$3,000 quarterly	End	10 years
2. 1.5%	8% monthly	$5,000 semi-annually	End	15 years
3. 0.25%	10% annually	$150 monthly	Beginning	25 years, 11 months
4. 4%	8.5% annually	$7,500 annually	Beginning	18 years
5. 3%	9.2% semi-annually	$4,000 annually	End	40 years

Solve each of the following constant growth annuities for

a. **The first annuity payment**
b. **The amount of the annuity payment corresponding to the payment number specified**

Rate of Growth	Value	Interest Rate	Payment Frequency	Payment Timing	Term	Payment Number
6. 0.15%	PV_{DUE} = \$100,000	5.4% annually	monthly	Beginning	10 years	49
7. 3%	PV_{ORD} = \$350,000	4.75% quarterly	annually	End	15 years	6
8. 0.8%	FV_{DUE} = \$500,000	7.75% monthly	quarterly	Beginning	25 years	90
9. 1.75%	FV_{ORD} = \$1,500,000	8.1% semi-annually	semi-annually	End	45 years	last

Applications

10. Yarianny wants to withdraw \$25,000 annually starting today for the next 20 years and will increase the withdrawals by 3.5% each year. If the annuity can earn 6% compounded semi-annually, how much money needs to be invested in the fund today?
11. Nikolay wants to make annual contributions to his RRSP for the next 25 years. He will increase each annual payment by 4.5%, and the RRSP can earn 9.3% compounded annually. If he wants to accumulate \$250,000, what is the amount of his first payment today?
12. A large municipality is saving up to build a new state-of-the-art publicly funded hockey arena to attract a National Hockey League (NHL) franchise. The municipality plans to invest \$20 million at the end of the budget year and will increase contributions by 4% each year as its operating budget is expected to rise in future years. If the investment can earn 6.6% compounded annually, how much money rounded to the nearest dollar will the municipality have in 10 years?
13. A five-year union contract calls for a company to provide \$1 million in annual bonuses starting at the beginning of the first year and declining by 10% per year due to tough expected market conditions in the future. If the company wants to meet this commitment by putting a lump sum today into an investment earning 4.9% compounded quarterly, how much should it invest?
14. Matthew bought a \$200,000 annuity earning 5.75% compounded monthly. It will pay him at the end of every month for the next 20 years. The annuity is designed to make a large payment initially and then decrease by 0.4% per payment. a. What is the amount of his first annuity payment? b. Halfway through the annuity, what will his payment be? c. What is the amount of his last annuity payment?
15. Cisco Systems plans a dividend of \$1.50 per share at the end of the year and is expected to increase the dividend by 7% each year for the next 35 years. If an investor requires an interest rate of 12% compounded semi-annually on her investments, what is the value of a Cisco Systems share for that investor today based on the 35-year time horizon?
16. An employee's pension fund is projected to have a value of \$900,000 when he retires. An actuary determines that the employee has a life expectancy of 17 years after retirement. The pension fund can earn 4.3% annually, and the month-end payments will constantly increase by 0.25%.
    a. What is the amount of the first payment?
    b. What is the amount of the last payment?
    c. What is the total amount of the annuity payments over the entire term?

Challenge, Critical Thinking, & Other Applications

17. Harlen is 24 years old. His financial plan is to contribute \$4,000 at the end of every year to his RRSP until age 60, growing each annual payment by 4.25%. At that point, he plans on retiring and using the money to fund 25 years of retirement. His month-end withdrawals will increase by 0.25% each month. The investment can earn 8.5% compounded annually and then 4.5% compounded annually in retirement.
    a. What is his total contribution to his RRSP?

   b. What is the amount of his first retirement payment?
   c. What are the total withdrawals he will make during retirement?

18. The company accountant needs to record the current value of a pension liability for one of the company's employees. The employee just retired at age 67 and will receive $40,000 at the beginning of every year, increasing by 4% each year. Corporate actuaries have pegged the employee's expected life span at an estimated 23 years past retirement. The pension fund can earn 6.2% compounded semi-annually.
    a. What liability amount should the accountant record?
    b. Rounded to two decimals, what percentage of that amount is a result of the growth in the employee's payments?
19. Recalculate the value of the Cisco Systems share in question 15 if the forecasted dividends are expected to continue into the infinite future. (Hint: use a very large value for N.) Comment on the difference obtained between your answer to question 15 and your answer to this question.
20. Examine the impact of different growth rates on your RRSP.
    a. Start with your contributions to save up for retirement. In all cases, assume a starting contribution of $3,000 at the end of every year for a term of 40 years earning 9% compounded annually. Determine the future value of your RRSP with annual growth rates of 1%, 2%, 3%, 4%, and 5%. Also calculate the total payments made to the RRSP along with the interest earned. Comment on the results.
    b. Now examine your withdrawals after you have retired. In all cases, assume a starting withdrawal of $50,000 at the end of every year for a term of 16 years earning 4.8% compounded annually. Determine the present value required to fund your RRSP with annual growth rates of 1%, 2%, 3%, 4%, and 5%. Comment on the results.

12.3: Perpetuities

Mechanics

Solve each of the following perpetuities for the missing value (identified with a ?).

Present Value	Interest Rate and Compounding Frequency	Payment Amount and Payment Frequency	Payment Timing
1. ?	4.85% annually	$25,000 annually	End
2. ?	8.4% semi-annually	$20,000 semi-annually	Beginning
3. $1,000,000	3.95% annually	? annually	Beginning
4. $2,400,000	4.4% quarterly	? quarterly	End
5. ?	5.35% quarterly	$10,000 semi-annually	End
6. $500,000	5.5% annually	? semi-annually	Beginning
7. $750,000	5% monthly	? annually	End
8. ?	6.1% semi-annually	$7,500 quarterly	Beginning

Applications

9. How much money is required today to fund a perpetuity earning 5.65% annually that needs to pay out $17,000 at the end of each year?
10. If $2,500,000 was invested at 4.8% compounded semi-annually, what payment at the end of every six months could be sustained in perpetuity?
11. The dean of the School of Business and Applied Arts at Red River College in Winnipeg wants to establish a scholarship program for the newly created finance major in its business administration stream. He wants to distribute five $2,000 scholarships annually starting immediately. How much money must he raise from college supporters if the perpetuity can earn 3.6% compounded monthly?

12. Samson just won the grand prize of $50 million in the Lotto Max lottery. If he invests the money into a perpetuity fund earning 5.5% compounded annually, what monthly payment can he expect to receive starting today?
13. An investor states that he would be willing to pay $25 for common shares to achieve his desired rate of return of 15% annually. What are the forecasted annual dividends starting one year from now?
14. In 1910, an Aboriginal group signed a treaty with the British Columbia government in which they agreed to receive $1,542 annually in perpetuity. The government of today would like to pay off this debt. If prevailing interest rates are 3% compounded annually, how much should the Aboriginal group be willing to accept as payment in full for the perpetuity?
15. The Coca-Cola Company is forecasted to have dividends of $0.50 per share quarterly for the next five years, followed by dividends of $0.70 per share quarterly in perpetuity. If an investor desires a 10% compounded annually rate of return, what amount would she be willing to pay per share?

Challenge, Critical Thinking, & Other Applications

16. Indigo's will states that $150,000 is to be set aside into a fund that will make annual payments to her grandson starting when he turns 18 years old. If Indigo dies when her grandson is six years old and the fund can earn 4.9% compounded quarterly, what annual payment will he receive in perpetuity?
17. In 2009, the Canadian federal government committed $1.5 million in annual operating funding for the Canadian Museum for Human Rights in Winnipeg, which is scheduled to open in 2014. If the government had funded the museum by setting up an ordinary perpetuity in 2009 that could earn prevailing interest rates of 3% compounded annually, how much money would have been required?
18. A rare example of perpetual bonds is the Consol bonds first issued by the British government in the 18th century. These perpetual bonds make end-of-quarter payments based on a semi-annual interest rate to the bondholders. If prevailing interest rates are 5%, what is the value of a bond today that pays $29.92 in total per year?
19. Explore the impact of the interest rate on the principal required to fund an ordinary perpetuity. If the annual perpetuity payment is $10,000, calculate the principal required at annual interest rates of 2%, 3%, 4%, 5%, and 6%. Comment on your findings.
20. Explore the impact of the interest rate on the ordinary perpetuity payment for a fixed principal. If the principal is $100,000, calculate the perpetuity payment at annual interest rates of 2%, 3%, 4%, 5%, and 6%. Comment on your findings.

12.4: Leases

Mechanics

For each of the following leases, determine the unknown variable (identified with a ?).

Purchase Price	Down Payment	Lease Term (Years)	Nominal Interest Rate and Compounding Frequency	Lease Payment and Payment Frequency	Residual Value
1. ?	$5,000.00	3	5% monthly	$450.00 monthly	$11,500.00
2. $39,544.35	$3,000.00	3	3.75% annually	$680.37 monthly	?
3. $45,885.00	$4,500.00	4	5.9% monthly	? quarterly	$9,750.00
4. $40,000.00	$0.00	5	? annually	$851.95 monthly	$0.00
5. $85,000	$15,000	1	9.99% monthly	$5,609.67 monthly	?
6. ?	$0.00	4	2.5% monthly	$26,500.00 quarterly	$0.00
7. $44,518.95	$7,200.00	7	? quarterly	$2,627.17 semi-annually	$14,230

Purchase Price	Down Payment	Lease Term (Years)	Nominal Interest Rate and Compounding Frequency	Lease Payment and Payment Frequency	Residual Value
8. $175,000	$10,000	6	13% semi-annually	? quarterly	$5,000

Applications

9. A Ford F150 is on a 60-month lease at 5.99% compounded annually. Monthly payments are $397.95, and a $5,000 down payment was made. The residual value is $5,525.00. What was the purchase price of the truck?
10. Pitney Bowes leases office equipment to other businesses. A client wants to lease $75,000 worth of equipment for 4½ years, at which point the equipment will have a residual value of $7,850. If Pitney Bowes requires a 15% compounded annually rate of return on its lease investments, what quarterly payment does it need to charge the client?
11. Franklin is the office manager for Cargill Limited. He just signed a six-year leasing contract on some new production equipment. The terms of the lease require monthly payments of $32,385. If the alternative source of financing would have an interest rate of 7.3% compounded semi-annually, what capitalized lease liability should be recorded on the Cargill Limited balance sheet today?
12. Harold is the production manager at Old Dutch Foods. He is considering leasing a new potato chip–making machine that will improve productivity. The terms of the 10-year lease require quarterly payments of $35,125 including interest at 6.99% compounded monthly. If the equipment is valued at $1,315,557.95 today, what is the estimated residual value of the equipment when the lease expires?
13. Simi is considering leasing some computer equipment and peripherals on a two-year lease. The purchase price of the equipment is $40,674.35 with monthly payments of $1,668.86 including interest. The residual value is estimated to be 7.5% of its original value. What quarterly compounded interest rate, rounded to two decimals in percent format, is Simi being charged?
14. Xerox's outstanding sales agent, Sebastien, proudly boasts that he just completed a leasing arrangement on some new copier equipment for a three-year term at 17% compounded annually. It requires the client to make monthly payments of $2,375, and the equipment has a residual value of $2,500. What was the leasing price of the copier equipment?
15. Unger Accounting Services leases its office and computer equipment. One year ago, it signed a three-year lease requiring quarterly payments of $1,600. Financing the equipment would have required a bank loan at 8.8% compounded monthly. How much has the capital leasing liability been reduced since the inception of the lease?
16. Jerilyn operates a large farm in Alberta. She wants to lease a new harvester for her fields, one that has a purchase price of $120,000 and an estimated residual value of $50,000 after five years. The farm equipment dealer offers her a lease with a 6% annually compounded interest rate. What will her annual payments be?

Challenge, Critical Thinking, & Other Applications

17. A rent-to-own transaction is similar to a lease, where payments are required up front and the residual value equals zero. 1. What effective rate of interest (rounded to two decimals) is being charged for a two-year rent-to-own contract on a Panasonic camcorder with a purchase price of $379 requiring monthly payments of $79.88? What total amount of interest is paid? 2. If the individual borrowed money on a high-interest credit card instead with a rate of 29.99% compounded daily, what would be the monthly payments? What total amount of interest is paid?
18. Fehrway Tours needs three extra touring vehicles for the next operating year because of a temporary increase in demand for its tour routes. Fehrway can buy the vehicles from Greyhound Canada for $175,000 each, or it can lease them for one year requiring monthly payments of $7,350 each based on a residual value of $110,000 each. If Fehrway purchases the vehicles, it requires financing at 6.6% compounded monthly and projects it could sell the vehicles for net revenue of $95,000 each after one year. Which option would you recommend that Fehrway pursue? How much better is the option in current dollars?

19. Da-Young needs to record the total capital lease liability of her company. Based on the following outstanding capital leases, what amount should appear on her balance sheet?
    - A five-year lease signed 2½ years ago requiring quarterly payments of $3,400 at 3.95% compounded annually and a residual value of $5,000.
    - A seven-year lease signed 3¼ years ago requiring monthly payments of $895 at 6.8% compounded quarterly.
    - A lease signed today for two years requiring semi-annual payments of $2,300 at 8.85% compounded annually with a residual value of $7,200.
    - Some equipment leased with a purchase price of $200,000 three years ago on a six-year lease requiring semi-annual payments of $18,686.62 at 4.35% compounded semi-annually with a residual value of $21,000.
20. Elena is shopping around for a one-year-old used Chevrolet Cobalt. The offers from three different used car dealerships are listed below. Which offer is financially best for Elena in current dollars? Calculate the present value and rank the three offers for Elena.
    - Dealer #1: If she places $3,500 down, her four-year monthly lease payments will be $217.48 at 4.85% compounded monthly, and she will need to pay $3,629 at the end to buy her car outright.
    - Dealer #2: If she places $3,000 down, her four-year monthly lease payments will be $237.53 at 4.65% compounded monthly and she will need to pay $3,400 at the end to buy her car outright.
    - Dealer #3: If she places $2,000 down, her three-year monthly lease payments will be $259.50 at 4.5% compounded monthly and she will need to pay $6,000 at the end to buy her car outright.

12.5: Application - How To Purchase A Vehicle

Because of the process required to make the vehicle purchase decision, this section offers only five exercises. You can solve these questions by the process presented using either formulas or the spreadsheet templates that accompany this text. For each of these questions, select the best method to acquire the vehicle based on the information provided. Identify the value of the payment under this best choice. Here are the options:

1. Lease from the Dealership
2. Lease from the Bank
3. Loan from the Dealership
4. Loan from the Bank

Mechanics

Note that all interest rates in the table are compounded annually. Note for the Banking Information that after calculating the dealer rates, you would go online and see if you can find something lower; the banking information below represents the results of your search.

Vehicle	Dealer Leasing Information	Dealer Purchase Information	Banking Information
1. Chevrolet Silverado	3-year term 5.3% $295.00/month $17,810 residual value	$2,000 cash rebate 5.79%	Lease Rate = 7.15% Loan Rate = 8.8% Savings Rate = 3%
2. Honda Civic	4-year term 2.4% $236.46/month $8,520 residual value	$2,500 cash rebate 7.9%	Lease Rate = 8% Loan Rate = 10.9% Savings Rate = 2.25%
3. Acura ZDX	4-year term 2.4% $699.00/month $22,955.90 residual value	$5,000 cash rebate 2.95%	Lease Rate = 6.6% Loan Rate = 8.3% Savings Rate = 3.15%

Vehicle	Dealer Leasing Information	Dealer Purchase Information	Banking Information
4. Dodge Grand Caravan	5-year term 4.99% $141.00 biweekly $13,005 residual value	$6,500 cash rebate 0%	Lease Rate = 6.9% Loan Rate = 9.5% Savings Rate = 4.8%
5. Toyota Matrix	4-year term 0.9% $401.90/month $10,917.90 residual value	$3,000 cash rebate 0.9%	Lease Rate = 5.9% Loan Rate = 5.95% Savings Rate = 2.95%

12.6: Application - Planning Your RRSP

Due to the nature of these questions, five Application questions, two Challenge questions, and one Spreadsheet question are offered.

Applications

1. When Criss turns 55, 35 years from now, he wants his RRSP savings to pay him the equivalent of $35,000 annually today with 2.4% annual inflation. He thinks his annual contributions can earn 9% compounded annually increasing by 4% per payment. Upon retirement, he wants to receive end-of-month payments for 30 years and estimates his account can earn 5.35% compounded annually. What is his first RRSP contribution payment at the end of the year?
2. After graduating college at age 22, Kandahar immediately gained employment in the accounting field. He thinks that his career path will follow the “standard” path and he will retire at age 65. He feels he could live comfortably off of $30,000 in today’s dollars, with expected inflation of 2.1% annually. In retirement, he plans on receiving month-end payments for 13 years with 3.75% compounded monthly interest. Suppose his investment earns 8% compounded quarterly and he will increase his contributions by 0.9% per payment. What is his first quarterly RRSP contribution three months from now?
3. Aisha and Richard are getting married, 33 years before they hit age 65. Neither has any RRSP savings to date. They figure that together in retirement they would like to earn the equivalent of $60,000 today plus 3% annual inflation. After retiring, they want to receive payments at the end of every three months for 15 years while the account earns an estimated 4.35% compounded semi-annually. If their RRSP can earn 8.25% compounded monthly and they plan on increasing contributions by 1.5% per interval, what first semi-annual payment is made today by each of them to achieve their joint goal?
4. When Vanessa retires in 24 years, she wants her retirement fund to pay her the equivalent of $50,000 annually in today’s funds for 19 years while earning interest at a forecasted 5.15% compounded semi-annually. Her monthly payments start immediately upon the date of retirement. Current savings in her RRSP total $100,000, and historically she has been earning 7.35% compounded annually on the investment. She will make end-ofquarter contributions to her RRSP growing at 0.75% per payment. If the rate of inflation is 3% annually, what is the amount of the first contribution?
5. Emma will make her first of 32 annual RRSP contributions today. Her financial adviser has indicated that she might earn approximately 6.5% compounded annually on her investment, and that she needs to increase each payment by 4%. After retiring at the beginning of the 33rd year from now, she wants month-end payments for 21 years that are the equivalent of $37,500 annually today while factoring in 2.55% annual inflation. Her retirement fund is expected to earn 4.05% compounded annually. She has current savings in her RRSP of $17,800. What is the amount of her first contribution?

Challenge, Critical Thinking, & Other Applications

6. Caitlin and Tu have three children to whom they will leave an inheritance of $100,000 each from the remaining balance in their retirement money. With 31 years until retirement, they have already managed to save $150,000 earning 6.9% compounded quarterly. They want to make month-end contributions increasing at 0.3% per contribution such that in retirement for 20 years they can receive on a month-end basis an annual income equivalent

to $70,000 today. Inflation is expected to hover around 2.75% annually. The retirement fund is expected to earn 4.25% compounded annually. What is the first contribution required to achieve their desired goals?

7. Wenli and Arjinder just celebrated the birth of their twins and are planning ahead for their education. In researching universities, they noted that the average tuition for an undergraduate student today is $5,138 and it is forecasted to rise by 4% annually into the foreseeable future. They plan that their education fund will make beginning-of-year payments for four years to each of their children when they turn 18 years old such that their education will be fully funded. Wenli and Arjinder will make semi-annual contributions rising by 2.5% each time. The education fund is expected to earn 7% compounded semi-annually, and when their kids turn 18 it will decrease to 5% compounded annually. What payment six months from now is required?
8. Two key factors play an important role in RRSP calculations. These are the rate of inflation and the growth rate of the RRSP contributions. Use the information found in question 2 to answer the following:
    a. Using annual inflation rates of 2%, 3%, 4%, and 5%, calculate the RRSP contributions required under each scenario. What do you observe?
    b. Leave the inflation at 2.1% annually and recalculate the RRSP contributions required using growth rates of 0.5%, 0.7%, 0.9%, and 1.1%. What do you observe?

Review Exercises

Mechanics

1. Four years from now, an annuity needs to pay out $1,000 at the end of every quarter for three years. Using an interest rate of 5% quarterly throughout, what amount of money must be invested today to fund the investment?
2. Marnie wants to save up $250,000 to pay cash for a home purchase 15 years from now. If her investment can earn 6.1% compounded monthly and she intends to grow each payment by 0.25%, what will be her first monthly payment one month from now?
3. Red Deer College wants to set up a scholarship for students in its business programs such that at the end of every year it could distribute a total of $50,000. If the perpetuity fund can earn 4.85% compounded semiannually, how much money will need to be raised to fund the scholarship?
4. Carradine Industries needs to set a two-year quarterly lease price on some new equipment it is offering to its clients. If the company requires a 17% rate of return, what price should it charge on equipment valued at $44,750 with no expected residual value?
5. Under a 48-month contract, Contact Marketing Inc. has been leasing $47,000 worth of servers and computer equipment for $1,292 per month for the past 23 months. What capitalized lease liability should be recorded on the balance sheet today if alternative financing could have been arranged for 9.4% compounded monthly?

Applications

6. Procter and Gamble shares are valued at $61.00 with perpetual year-end dividends of 3.1639%. What dividend payment would a holder of 750 shares receive in perpetuity assuming the share price and dividend rate remain unchanged?
7. A Toyota RAV4 Limited 4WD V6 is advertised with low monthly lease payments of $488.86 for 48 months. The terms of the lease specify that the MSRP of the vehicle is $38,900 with a residual value of $17,502.80. A $6,250 down payment is required. What monthly compounded lease rate is Toyota charging?
8. Jean-Rene wants to make a lump-sum deposit today such that at the end of every three months for the next five years he can receive a payment starting at $2,500 and increasing by 1% each time thereafter. At the end of the term, an additional lump-sum payment of $10,000 is required. If the annuity can earn 8.75% compounded semi-annually, what lump sum should he deposit today?
9. A lump sum of $20,000 is invested at 6.25% compounded monthly for 18 years. It then pays out $2,500 at the end of every month while earning 5.05% compounded monthly. How many payments can the annuity sustain?
10. The marketing manager for Infiniti places an advertisement for the new Infiniti QX56. Suppose the MSRP of the vehicle is $75,050 with a residual value after 48 months of $29,602. What monthly lease payment should she

advertise if a $5,000 down payment is required and Infiniti Financial Services requires 5.9% compounded annually on all leases?

11. Kraft Foods is planning for the replacement of some major production equipment. It will invest $1.475 million today at 5.95% compounded annually. The equipment purchase will be financed such that Kraft makes payments of $250,000 at the end of every quarter for two years at 6.65% compounded monthly. To the nearest day, how long does Kraft Foods need to wait to make the equipment purchase?
12. The common shares of The Coca-Cola Company are forecast to pay $1.55 per share at the end of the next four years, and then $2.05 annually in perpetuity. If the market rate of return on such shares is 2.89%, what price should an investor be willing to pay today?
13. A Hyundai dealership advertises a Hyundai Genesis for a low 60-month lease payment of $407.38 monthly at 0% interest with no down payment required. The residual value of the car is $16,316. A cash rebate of $4,000 is available. The lowest lease bank rate is 5.95% compounded monthly and savings accounts are paying 2.5% compounded annually. From the dealership, purchase financing is available for 1.9% compounded annually. Alternatively the consumer can place the car on their home equity line of credit at a rate of 3.5% compounded monthly. What is the lowest cost method to purchase this vehicle (lease or finance from either the dealer or the bank)?
14. Klara is 25 years old and is planning for her retirement. She wants to consider both options of retiring at age 60 or at age 65, and has no savings into her RRSP yet. She figures she would like to earn in retirement the equivalent of $25,000 annually today. In retirement, she estimates the account would earn 4.35% compounded annually until age 82 before being depleted, all the while receiving end-of-month payments. She believes her RRSP can earn 8.4% compounded quarterly and she wants to grow her end-of-month contributions by 0.4% each. Supposing that annual inflation is expected to average 3%, what percentage larger will her first contribution be if she wants to retire at age 60 instead of age 65?

Challenge, Critical Thinking, & Other Applications

15. Sanchez is eligible to receive retirement benefits of $901.01 at age 65, but decides to start receiving them when he turns 60 instead. This incurs a 36% penalty on his benefits. By taking his retirement benefits at age 60, he will realize a monthly saving of $184.80 in retirement fund contributions. Both the benefits and contributions savings are expected to rise monthly by 0.15%. He will contribute the sum of these two amounts into his RRSP, which earns 7.35% compounded monthly until age 65, when he retires. In retirement, he will use the accumulated savings, which are expected to earn 4.5% compounded quarterly, to top up his retirement benefit payments to his pre-penalty amount. The monthly benefits are expected to rise monthly by 0.15%.
    a. Calculate the total of the monthly benefits and savings at age 60.
    b. Calculate the future value of the monthly savings and benefits at age 65.
    c. Calculate the amount of the initial penalty taken at age 60 and grow the penalty by 1.8% per year from age 60 to age 65.
    d. Take the amount of money in the fund at age 65 (from part (b)) along with the payments required to top up his retirement benefits to their regular level (from part (c)) to calculate the amount of money remaining in the fund at age 70.
16. A Canadian college plans to implement a new business major in five years. To support the new program, it wants to offer 15 annual $2,500 scholarships at the beginning of each school year in perpetuity. If the scholarship fund can earn 4.65% compounded annually, what amount of money does the college need to raise today to fund the program?
17. Luxmi wants to celebrate the birth of her first grandson by investing money for his future education. She estimates the annual cost of postsecondary education today to be $6,000 per year, and that the typical degree takes four years. She wants to invest a lump sum today that could sustain these payments with inflation starting 18 years from today. If annual inflation is 3.7%, and her investment can earn 6.6% compounded monthly, what amount does she need to invest today?

18. AVCO Financial is under contract with a national retail chain to purchase its loan contracts on its date of sale. Under a special promotion, the retail chain allows a customer to defer her payments. If AVCO purchases the contract for $5,276.83 at its contractual rate of 21% compounded monthly and the consumer is required to make 30 month-end payments of $425, in how many months will AVCO receive its first payment?
19. The Kenna Connect Marketing Group relies heavily on advanced technology to drive its business. A request for proposals on the next two-year computer equipment lease has resulted in four companies submitting bids:
    - Company #1: Quarterly lease payments of $125,000 at 8.5% compounded quarterly with a residual value of $25,000.
    - Company #2: Quarterly lease payments of $115,000 at 7.95% compounded quarterly with a residual value of $40,000 and a down payment of $80,000 required.
    - Company #3: Monthly lease payments of $40,000 at 8.25% compounded monthly with a residual value of $35,000 and a down payment of $50,000 required.
    - Company #4: Biweekly lease payments of $19,400 at 7.3% compounded monthly with a residual value of $42,000. Based on these bids, which company should Kenna Connect pursue the lease with and what is its cost in today's dollars?
20. Jacques Cousteau just won the $10 million Powerball lottery. He has been offered the following choices on how to collect his winnings:
    - Option 1: A one-time lump-sum payment of $4,289,771.59 today
    - Option 2: A five-year deferred annuity followed by year-end payments of $400,000 for 25 years at 5% compounded annually throughout
    - Option 3: A 35-year constant growth annuity with annual payments of $165,392.92 starting today and growing at 3% per year, earning interest at 5% compounded annually
    - Option 4: Annual payments of $207,150 in perpetuity starting today earning 5% compounded annually Which option is the best financial choice? How much better in today's dollars is this option than the worst option?

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CHAPTER OVERVIEW

13: Understanding Amortization and its Applications

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13.1: Calculating Interest and Principal Components
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13.3: Amortization Schedules
13.4: Special Application - Mortgages
13.S: Summary
13.CS: Case Study - Cash Or Credit?
13.E: Understanding Amortization and its Applications (Exercises)

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13.0: Introduction

It would be nice to pay cash for everything that you acquire; however, goods such as housing, automobiles, furniture, and electronics require more money than most Canadians have lying around. If you tried saving up and paying cash for these large purchases, it would take a long time to accumulate enough funds. Take the example of a new home. If you saved $300 per month at 5% compounded annually, it would take approximately 30 years to save up $250,000 in cash, and in the meantime, of course, you would still need to pay for somewhere to live.

So it is not necessarily bad to borrow money at some point. However, to borrow wisely you had better understand where your money is going. Would it shock you to learn that in a typical mortgage arrangement most homeowners pay approximately double for their homes? That is, $250,000 for the home and $250,000 in interest!

Businesses borrow money for many of the same reasons as consumers. Loans and mortgages are commonplace. Certain business activities need to be financed upfront. For example, new products must be researched and developed before a single unit can be sold. This requires investments that will not be reimbursed until the products turn a profit. All company debts must be accurately recorded onto balance sheets to reflect the balances owing. Income statements must appropriately track interest expenses or earnings. Because businesses can deduct their interest expenses against corporate taxable income and lower their taxes, it is important to see what proportion of their loan payments is going to interest.

When you take out a mortgage for yourself or your business, where does your money go? You need a chart of your loan payments showing how much interest the bank charges and how much is applied against your principal.

This chapter takes you through calculating the principal and interest components of any single payment or series of payments for both loans and investment annuities. Recall from Chapter 11 that you estimated the final payment on a loan; you will now see how to calculate that payment precisely. Finally, you will be guided through the largest transactions you personally are likely to make: basic mortgages and renewals.

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13.1: Calculating Interest and Principal Components

How much of the principal do you pay off when you make a loan payment? One year ago you purchased your $250,000 dream home on a 25-year mortgage at a fixed 5% compounded semi-annually interest rate. With monthly contributions of $1,454.01, or $17,448.12 in total for the past year, you figure you must have put a serious dent in the balance owing.

You get a rude shock when you inspect your mortgage statement, as you realize the balance owing is not what you expected. Your remaining balance is $244,806.89, reflecting a principal reduction of only $5,193.11! The other 70% of your hard-earned money, amounting to $12,255.01, went solely toward the bank's interest charges.

Many people do not fully understand how their loan payments are portioned out. Over the full course of the 25-year mortgage you will pay $186,204.46 in interest charges at 5% compounded semi-annually, or approximately 74.5% of the home's price tag. That is a total of $436,204.46 paid on a $250,000 home. What if interest rates rise? At a more typical 7% semi-annual rate, you would owe $275,311.51 in total interest, or 110% of the value of your home.

Knowing these numbers, what can you do about them? Term, interest rates, payment amounts, and payment frequency all affect the amount of interest you pay. What if you made one extra mortgage payment per year? Did you know that it would only take approximately 21 years instead of 25 years to own your home? Instead of paying $186,204.46 in interest you would pay only $156,789.33, a savings of almost $30,000!

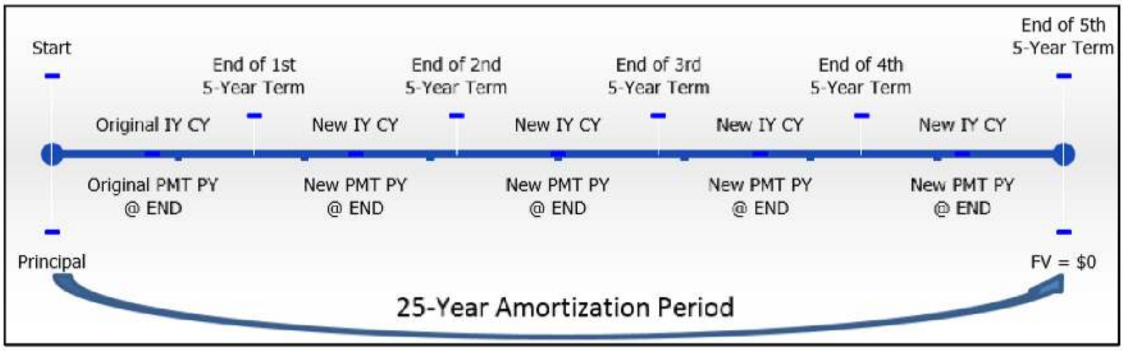
These calculations should make it clear that both businesses and consumers need to understand the interest and principal components of annuity payments. This section shows you how to calculate principal and interest components both for single payments and for a series of payments.

What Is Amortization?

Amortization is a process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. Two components make up the agreed-upon time component:

1. **Amortization Term**. The amortization term is the length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged. Thus, if the agreement is for monthly payments at a 5% fixed rate over five years, it is binding for the entire five years. Or if the agreement is for quarterly payments at a variable rate of prime plus 2% for three years, then interest is calculated on this basis throughout the three years.
2. **Amortization Period**. The amortization period is the length of time it will take for the principal to be reduced to zero. For example, if you agree to pay back your car loan over six years, then after six years you reduce your principal to zero and your amortization period is six years.

In most relatively small purchases, the amortization term and amortization period are identical. For example, a vehicle loan has an agreed-upon interest rate and payments for a fixed term. At the end of the term, the loan is fully repaid. However, larger purchases such as real estate transactions typically involve too much money to be repaid under short time frames. Financial institutions hesitate to agree to amortization terms of much more than five to seven years because of the volatility and fluctuations of interest rates. As a result, a term of five years may be established with an amortization period of 25 years. When the five years elapse, a new term is established as agreed upon between the borrower and lender. The conditions of the new term reflect prevailing interest rates and a payment plan that continues to extinguish the debt within the original amortization period.



The figure shows the timeline for a 25-year mortgage in which the borrower establishes five sequential five-year terms throughout the 25-year amortization period to extinguish the debt. With regard to amortization, all loans take the structure of either simple ordinary annuities or general ordinary annuities unless otherwise stated. Therefore, this textbook will primarily focus on ordinary annuities.

Calculating Interest and Principal Components for a Single Payment

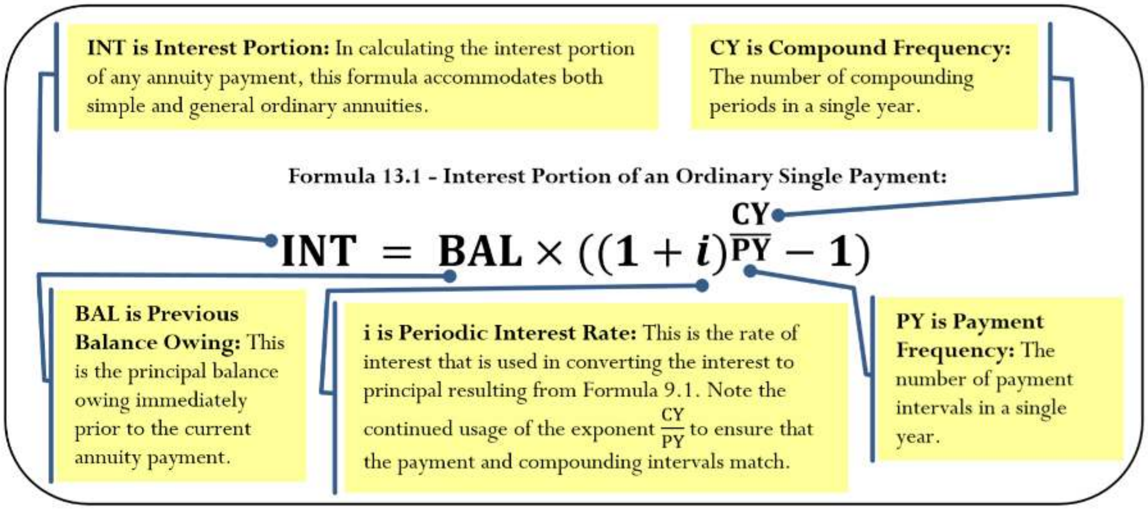
At any point during amortization you can precisely calculate how much any single payment contributes toward principal and interest. Businesses must separate the principal and interest components for two reasons:

1. **Interest Expense**. Any interest paid on a debt is an accounting expense that must be reported in financial statements. In addition, interest expenses have tax deduction implications for a business.
2. **Interest Income**. Any interest that a company receives is a source of income. This must be reported as revenue in its financial statements and is subject to taxation rules.

The Formula

To calculate the interest and principal components of any annuity payment, follow this sequence of two formulas.

1. Calculate the interest portion of the payment (Formula 13.1).
2. Calculate the principal portion of the payment (Formula 13.2).



INT is Interest Portion: In calculating the interest portion of any annuity payment, this formula accommodates both simple and general ordinary annuities.

CY is Compound Frequency: The number of compounding periods in a single year.

Formula 13.1 - Interest Portion of an Ordinary Single Payment:

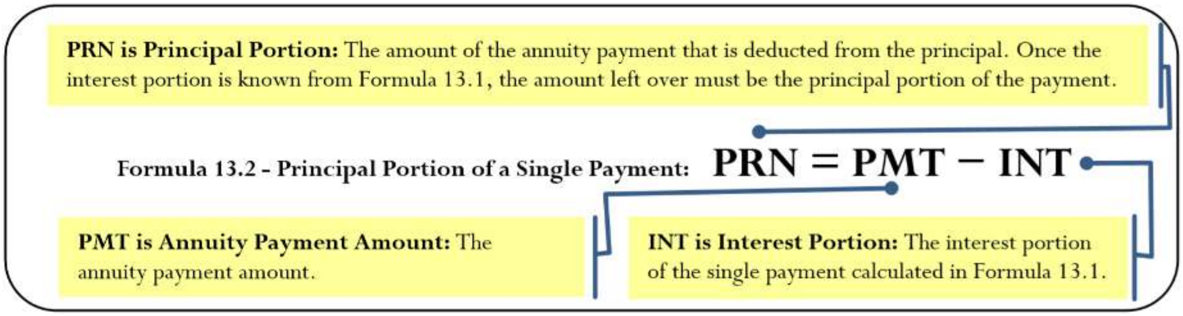
$$\mathrm{INT} = \mathrm{BAL} \times \left((1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}} - 1\right)$$

BAL is Previous Balance Owing: This is the principal balance owing immediately prior to the current annuity payment.

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal resulting from Formula 9.1. Note the continued usage of the exponent $\frac{\mathrm{CY}}{\mathrm{PY}}$ to ensure that the payment and compounding intervals match.

PY is Payment Frequency: The number of payment intervals in a single year.

Formula 13.1



PRN is Principal Portion: The amount of the annuity payment that is deducted from the principal. Once the interest portion is known from Formula 13.1, the amount left over must be the principal portion of the payment.

Formula 13.2 - Principal Portion of a Single Payment: $\mathrm{PRN} = \mathrm{PMT} - \mathrm{INT}$

PMT is Annuity Payment Amount: The annuity payment amount.

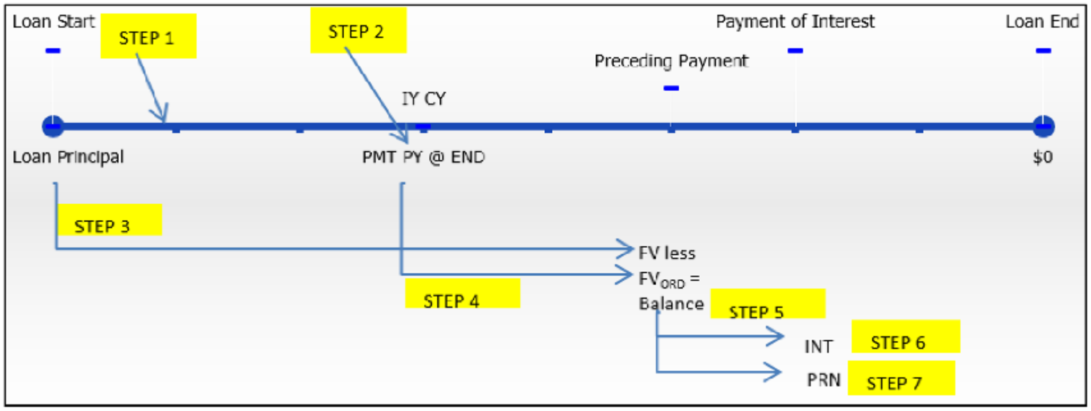
INT is Interest Portion: The interest portion of the single payment calculated in Formula 13.1.

Formula 13.2

How It Works

Follow these steps to calculate the interest and principal components for a single annuity payment:

Step 1: Draw a timeline (seen below). Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD}. The annuity payment amount may or may not be known.



Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, solve for it using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round the payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the payment being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments). For example, when you calculate the interest and principal portions for the 22nd payment, you need to know the balance immediately after the 21st payment.

Step 4: Calculate the future value of all annuity payments already made. Use Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). For example, if you need to calculate the interest and principal portions for the 22nd payment, you need to know the future value of the first 21 payments.

Step 5: Calculate the balance (BAL) prior to the payment by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date.

Step 6: Calculate the interest portion of the current annuity payment using Formula 13.1.

Step 7: Calculate the principal portion of the current annuity payment using Formula 13.2.

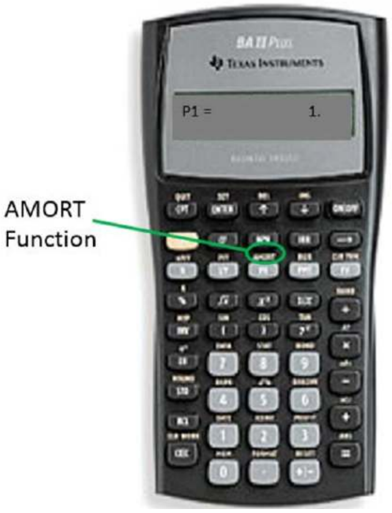
Important Notes

Investment Annuities

The formulas and techniques being discussed in this section also apply to any type of investment annuity from which annuity payments are received. For example, most people receive annuity payments from their accumulated RRSP savings when in retirement. In these cases, view the investment as a loan to the financial institution at an agreed-upon interest rate. The financial institution then makes annuity payments to the retiree to extinguish its debt at some future point; these payments consist of the principal and interest being earned.

Your BAII Plus Calculator.

The function that calculates the interest and principal components of any single payment on your BAII Plus calculator is called AMORT. It is located on the 2nd shelf above the PV button.



The Amortization window has five variables (use ↓ or ↑ to scroll through them). The first two, P1 and P2, are data entry variables. The last three, BAL, PRN, and INT, are output variables.

- P1 is the starting payment number. The calculator works with a single payment or a series of payments.
- P2 is the ending payment number. This number is the same as P1 when you work with a single payment. When you work with a series of payments later in this section, you set it to a number higher than P1.
- BAL is the principal balance remaining after the P2 payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.

To use the Amortization function, the commands are as follows:

1. You must enter all seven time value of money variables accurately ($N, I/Y, PV, PMT, FV, P/Y$, and C/Y). If PMT was computed, you must re-enter it with only two decimals and the correct cash flow sign.
2. Press 2nd AMORT.
3. Enter a value for P1 and press Enter followed by ↓.
4. Enter a value for P2 and press Enter followed by ↓. Note that the higher the numbers entered in P1 or P2, the longer it takes the calculator to compute the outputs. It is possible that your calculator will go blank for a few moments before displaying the outputs.
5. Using the ↓ and ↑, scroll through BAL, PRN, and INT to read the output.

Things To Watch Out For

A common misunderstanding when using the AMORT function on the calculator occurs when inputting the values for P1 and P2. Many people think that when they solve for a single payment they need to set these values one apart. For example, if they are looking for the 22nd payment they imagine that P1 = 21 and P2 = 22. This is incorrect, as the calculator would then compute the total values for both payments 21 and 22.

If you are interested in a single payment, you must set P1 and P2 to the exact same value. In the example, if you want the 22nd payment then both P1 = 22 and P2 = 22.

? Exercise 13.1.1: Give It Some Thought

a. For any loan, if you calculated the interest portions of the second payment and the tenth payment, which payment has a smaller interest portion?
b. For any loan, if you calculated the principal portions of the fifth payment and the twelfth payment, which payment has a smaller principal portion?
c. Holding all other variables constant, if Loan A had an interest rate of 4% while Loan B had an interest rate of 6%, which loan has the higher interest portion on any payment?

Answer

a. The tenth payment, since the principal is much smaller by then so less interest is charged.
b. The fifth payment, since the balance is higher so more interest is charged.
c. Loan B, since the interest rate is higher.

Example 13.1.1: Interest and Principal of a Loan Payment

The accountant at the accounting firm of Nichols and Burnt needs to separate the interest and principal on the tenth loan payment. The company borrowed $10,000 at 8% compounded quarterly with month-end payments for two years.

Solution

Note that this is an ordinary general annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the tenth payment on the two-year loan.

What You Already Know

Step 1:

The information about the accounting firm's loan are in the timeline. PV_{ORD} = $10,000, IY = 8%, CY = 4, PY = 12, Years = 2, FV = $0

How You Will Get There

Step 2:

PMT is unknown. Apply Formulas 9.1, 11.1, and 11.4.

Step 3:

Calculate the future value of the loan principal using Formulas 9.2 and 9.3.

Step 4:

Calculate the future value of the first nine payments using Formulas 11.1 and 11.2.

Step 5:

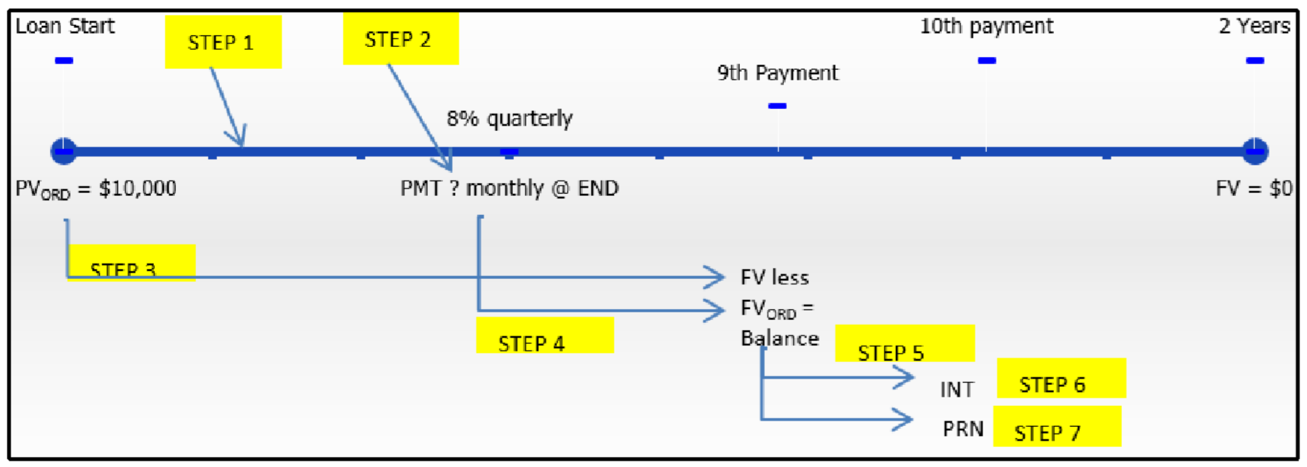
Calculate the principal balance after nine payments through $BAL = FV - FV_{ORD}$.

Step 6:

Calculate the interest portion by using Formula 13.1.

Step 7:

Calculate the principal portion by using Formula 13.2.


Loan Start
STEP 1
STEP 2
10th payment
2 Years
9th Payment
8% quarterly
PV$_{ORD}$ = $10,000
PMT ? monthly @ END
FV = $0
STEP 3
FV less
FV$_{ORD}$ =
Balance
STEP 4
STEP 5
INT
STEP 6
PRN
STEP 7

Perform

Step 2:

$$i=8\%/4=2\%; N=12\times 2=24 \text{ payments}$$

$$\$10,000=PMT\left[\frac{1-\left[\frac{1}{(1+0.02)^{\frac{4}{12}}}\right]^{24}}{(1+0.02)^{\frac{4}{12}}-1}\right]$$

$$PMT=\frac{\$10,000}{\left[\frac{1-\left[\frac{1}{(1+0.02)}^{\frac{4}{12}}\right]^{24}}{(1+0.02)^{\frac{4}{12}}-1}\right]}=\frac{\$10,000}{\left[\frac{0.146509}{0.006622}\right]}=\$452.03$$

Step 3:

$$N=4\times\frac{9}{12}=3 \text{ compounds; } FV=\$10,000(1+0.02)^3=\$10,612.08$$

Step 4:

$$N=12\times\frac{9}{12}=9 \text{ payments}$$

$$FV_{ORD}=\$452.03\left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^9-1}{(1+0.02)^{\frac{4}{12}}-1}\right]=\$4,177.723934$$

Step 5:

$$BAL=\$10,612.08-\$4,177.723934=\$6,434.356066$$

Step 6:

$$INT=\$6,434.356066\times\left((1+0.02)^{\frac{4}{12}}-1\right)=\$42.612871$$

Step 7:

$$PRN=\$452.03-\$42.612871=\$409.42$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	8	10000	Answer: -452.032375 Rekeyed as: -452.03	0	122	4

P1	P2	BAL (output)	PRN (output)	INT (output)
10	10	6,024.938937	409.417128	42.612871

The accountant for Nichols and Burnt records a principal reduction of $409.42 and an interest expense of $42.61 for the tenth payment.

Example 13.1.2: Interest and Principal of an Investment Annuity Payment

Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments to him. For his fifth payment, he needs to know how much of his payment came from his principal and how much interest was earned on the investment.

Solution

Note that this is an ordinary simple annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the fifth payment on the five-year investment annuity.

What You Already Know

Step 1:

You know the following about the investment annuity, as illustrated in the timeline. PV_{ORD} = \$50,000, IY = 5%, CY = 4, PY = 4, Years = 5, FV = \$0

How You Will Get There

Step 2:

PMT is unknown. Apply Formulas 9.1, 11.1, and 11.4.

Step 3:

Calculate the future value of the loan principal using Formulas 9.2 and 9.3.

Step 4:

Calculate the future value of the first four payments using Formulas 11.1 and 11.2.

Step 5:

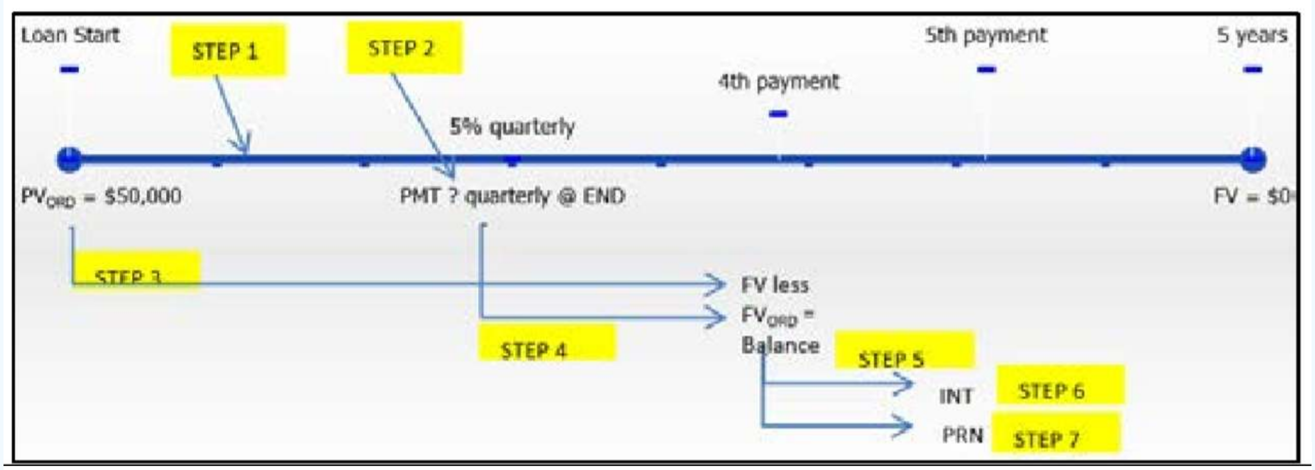
Calculate the principal balance after four payments through $BAL = FV - FV_{ORD}$.

Step 6:

Calculate the interest portion by using Formula 13.1.

Step 7:

Calculate the principal portion by using Formula 13.2.


Loan Start
STEP 1
STEP 2
5th payment
5 years
4th payment
5% quarterly
PV$_{ORD}$ = \$50,000
PMT ? quarterly @ END
FV = \$0
STEP 3
FV less
FV$_{ORD}$ =
Balance
STEP 4
STEP 5
INT
STEP 6
PRN
STEP 7

Perform

Step 2:

$$i = 5\%/4 = 1.25\%;\ N = 4 \times 5 = 20 \text{ payments}$$

$$\$50{,}000 = PMT\left[1-\left[\frac{1}{(1+0.0125)^{\frac{4}{4}}}\right]^{20}\right]$$

$$\$50,000 = PMT\left[\frac{}{(1+0.0125)^{\frac{4}{4}}-1}\right]$$

$$PMT = \frac{\$50,000}{\left[\frac{1-\left[\frac{1}{(1+0.0125)^{\frac{4}{4}}}\right]^{20}}{\left[(1+0.0125)^{\frac{4}{4}}\right]-1}\right]} = \frac{\$50,000}{\left[\frac{0.219991}{0.0125}\right]} = \$2,841.02$$

Step 3:

$$N = 4\times 1 = 4 \text{ compounds; } FV = \$50,000(1+0.0125)^{4} = \$52,547.26685$$

Step 4:

$$N = 4\times 1 = 4 \text{ payments}$$

$$FV_{ORD} = \$2,841.02\left[\frac{\left[(1+0.0125)^{\frac{4}{4}}\right]^{4}-1}{(1+0.0125)^{\frac{4}{4}}-1}\right] = \$11,578.93769$$

Step 5:

$$BAL = \$52,547.26685 - \$11,578.93769 = \$40,968.32916$$

Step 6:

$$INT = \$40,968.32916\times\left((1+0.0125)^{\frac{4}{4}}-1\right) = \$512.104114$$

Step 7:

$$PRN = \$2,841.02 - \$512.104114 = \$2,328.92$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	Answer: 2,841.019482 Rekeyed as: 2,841.01	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
5	5	-38,639.41327	2,328.915886	512.104114

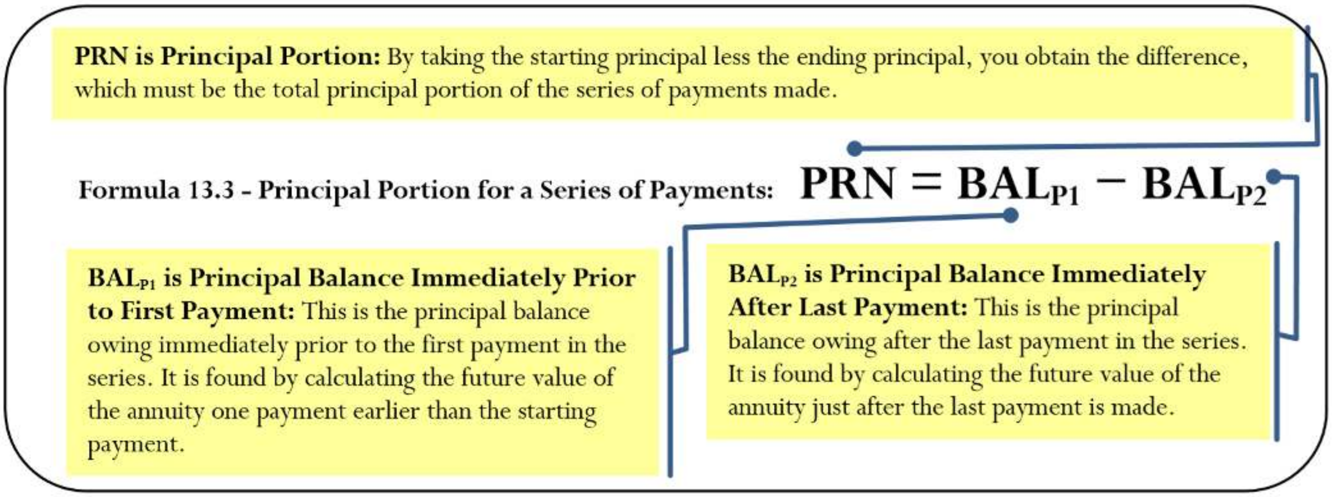
On Baxter's fifth payment of $2,841.02, he has $2,328.92 deducted from his principal and the remaining $512.10 comes from the interest earned on his investment.

Calculating Interest and Principal Components for a Series of Payments

Many times in business, you need to know the principal and interest portions for a series of annuity payments. For example, when completing tax forms a company needs the total loan interest paid annually. If the loan payments are monthly, using Formula 13.1 and Formula 13.2 requires you to perform the calculations 12 times (once for each payment) to arrive at the total interest paid. Clearly, that is time consuming and tedious. In this section, you learn new formulas and a process for calculating the principal and interest portions involving a series of payments.

The Formula

Formulas 13.3 and 13.4 are used to determine the interest and principal components for a series of annuity payments.



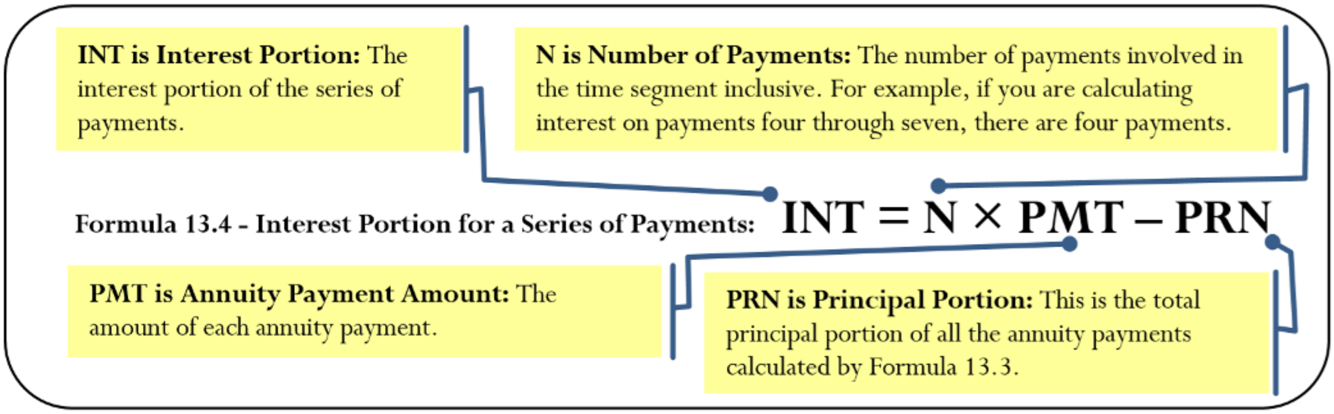
PRN is Principal Portion: By taking the starting principal less the ending principal, you obtain the difference, which must be the total principal portion of the series of payments made.

Formula 13.3 - Principal Portion for a Series of Payments: $\mathrm{PRN} = \mathrm{BAL}_{\mathrm{P1}} - \mathrm{BAL}_{\mathrm{P2}}$

$\mathrm{BAL}_{\mathrm{P1}}$ is Principal Balance Immediately Prior to First Payment: This is the principal balance owing immediately prior to the first payment in the series. It is found by calculating the future value of the annuity one payment earlier than the starting payment.

$\mathrm{BAL}_{\mathrm{P2}}$ is Principal Balance Immediately After Last Payment: This is the principal balance owing after the last payment in the series. It is found by calculating the future value of the annuity just after the last payment is made.

Formula 13.3



INT is Interest Portion: The interest portion of the series of payments.

N is Number of Payments: The number of payments involved in the time segment inclusive. For example, if you are calculating interest on payments four through seven, there are four payments.

Formula 13.4 - Interest Portion for a Series of Payments: $\mathrm{INT} = \mathrm{N} \times \mathrm{PMT} - \mathrm{PRN}$

PMT is Annuity Payment Amount: The amount of each annuity payment.

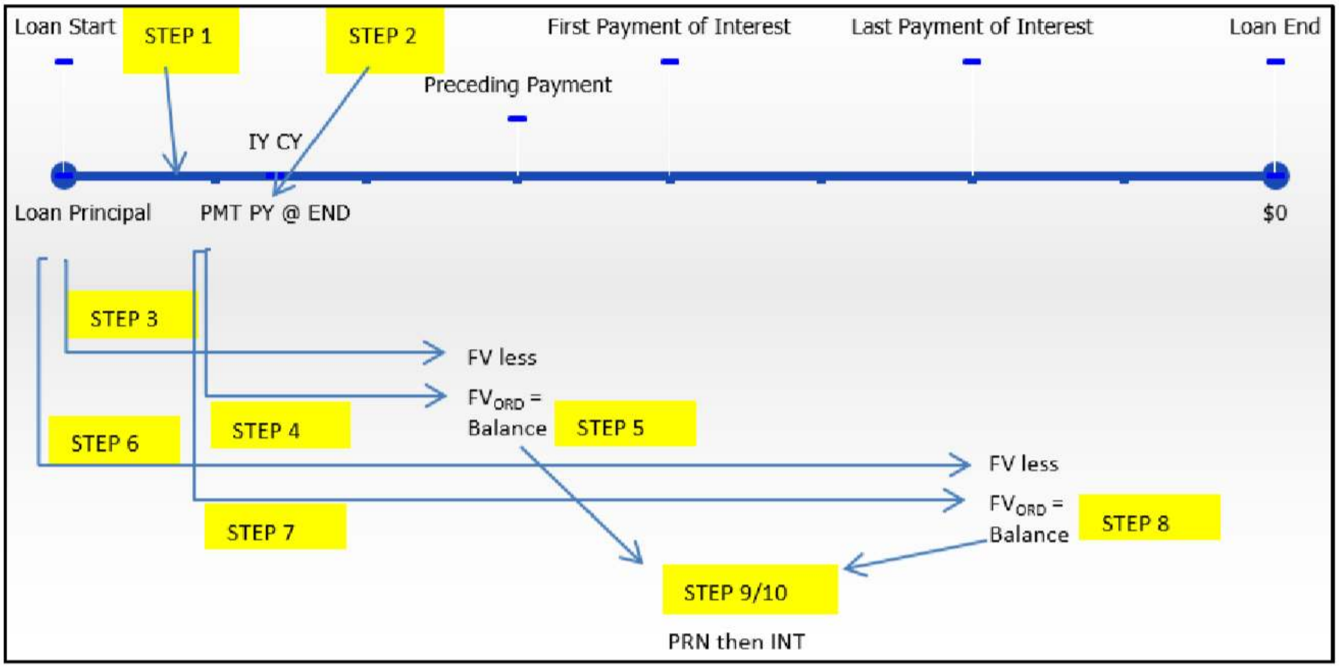
PRN is Principal Portion: This is the total principal portion of all the annuity payments calculated by Formula 13.3.

Formula 13.4

How It Works

Follow these steps to calculate the interest and principal components for a series of annuity payments:

Step 1: Draw a timeline. Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD}. The annuity payment amount may or may not be known.



Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, solve for it using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round the payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the series of payments being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments). For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the balance immediately after the 21st payment.

Step 4: Calculate the future value of all annuity payments already made prior to the first payment in the series. Use Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the future value of the first 21 payments.

Step 5: Calculate the balance (BAL) prior to the series of payments by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date. Do not round this number.

Steps 6 to 8: Repeat steps 3 to 5 to calculate the future value of the original principal immediately after the last payment in the series is made. For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the balance immediately after the 25th payment.

Step 9: Calculate the principal portion of the series of payments using Formula 13.3.

Step 10: Calculate the interest portion of the series of payments using Formula 13.4.

Important Notes

Working with a series of payments on the BAII Plus calculator requires you to enter the first payment number into the P1 and the last payment number into the P2. Thus, if you are looking to calculate the interest and principal portions of payments four through seven, set P1 = 4 and P2 = 7. In the outputs, the BAL window displays the balance remaining after the last payment entered (P2 = 7), and the PRN and INT windows display the total principal interest portions for the series of payments.

Things To Watch Out For

A common mistake occurs in translating years into payment numbers. For example, assume payments are monthly and you want to know the total interest paid in the fourth year. In error, you might calculate that the fourth year begins with payment 36 and ends with payment 48, thus looking for payments 36 to 48. The mistake is to fail to realize that the 36th payment is actually the last payment of the third year. The starting payment in the fourth year is the 37th payment. Hence, if you are concerned only with the fourth year, then you must look for the 37th to 48th payments.

There are two methods to calculate the correct payment numbers:

1. Calculate the payment at the end of the year in question, then subtract the payment frequency less one ($PY - 1$) to arrive at the first payment of the year. In the example, the last payment of the fourth year is 48. With monthly payments, or PY = 12, then 48 − (12 − 1) = 37, which is the first payment of the fourth year.
2. You could determine the last payment of the year prior to the year of interest and add one payment to it. Thus, the end of the third year is payment #36, so the first payment of the fourth year is 36 + 1 = 37. The last payment of the fourth year remains at payment 48.

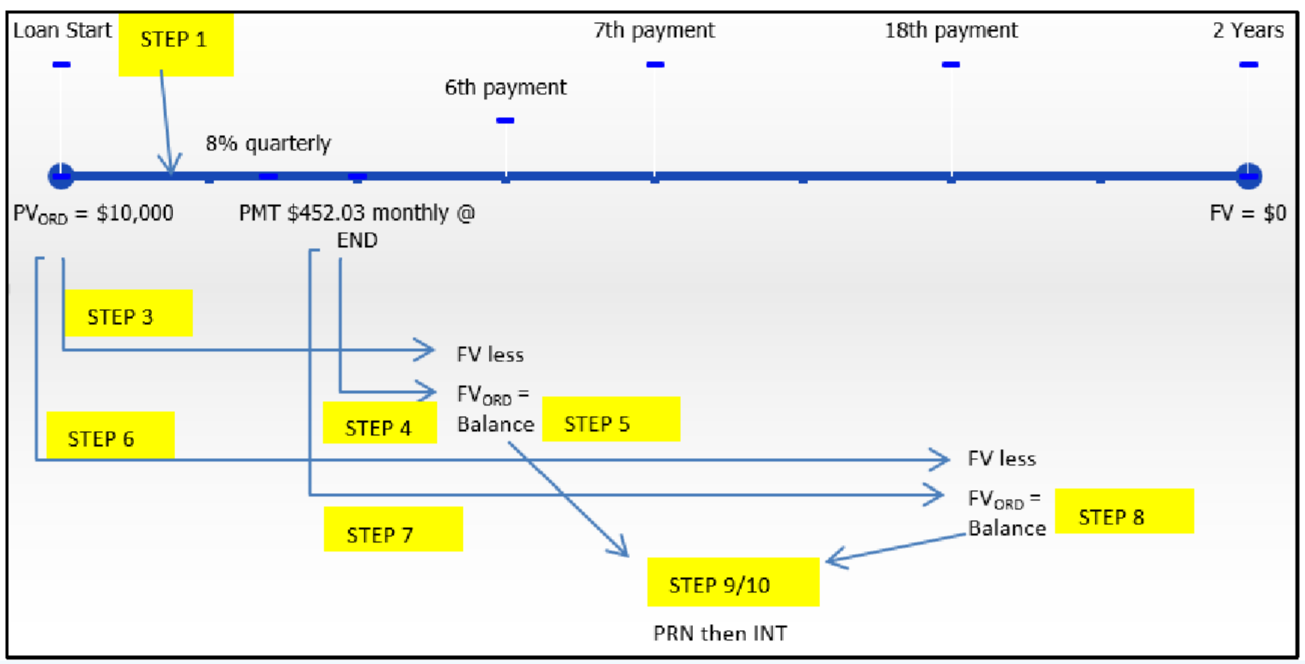
✓ Example 13.1.3: Interest and Principal of a Series of Loan Payments

Revisit Example 13.1.1. The accountant at the accounting firm of Nichols and Burnt is completing the tax returns for the company and needs to know the total interest expense paid during the tax year that encompassed payments 7 through 18 inclusively. Remember, the company borrowed $10,000 at 8% compounded quarterly with month-end payments for two years.

Solution

Note that this is an ordinary general annuity. Calculate the total principal portion (PRN) and the total interest portion (INT) of the 7th to the 18th payments on the two-year loan.

What You Already Know


Loan Start
STEP 1
7th payment
18th payment
2 Years
6th payment
8% quarterly
PV$_{ORD}$ = $10,000
PMT $452.03 monthly @ END
FV = $0
STEP 3
FV less
FV$_{ORD}$ = Balance
STEP 4
STEP 5
STEP 6
FV less
FV$_{ORD}$ = Balance
STEP 8
STEP 7
STEP 9/10
PRN then INT

Step 1:

The following information about the accounting firm's loan is known, as illustrated in the timeline.

PV_{ORD} = $10,000, IY = 8%, CY = 4, PMT = $452.03, PY = 12, Years = 2, FV = $0

How You Will Get There

Step 2:

PMT is known. Skip this step.

Step 3:

Calculate the future value of the loan principal prior to the first payment in the series using Formulas 9.2 and 9.3.

Step 4:

Calculate the future value of the first six payments using Formulas 11.1 and 11.2.

Step 5:

Calculate the principal balance prior to the 7th payment through $BAL_{P1} = FV - FV_{ORD}$.

Steps 6 to 8:

Repeat steps 3 to 5 for the 18 payment to calculate BAL_{P2}.

Step 9:

Calculate the principal portion by using Formula 13.3.

Step 10:

Calculate the interest portion by using Formula 13.4.

Perform

Step 3:

Recall $i = 2\%; N = 4 \times \frac{6}{12} = 2$ compounds; $FV = \$10,000(1+0.02)^2 = \$10,404.00$

Step 4:

$$N = 12 \times \frac{6}{12} = 6 \text{ payments}$$

$$FV_{ORD} = \$452.03 \left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^6 - 1}{(1+0.02)^{\frac{4}{12}-1}} \right] = \$2,757.483449$$

Step 5:

$$BAL_{P1} = \$10,404.00 - \$2,757.483449 = \$7,646.516551$$

Step 6:

$$N = 4 \times \frac{18}{12} = 6 \text{ compounds; } FV = \$10,000(1+0.02)^6 = \$11,261.62419$$

Step 7:

$$N = 12 \times \frac{18}{12} = 18 \text{ payments}$$

$$FV_{ORD} = \$452.03 \left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^{18} - 1}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \$8,611.157995$$

Step 8:

$$BAL_{P2} = \$11,261.62419 - \$8,611.157995 = \$2,650.466195$$

Step 9:

$$PRN = \$7,646.516551 - \$2,650.466195 = \$4,996.05$$

Step 10:

N = 7th through 18th payment inclusive = 12 payments;

$$INT = 12 \times \$452.03 - \$4,996.06 = \$5,424.36 - \$5,064.96 = \$428.30$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	8	10000	-452.03	0	12	4

P1	P2	BAL (output)	PRN (output)	INT (output)
7	18	2,650.466197	4,996.050354	428.309646

For the tax year covering payments 7 through 18, total payments of \$5,424.36 are made, of which \$4,996.05 was deducted from principal while \$428.31 went to the interest charged.

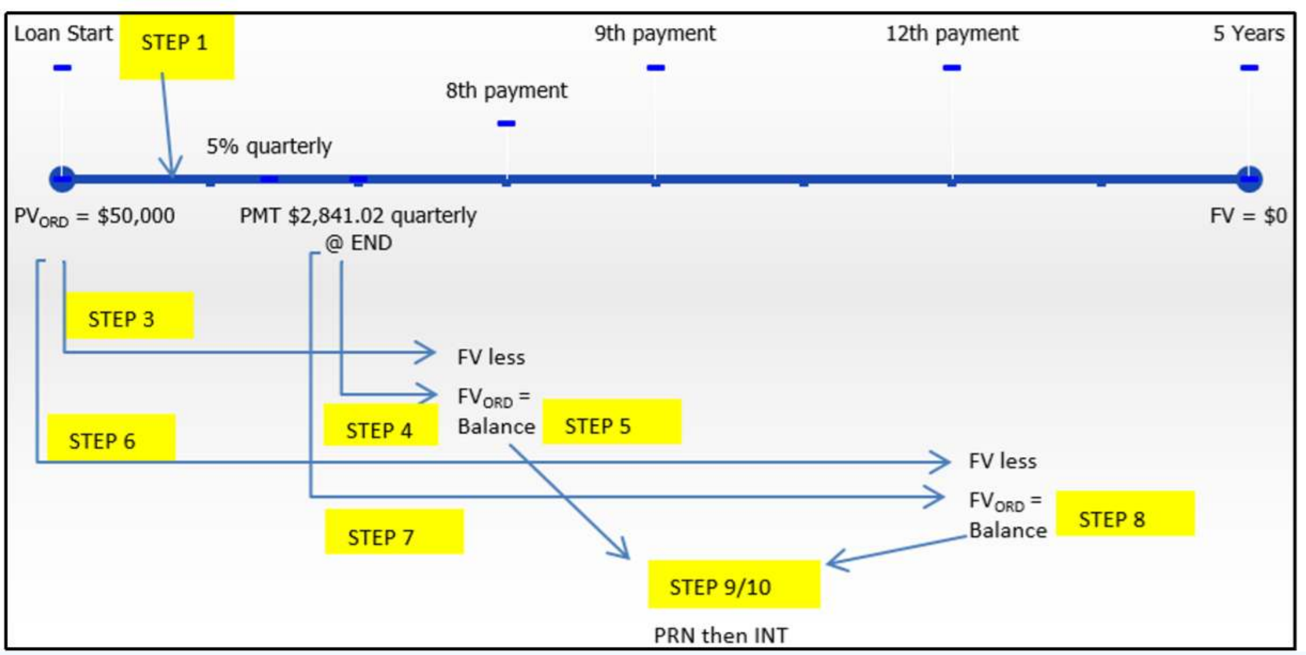
Example 13.1.4: Interest and Principal of a Series of Investment Annuity Payments

Revisit Example 13.1.2, in which Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments to him. For his third year, he needs to know how much of his payments came from his principal and how much was interest earned on the investment.

Solution

Note that this is an ordinary simple annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the third-year payments for the five-year investment annuity. This is the 9th through the 12th payments inclusive.

What You Already Know


Loan Start
STEP 1
9th payment
12th payment
5 Years
8th payment
5% quarterly
PV_ORD = $50,000
PMT $2,841.02 quarterly @ END
FV = $0
STEP 3
FV less
FV_ORD =
Balance
STEP 4
STEP 5
STEP 6
FV less
FV_ORD =
Balance
STEP 8
STEP 7
STEP 9/10
PRN then INT

Step 1:

The following information about the investment annuity is known, as illustrated in the timeline.

PV_{ORD} = \$50,000, IY = 5%, CY = 4, PMT = \$2,841.02, PY = 4, Years = 5, FV = \$0

How You Will Get There

Step 2:

PMT is known. Skip this step.

Step 3:

Calculate the future value of the loan principal prior to the first payment in the series using Formulas 9.2 and 9.3.

Step 4:

Calculate the future value of the first eight payments using Formulas 11.1 and 11.2.

Step 5:

Calculate the principal balance prior to the ninth payment through $BAL_{P1} = FV - FV_{ORD}$.

Steps 6 to 8:

Repeat steps 3 to 5 for the 12th payment to calculate BAL_{P2}.

Step 9:

Calculate the principal portion by using Formula 13.3.

Step 10:

Calculate the interest portion by using Formula 13.4.

Perform

Step 3:

$$\text{Recall } i = 1.25\%; N = 4 \times 2 = 8 \text{ compounds; } FV = \$50,000(1+0.0125)^8 = \$55,224.30506$$

Step 4:

$$N = 4 \times 2 = 8 \text{ payments}$$

$$FV_{ORD} = \$2,841.02\left[\frac{\left[(1+0.0125)^{\frac{4}{4}}\right]^8 - 1}{(1+0.0125)^{\frac{4}{4}} - 1}\right] = \$23,747.76825$$

Step 5:

$$BAL_{Pl} = \$55,224.30506 - \$23,747.76825 = \$31,476.53681$$

Step 6:

$$N = 4 \times 3 = 12 \text{ compounds; } FV = \$50,000(1+0.0125)^{12} = \$58,037.72589$$

Step 7:

$$N = 4 \times 3 = 12 \text{ payments}$$

$$FV_{ORD} = \$2,841.02\left[\frac{\left[(1+0.0125)^{\frac{4}{4}}\right]^{12} - 1}{(1+0.0125)^{\frac{4}{4}} - 1}\right] = \$36,536.544$$

Step 8:

$$BAL_{P2} = \$58,037.72589 - \$36,536.544 = \$21,501.18189$$

Step 9:

$$PRN = \$31,476.53681 - \$21,501.18189 = \$9,975.35$$

Step 10:

N = 9th through 12th payment inclusive = 4 payments;

$$INT = 4 \times \$2,841.02 - \$9,975.35 = \$11,364.08 - \$9,975.35 = \$1,388.73$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	2,841.02	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
9	12	-21,501.18189	9,975.354914	1,388.725086

In the third year, Baxter receives a total of \$11,364.08 in payments, of which \$9,975.35 is deducted from the principal and \$1,388.73 represents the interest earned on the investment.

Contributors and Attributions

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13.2: Calculating the Final Payment

If you have ever paid off a loan you may have noticed that your last payment was a slightly different amount than your other payments. Whether you are making monthly insurance premium payments, paying municipal property tax installments, financing your vehicle, paying your mortgage, receiving monies from an investment annuity, or dealing with any other situation where an annuity is extinguished through equal payments, the last payment typically differs from the rest, by as little as one penny or up to a few dollars. This difference can be much larger if you arbitrarily chose an annuity payment as opposed to determining an accurate payment through time value of money calculations.

Why is it important for this final payment to differ from all of the previous payments? From a consumer perspective, you do not want to pay a cent more toward a debt than you have to. In 2011, the average Canadian is more than $100,000 in debt across various financial tools such as car loans, consumer debt, and mortgages. Imagine if you overpaid every one of those debts by a dollar. Over the course of your lifetime those overpayments would add up to hundreds or even thousands of dollars.

On the business side, particularly where companies are collecting debts from customers, two main issues are legality and profitability:

1. **Legality**. A business can legally collect only the exact amount of money its consumers owe and not a penny more. If a business collects more money than it is owed, it is legally obliged to reimburse the consumer or it will face legal repercussions.
2. **Profitability**. RBC has over $200 billion in outstanding loans. What if it told every customer who borrowed money last year that they did not have to worry about paying off that final nickel on their loan? This decision would forego millions in revenue. Shareholders would be very unhappy. Giving up even the tiniest amounts of revenue can aggregate into meaningful losses.

Section 11.5 discussed an approximation technique for the final annuity. It is now time to be precise in this calculation. In the current section, you will see why the final payment differs and how to calculate its exact amount. You will also calculate the principal and interest components for a series of payments involving the final payment.

Why Is the Final Payment Different?

Section 11.4 introduced the calculations to determine the annuity payment. Observe that you always needed to round a non-terminating annuity payment to two decimals. It is rare for a calculated annuity payment not to require rounding. The rounding up or down of the annuity payment forms the basis for adjusting the final payment. Observe the implication of each rounding procedure as summarized in this table.

How the Payment Was Rounded	Principal Implications	Interest Implications	Final Payment Implications
Upward	Overpayment	Slight decrease	Needs to be reduced in an amount equal to the overpayment and savings on the interest.
Downward	Underpayment	Slight increase	Needs to be increased in an amount equal to the underpayment and extra charges on the interest.

1. **Annuity Payment Rounded Up**. If the calculated annuity payment is exactly PMT = $999.995, this payment is rounded to two decimals and payments of $1,000 are made. With each annuity payment you are then overpaying the debt by $1,000 − $999.995 = $0.005. Nominally, this means that every two payments made result in an overpayment of $0.01 toward the debt. If you make 20 such payments, then you nominally overpay the debt by 20 × $0.005 = $0.10. Therefore, when it comes to the final payment you need to compensate for all of the overpayments

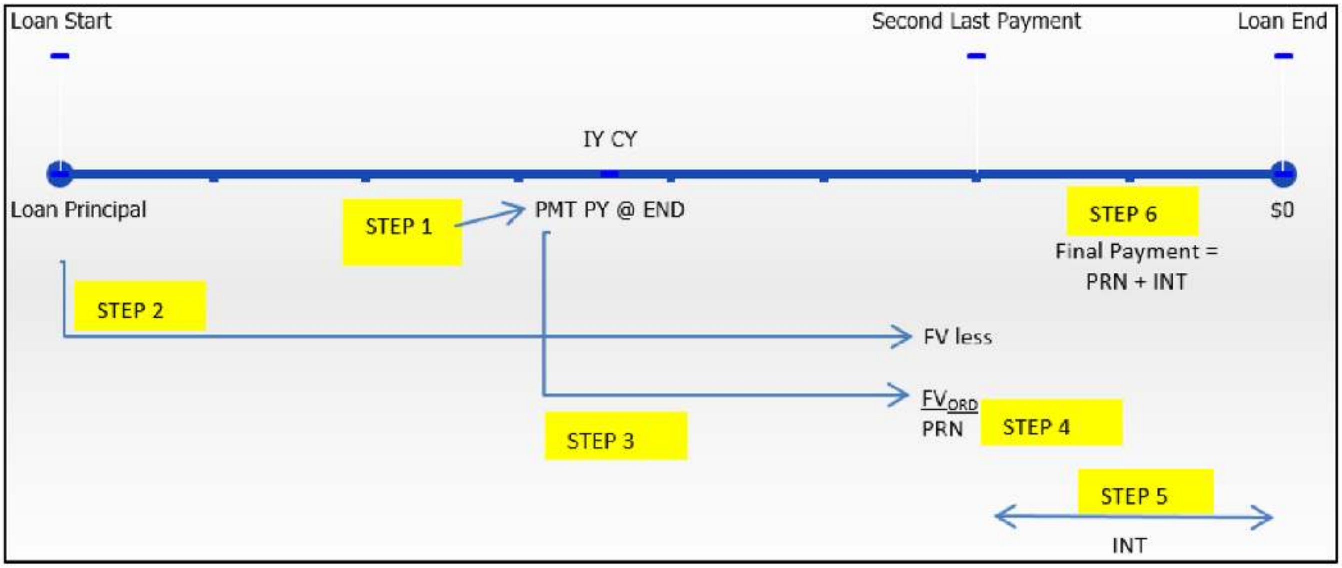
made, reducing the final payment by $0.10. And since the principal is slightly smaller at all times as a result of the overpayment, an additional adjustment may be needed because of less interest being calculated.

2. **Annuity Payment Rounded Down**. The same principles apply when the annuity payment is rounded down. If you calculated PMT = $1,000.0025, your payments of $1,000 are underpaying the debt by $1,000.0025 − $1,000 = $0.0025. Over the course of 20 payments you nominally underpay by 20 × $0.0025 = $0.05, for which you must increase the final payment by this amount. As well, since the principal is slightly larger at all times as a result of the underpayment, an additional adjustment may be needed because of more interest being calculated.

For example, take a $200,000 loan for 25 years at 6% compounded semi-annually with monthly payments. The calculated PMT = $1,279.613247 becomes a rounded down payment of PMT = $1,279.61. Each payment is $0.003247 underpaid. As a result, the final 300th payment is $2.23 more, representing $0.003247 × 300 = $0.97 of original principal underpayment plus $1.26 of extra interest on the outstanding principal.

How It Works

The following six steps are needed to calculate the final payment. These steps are designed to integrate with the next section, where the principal and interest components on a series of payments involving the final payment are calculated.



Step 1: Draw a timeline to represent the annuity. A typical timeline format appears in the figure above. Identify all seven time value of money variables. If all are known, proceed to step 2. Most commonly, PMT is unknown. Solve for it using Formulas 9.1 (Periodic Interest Rate), 11.1 (Number of Annuity Payments), and 11.4 (Ordinary Annuity Present Value), rearranging for PMT. Round the PMT to two decimals.

Step 2: Calculate the future value of the original principal at $N-1$ payments. Obtain the periodic interest rate using Formula 9.1 (Periodic Interest Rate) if you have not already calculated it in step 1. Then use Formulas 9.2 (Number of Compound Periods for Single Payments) and 9.3 (Compound Interest for Single Payments). For example, if your final payment is the 24th payment, you need the balance remaining after the 23rd payment.

Step 3: To calculate the future value of all annuity payments $(N-1)$ already made, apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). Remember that if the final payment is the 24th payment, then only 23 payments have already occurred.

Step 4: Subtract the future value of the payments from the future value of the original principal (step 2 − step 3) to arrive at the principal balance remaining immediately prior to the last payment. This is the principal owing on the account and therefore is the principal portion (PRN) for the final payment. The final payment must reduce the annuity balance to zero!

Step 5: Calculate the interest portion (INT) of the last payment using Formula 13.1 on the remaining principal.

Step 6: Add the principal portion from step 4 to the interest portion from step 5. The sum is the amount of the final payment.

Important Notes

The calculator determines the final payment amount using the AMORT function described in Section 13.1. To calculate the final payment:

1. You must accurately enter all seven time value of money variables ($N, I/Y, PV, PMT, FV, P/Y$, and C/Y). If PMT was calculated, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
2. Press 2nd AMORT.
3. Enter the payment number for the final payment into P1 and press Enter followed by ↓.
4. Enter the same payment number for P2 and press Enter followed by ↓.
5. In the BAL window, note the balance remaining in the account after the last payment is made. Watch the cash flow sign to properly interpret what to do with it! The sign matches the sign of your PV. The next table summarizes how to handle this balance.

Type of Transaction	Positive BAL	Negative BAL
Loan	Increase final payment	Decrease final payment
Investment Annuity	Decrease final payment	Increase final payment

- **Loans**. For a loan for which PV is entered as a positive cash flow and hence PMT is a negative cash flow, a positive balance means you are borrowing it. Thus, you need to increase the final payment by this amount to pay off the loan. A negative balance means you overpaid and the bank owes you. Thus, you need to decrease the final payment by this amount.
- **Investment Annuities**. For an investment annuity where PV is entered as a negative cash flow and hence PMT is a positive cash flow, a negative balance means you still have money invested, so you should add it to your final payment to get it back. A positive balance means you have been paid too much, so you need to decrease your final payment by this amount.
- A helpful key sequence shortcut to arrive at the final payment is to have BAL on your display and then press − RCL PMT =
- This sequence automatically adjusts the payment accordingly for both loans and investment annuities. Manually round the answer to two decimals when the calculation is complete.

6. If you are interested in the PRN or INT portions of the final payment, the INT output is correct. However, the PRN output is incorrect since the calculator has not adjusted the final payment. You must adjust the PRN output in the same manner and amount as the final payment (by adding or subtracting the BAL remaining).

Things To Watch Out For

In Section 11.5 you calculated the N, and if it turned out to be a decimal number you then approximated the final payment by taking solely the decimal and multiplying it by the annuity payment. Remember that this was only an approximation and not actually the correct answer. For example, revisiting Example 11.5.1, it was determined that Samia could sustain 293.660180 payments of $3,000. The estimated final payment was 0.660180 × $3,000 = $1,980.54. In applying this chapter's technique, you will find that the precise and correct amount of the final payment is exactly $1,982.00.

You do not achieve the same answer through the approximation technique, because N does not represent a portion of the payment amount. N represents the portion of the next payment period before the account runs out of money. Recall that in all of the time value formulas N is an exponent, meaning that you must determine the final payment exponentially and not through multiplication.

Paths To Success

An alternative method to adjust the final payment is to calculate the future value of the rounding error on the payment. Recall the $200,000 loan for 25 years at 6% compounded semi-annually with monthly payments. The calculated PMT = $1,279.613247 was rounded down to PMT = $1,279.61. Each payment is $0.003247 underpaid. If you move the $0.003247 underpayment to the 300th final payment this produces a future value of

$$FV_{ORD} = \$0.003247\left[\frac{\left[(1+0.03)^{\frac{2}{12}}\right]^{300}-1}{(1+0.03)^{\frac{2}{12}}-1}\right] = \$2.23$$

The underpayment is $2.23, which then means the final payment is increased to $1,279.61 + $2.23 = $1,281.84. Note the one disadvantage of this technique is that you are unable to determine the interest and principal components of that final payment. Calculating these components requires you to apply the six-step procedure discussed above.

? Exercise 13.2.1: Give It Some Thought

Consider the following five statements and then answer the two questions that follow.

a. The final payment is exactly the same as any other annuity payment.
b. The final payment is smaller, reflecting an overpayment of principal of $0.04 plus any lower interest charges saved.
c. The final payment is smaller, reflecting an underpayment of principal of $0.04 plus any higher interest charges earned.
d. The final payment is larger, reflecting an overpayment of principal of $0.04 plus any lower interest charges saved.
e. The final payment is larger, reflecting an underpayment of principal of $0.04 plus any higher interest charges earned.

1. If the annuity payment is rounded up by $0.004 per payment and involves 10 payments, which statement is correct?
2. If the annuity payment is rounded down by $0.004 per payment and involves 10 payments, which statement is correct?

Answer

1. b. When rounding up, principal is reduced faster (overpayment) and more interest is saved.
2. e. When rounding down, principal is reduced more slowly (underpayment) and more interest is earned

✓ Example 13.2.1: Final Payment on a Loan

Recall Example 13.1.1, in which Nichols and Burnt borrowed $10,000 at 8% compounded quarterly with month-end payments of $452.03 for two years. The accountant now needs to record the final payment on the loan with correct portions assigned to principal and interest.

Solution

Calculate the principal portion (PRN) and the interest portion (INT) of the final payment on the two-year loan, along with the amount of the final payment itself (PMT).

What You Already Know

Step 1:

The following information about the accounting firm's loan is known, as illustrated in the timeline.

PV_{ORD} = $10,000, IY = 8%, CY = 4, PMT = $452.03, PY = 12, Years = 2, N = 12 × 2 = 24, FV = $0

How You Will Get There

Step 2:

Calculate the future value of the loan principal at the time of the 23rd payment using Formulas 9.1, 9.2, and 9.3.

Step 3:

Calculate the future value of the first 23 payments using Formulas 11.1 and 11.2.

Step 4:

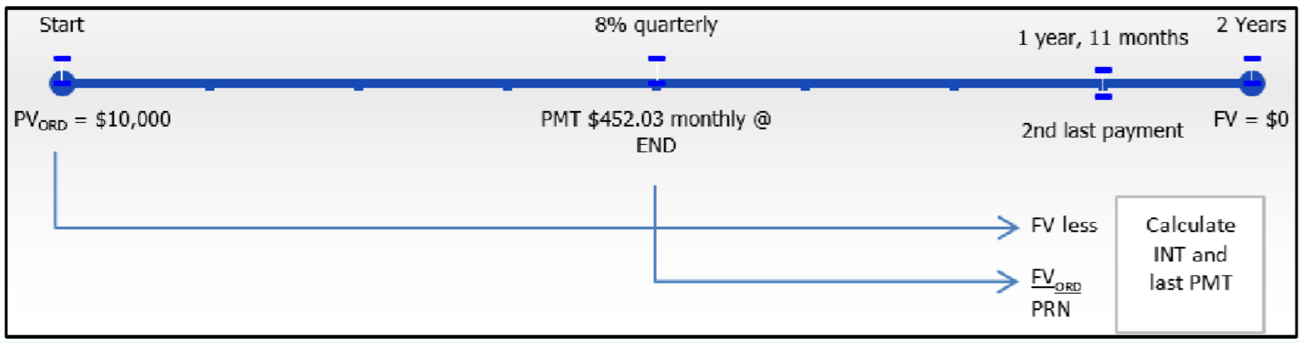
Calculate the principal balance remaining after 23 payments through $BAL = FV - FV_{ORD}$. Note that $BAL = PRN$ portion of the final payment.

Step 5:

Calculate the interest portion by using Formula 13.1.

Step 6:

Calculate the final payment by totaling steps 4 and 5 above.


Start
8% quarterly
1 year, 11 months
2 Years
PV$_{ORD}$ = $10,000
PMT $452.03 monthly @ END
2nd last payment
FV = $0
FV less
FV$_{ORD}$
PRN
Calculate INT and last PMT

Perform

Step 2:

$$i = 8\%/4 = 2\%; N = 4 \times 1\frac{11}{12} = 7.\overline{6} \text{ compounds}$$

$$FV = \$10,000(1+0.02)^{7.\overline{6}} = \$11,639.50872$$

Step 3:

$$N = 12 \times 1\frac{11}{12} = 23$$

$$FV_{ORD} = \$452.03\left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^{23}-1}{(1+0.02)^{\frac{4}{12}}-1}\right] = \$11,190.39151$$

Step 4:

$$BAL = \$11,639.50872 - \$11,190.39151 = \$449.117209$$

Step 5:

$$INT = \$449.117209 \times \left((1+0.02)^{\frac{4}{12}}-1\right) = \$2.974372$$

Step 6:

$$\text{Final } PMT = \$449.117209 + \$2.974372 = \$452.09$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	8	10000	-452.03	0	12	4

P1	P2	BAL (output)	PRN (output)	INT (output)
24	24	0.061582 *added to payment of $452.03=$452.09	449.055627 *added BAL=$449.12	2.974372

The accountant for Nichols and Burnt should record a final payment of \$452.09, which consists of a principal portion of \$449.12 and an interest portion of \$2.97.

✓ Example 13.2.2: Final Payment on an Investment Annuity

Recall Example 13.1.2, in which Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments of \$2,841.02 to him. He needs to know the amount of his final payment, along with the principal and interest components.

Solution

Calculate the principal portion (PRN) and the interest portion (INT) of the final payment on the five-year investment annuity, along with the amount of the final payment itself (PMT).

What You Already Know

Step 1:

You know the following about the investment annuity, as illustrated in the timeline.

PV_{ORD} = \$50,000, IY = 5%, CY = 4, PMT = \$2,841.02, PY = 4, Years = 5, N = 4 × 5 = 20, FV = \$0

How You Will Get There

Step 2:

Calculate the future value of the investment at the time of the 19th payment using Formulas 9.1, 9.2, and 9.3.

Step 3:

Calculate the future value of the first 19 payments using Formulas 11.1 and 11.2.

Step 4:

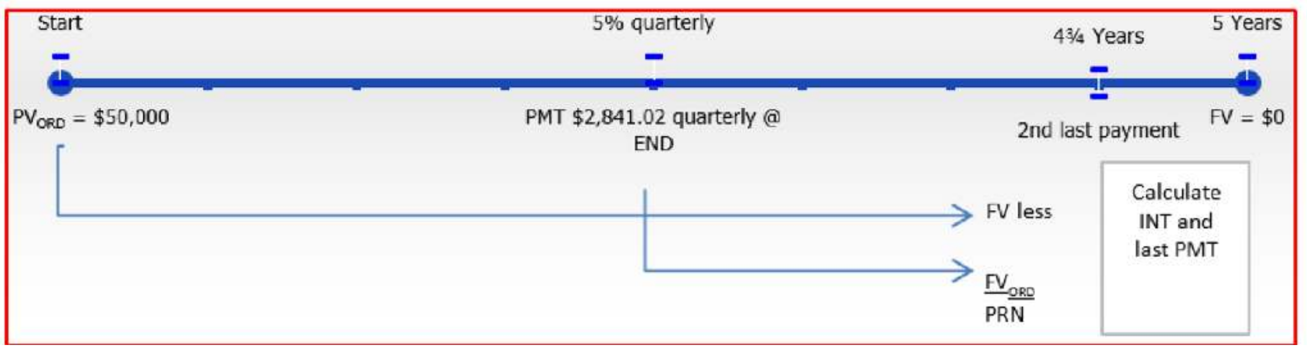
Calculate the principal balance remaining after 19 payments through $BAL = FV - FV_{ORD}$. Note that this is the PRN portion of the final payment.

Step 5:

Calculate the interest portion by using Formula 13.1.

Step 6:

Calculate the final payment by totaling steps 4 and 5 above.


Start
5% quarterly
4¾ Years
5 Years
PV_{ORD} = \$50,000
PMT \$2,841.02 quarterly @ END
2nd last payment
FV = \$0
FV less
FV_{ORD}
PRN
Calculate INT and last PMT

Step 2:

$$i = 5\%/4 = 1.25\%; N = 4 \times 4\tfrac{3}{4} = 19 \text{ compounds}$$

$$FV = \$50,000(1+0.0125)^{19} = \$63,310.48058$$

Step 3:

$$N = 4 \times 4\tfrac{3}{4} = 19 \text{ payments}$$

$$\left[\left[(1+0.0125)^{\frac{4}{4}}\right]^{19} - 1 \right]$$

$$FV_{ORD} = \$2,841.02\left[\frac{}{(1+0.0125)^{\frac{4}{4}}-1}\right] = \$60,504.54645$$

Step 4:

$$BAL = \$63,310.48058 - \$60,504.54645 = \$2,805.934127$$

Step 5:

$$INT = \$2,805.934127 \times \left((1+0.0125)^{\frac{4}{4}}-1\right) = \$35.074176$$

Step 6:

$$\text{Final } PMT = \$2,805.934127 + \$35.074176 = \$2,841.01$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	2841.02	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
20	20	0.011696 *subtracted from payment of $2,841.02=$2841.01	2,805.945823 *subtract BAL=$2,805.93	35.074176

Baxter will receive a final payment of $2,841.01 consisting of $2,805.93 in principal plus $35.08 in interest.

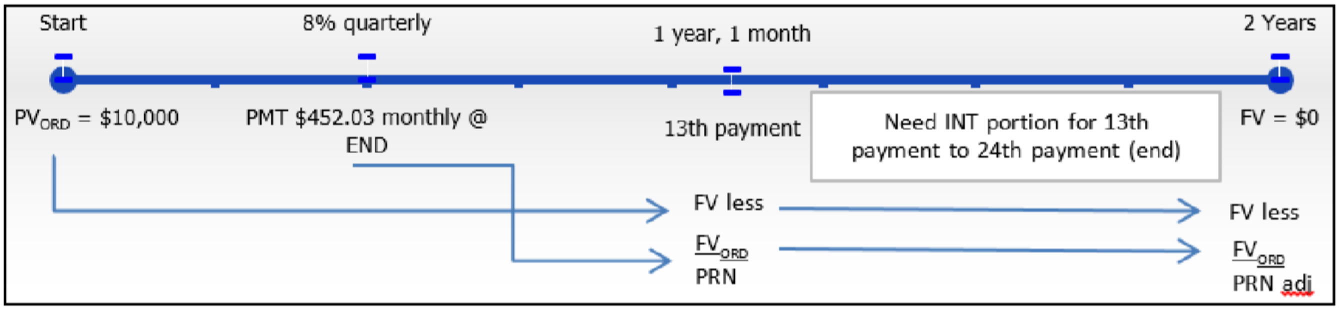
Calculating Principal and Interest Portions for a Series Involving the Final Payment

Now that you know how to calculate the last payment along with its interest and principal components, it is time to extend this knowledge to calculating the principal and interest portions for a series of payments that involve the final payment.

How It Works

For a series of payments, you follow essentially the same steps as in Section 13.1; however, you need a few minor modifications and interpretations:

Step 1: Draw a timeline. Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD}. The annuity payment amount may or may not be known.



Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, then solve for the annuity payment using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round this payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the series of payments being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for a Single Payment), and 9.3 (Compound Interest for Single

Payments).

Step 4: Calculate the future value of all annuity payments already made prior to the first payment in the series. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).

Step 5: Calculate the balance (BAL) prior to the series of payments by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The result of this step determines the amount of principal remaining in the account. This is the PRN for the series of payments, since the remaining payments must reduce the principal to zero.

Step 6: Calculate the future value of the original principal immediately at the end of the timeline. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments).

Step 7: Calculate the future value of all annuity payments, including the unadjusted final payment. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).

Step 8: Calculate the balance (BAL) after the series of payments by subtracting step 7 (the future value of the payments) from Step 6 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date. Do not round this number. The result of this step determines the amount of overpayment or underpayment, which you must then adjust in the next step.

Step 9: Calculate the interest portion using Formula 13.4, but modify the final amount by the result from step 8. Hence, Formula 13.4 looks like

$$INT = N \times PMT - PRN + \text{(balance from step 8)}$$

✓ Example 13.2.3: Principal and Interest for a Series Involving the Final Payment

Revisit Example 13.1.1. The accountant at the accounting firm of Nichols and Burnt is completing the tax returns for the company and needs to know the total principal portion and interest expense paid during the tax year encompassing payments 13 through 24 inclusively. Recall that the company borrowed $10,000 at 8% compounded quarterly, with month-end payments of $452.03 for two years.

Solution

Calculate the total principal portion (PRN) and the total interest portion (INT) of the 13th to 24th payments on the two-year loan. This involves the final payment since the 24th payment is the last payment, requiring usage of the adapted steps discussed in this section.

What You Already Know

Step 1:

The following information about the accounting firm's loan is known, as illustrated in the timeline.

PV_{ORD} = $10,000, IY = 8%, CY = 4, PMT = $452.03, PY = 12, Years = 2, N = 2 × 12 = 24, FV = $0

How You Will Get There

Step 2:

Skip this step, since PMT is known.

Step 3:

Calculate the future value of the loan principal after the 12th payment using Formulas 9.1, 9.2, and 9.3.

Step 4:

Calculate the future value of the first 12 payments using Formulas 11.1 and 11.2.

Step 5:

Calculate the balance remaining after 12 payments through $BAL_{P1} = FV - FV_{ORD}$. Note that for payments 13 through 24, $PRN = BAL_{P1}$.

Step 6:

Calculate the future value of the loan principal after the 24th payment using Formulas 9.2, and 9.3.

Step 7:

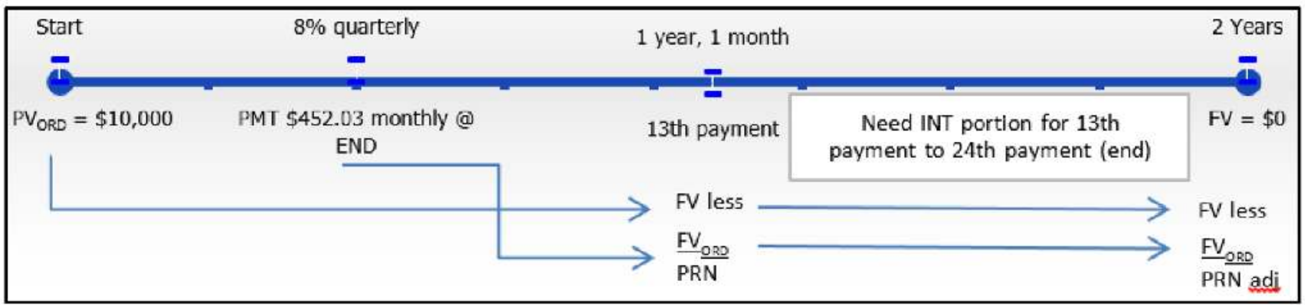
Calculate the future value of all 24 payments using Formulas 11.1 and 11.2.

Step 8:

Calculate the balance on the loan after all payments through $BAL_{P2} = FV - FV_{ORD}$. Note that this amount is used to adjust step 9.

Step 9:

Calculate the interest portion by using the adjusted Formula 13.4.


Start
8% quarterly
1 year, 1 month
2 Years
PV_ORD = $10,000
PMT $452.03 monthly @ END
13th payment
Need INT portion for 13th payment to 24th payment (end)
FV = $0
FV less
FV_ORD
PRN
FV less
FV_ORD
PRN adj

Perform

Step 3:

$$i = 8\%/4 = 2\%; N = 4 \times 1 = 4 \text{ compounds}$$

$$FV = \$10,000(1+0.02)^4 = \$10,824.3216$$

Step 4:

$$N = 12 \times 1 = 12 \text{ payments}$$

$$FV_{ORD} = \$452.03\left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^{12} - 1}{(1+0.02)^{\frac{4}{12}} - 1}\right] = \$5,626.36923$$

Step 5:

$$BAL_{Pl} = \$10,824.3216 - \$5,626.36923 = \$5,1923 = \$5,197.95237; PRN = \$5,197.95$$

Step 6:

$$N = 4 \times 2 = 8 \text{ compounds}$$

$$FV = \$10,000(1+0.02)^8 = \$11,716.59381$$

Step 7:

$$N = 4 \times 2 = 8 \text{ payments}$$

$$FV_{ORD} = \$452.03\left[\frac{\left[(1+0.02)^{\frac{4}{12}}\right]^{24} - 1}{(1+0.02)^{\frac{4}{12}} - 1}\right] = \$11,716.53223$$

Step 8:

$$BAL_{P2} = \$11,716.59381 - \$11,716.53223 = \$0.061582$$

Step 9:

N = 13th through 24th payment inclusive = 12 payments;

$$INT = 12 \times \$452.03 - \$5,197.95237 + \$0.061582 = \$5,424.36 - \$5,197.89 = \$226.47$$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	8	10000	-452.03	0	12	4

P1	P2	BAL (output)	PRN (output)	INT (output)
13	24	0.0615825	5,197.890788 *add BAL=$5,197.90	226.469212

For the tax year covering payments 13 through 24, total payments of $5,424.42 are made, of which $5,197.95 goes toward principal while $226.47 is the interest charged.

Paths To Success

Amortization by definition involves the repayment of loans, which are almost always ordinary in nature. What are the implications if you have an investment annuity due, or the rare occurrence of a loan due?

1. **Step-by-Step Procedures**. Whether you are dealing with ordinary annuities or annuities due, all of the processes and procedures remain unchanged.
2. **Formulas**. Make the appropriate substitutions from FV_{ORD} to FV_{DUE}.
3. **Excel**. The annuity type needs to change from 'Ordinary' to 'Due' (in cell C15 of the data entry screen in the template).
4. **Time Value of Money Button Calculator Settings**. When the calculator is loaded with the time value of money variables, ensure that BGN mode has been set and that if the PMT has been computed you have re-entered it with two decimals.
5. **AMORT Function on the BAII Plus Calculator**. A special adaptation is required but is not introduced at this time. This special adaptation will be discussed in Section 13.3 on amortization schedules due.

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13.3: Amortization Schedules

In the previous two sections, you have been working on parts of an entire puzzle. You have calculated the interest and principal portions for either a single payment or a series of payments. Additionally, you calculated the final payment amount along with its principal and interest components. The next task is to put these concepts together into a complete understanding of amortization. This involves developing a complete amortization schedule for an annuity (loan or investment annuity). Additionally, you will create partial amortization schedules that depict specific ranges of payments for a particular annuity.

These concepts primarily apply to accounting. Accountants must record debts properly on balance sheets, make proper journal entries into the accounting books, and track interest expenses clearly. Depreciation of fixed assets uses similar amortization processes and schedules. Bond premiums and discounts (discussed in Chapter 14) also use amortization.

On the personal side, these schedules help you understand any of your loans, mortgages, or investment annuities. Sometimes seeing the true amount of interest you are paying may motivate you to pay the debt off faster. Courts also use these schedules to settle legal matters such as alimony payments.

The Complete Amortization Schedule

An amortization schedule shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. As the title suggests, it provides a complete understanding of where the money goes.

How It Works

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
0–Start	N/A	N/A	N/A	(3)
1	(4)	(5)	(6)	(7)
etc.				
Last payment	(11)	(10)	(9)	$0.00
Total	*if needed	(13)	*equals original loan balance	N/A

Follow these steps to develop a complete amortization schedule:

Step 1: Identify all of the time value of money variables ($N, IY, PV_{ORD}, PMT, FV, PY, CY$). If either N or PMT is unknown, solve for it using an appropriate formula. Remember to round PMT to two decimals.

Step 2: Set up an amortization schedule template as per the table above. Cells of the table are marked with the step numbers that follow. The number of payment rows in the table equals the number of payments in the annuity.Note that in the instance of a general annuity, the table does not distinguish between earned interest and accrued interest. The schedule reflects only the total of earned interest and interest accrued at the time of a payment.

Step 3: On the first row, fill in the original principal of the annuity, or PV_{ORD}.

Step 4: Fill in the rounded annuity payment (PMT) all the way down the column except for the final payment row.

Step 5: Calculate the interest portion of the current annuity payment using Formula 13.1. Round the number to two decimals for the table but retain the decimals for future calculations.

Step 6: Calculate the principal portion of the current annuity payment using Formula 13.2. The interest component is the unrounded interest number from step 5. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 7: Calculate the new principal balance remaining by taking the previous un-rounded balance on the line above and subtracting the un-rounded principal portion (PRN) of the current payment. This is Formula 13.3 rearranged such that $BAL_{P2} = BAL_{P1} - PRN$. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 8: Repeat steps 5 through 7 for each annuity payment until you reach the final payment.

Step 9: For the final payment, the principal portion is exactly equal to the previous balance remaining on the line above. Round the result to two decimals for the table but retain the decimals for future calculations. Enter "$0.00" in the Principal Balance Remaining.

Step 10: Calculate the interest portion of the final annuity payment using Formula 13.1. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 11: Calculate the final payment by adding the un-rounded principal and interest portions of the final payment together. Round this number to two decimals.

Step 12: Since all numbers are rounded to two decimals throughout the table, check the table for the "missing penny," as discussed below. Always ensure that for each row the previous balance minus current principal equals the new balance. Then check that the principal portion plus interest portion equals the annuity payment amount. Make any penny adjustments as needed.

Step 13: Sum the interest portion column in the schedule. Note that no total is necessary for the principal portion since it equals the original principal of the annuity!

Important Notes

The "Missing Penny"

In the creation of the amortization schedule, you always round the numbers off to two decimals since you are dealing with currency. However, as per the rules of rounding, you do not round any numbers in your calculations until you reach the end of the amortization schedule and the annuity has been reduced to zero.

As a result, you have a triple rounding situation involving the balance along with the principal and interest components on every line of the table. What sometimes happens is that a "missing penny" occurs and the schedule needs to be corrected as per step 12 of the process above. In other words, calculations will sometimes appear to be off by a penny. You can identify the "missing penny" when one of the two standard calculations using the rounded numbers from the schedule becomes off by a penny:

1. Previous $BAL - PRN =$ current BAL (step 7)

or

2. $PMT - INT = PRN$ (step 6)

It is important to remember, though, that in actuality there is no "missing penny" in these calculations. It shows up only because you are rounding off numbers. If you were not rounding numbers, this "missing penny" would never occur.

In these instances of the "missing penny", you adjust the schedule as needed to ensure that the math works properly at all times. The golden rule, though, is that the balance in the account (BAL) is always correct and should NEVER be adjusted. Follow this order in making any adjustments:

1. Adjust the PRN if necessary such that the previous $BAL - PRN =$ current BAL.
2. Then adjust INT if necessary such that $PMT - PRN = INT$.

Usually these adjustments come in pairs, meaning that if you need to adjust the PRN up by a penny, somewhere later in the schedule you will need to adjust the PRN down by a penny. Ultimately, these changes in most circumstances have no impact on the total interest (INT) or total principal (PRN) components, since the "missing penny" is nothing more than a rounding error within the schedule.

Your BAII Plus Calculator

The calculator speeds up the repetitive calculations required in the amortization schedule. To create the schedule using the calculator, adapt the steps as follows:

Step 1: Load the calculator with all seven time value of money variables, solving for any unknowns. Ensure that PMT is keyed in with two decimals, and obey the cash flow sign convention.

Steps 2–4: Unchanged.

Steps 5–7: Open the AMORT function. Set the P1 = 1 and P2 = 1. In the appropriate column, record the BAL, PRN, and INT rounded to two decimals.

Step 8: Repeat steps 5–7 by increasing the payment number (P1 and P2) by one each time. Ensure P1 = P2 at all times.

Step 9: Unchanged.

Step 10: Set the P1 and P2 to the final payment number. Record the INT amount.

Steps 11–12: Unchanged.

Step 13: Set the P1 = 1 and P2 = final payment number. Record the INT amount.

✓ Example 13.3.1: Payment Plan on a Dishwasher

Tamara purchased a new dishwasher from The Bay for $895.94. By placing it on her Bay credit card, she can pay off the dishwasher through a special six-month payment plan promotion that charges her 5.9% compounded monthly. Construct the complete amortization schedule for Tamara and total her interest charges.

Solution

Construct a complete amortization schedule for the dishwasher payments along with the total interest paid.

What You Already Know

Step 1:

The timeline for Tamara's purchase appears in the timeline.

PV_{ORD} = $895.94, IY = 5.9%, CY = 12, PY = 12, Years = 0.5, FV = $0

How You Will Get There

Step 1 (continued):

Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.4.

Step 2:

Set up the amortization table.

Steps 3 and 4:

Fill in the original principal and payment column.

Steps 5 to 8:

For each line use Formula 13.1 and Formula 13.2 and the rearranged Formula 13.3.

Steps 9 to 11:

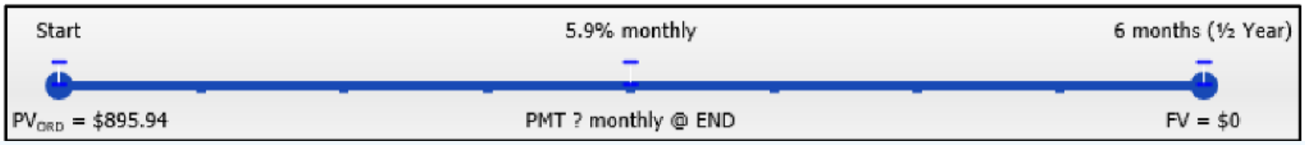
Fill in the principal and balance remaining, calculate the interest using Formula 13.1, and determine the final payment by adding the interest and principal components together.

Step 12:

Check for the "missing penny."

Step 13:

Sum the interest portion.



Perform

Step 1 (continued):

$i = 5.9\%/12 = 0.491\overline{6}\%; N = 12 \times 1/2 = 6$ payments

$$\$895.94 = PMT\left[\frac{1-\left[\frac{1}{(1+0.004916\overline{6})^{\frac{12}{12}}}\right]^{6}}{(1+0.004916\overline{6})^{\frac{12}{12}}-1}\right]$$

$$PMT = \frac{\$895.94}{\left[\frac{1-0.971001}{0.004916\overline{6}}\right]} = \frac{\$895.94}{[5.89808]} = \$151.90$$

Steps 2 to 11 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
0–Start				$895.94
1	$151.90	(1) $4.41	(2) $147.49	(3) $748.45
2	$151.90	$3.68	$148.22	$600.22
3	$151.90	$2.95	$148.95	$451.28
4	$151.90	$2.22	$149.68	$301.59
5	$151.90	$1.48	$150.42	$151.18
6	(4) $151.90	$0.74	$151.18	$0.00
Total				

(1) $INT = \$895.94 \times \left((1+0.004916\overline{6})^{\frac{12}{12}}-1\right) = \$4.405038$

(2) $PRN = \$151.90 - \$4.405038 = \$147.494961$

(3) $BAL_{P2} = \$895.94 - \$147.494961 = \$748.445038$

(4) Final Payment $= \$151.177613 + \$0.742386 = \$151.92$

Steps 12 to 13: Adjust for the "missing pennies" (noted in red) and total the interest.

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
0–Start				$895.94
1	$151.90	(1) $4.41	(2) $147.49	(3) $748.45
2	$151.90	$3.67	$148.23	$600.22
3	$151.90	$2.96	$148.94	$451.28
4	$151.90	$2.21	$149.69	$301.59

5	$151.90	$1.49	$150.41	$151.18
6	(4) $151.90	$0.74	$151.18	$0.00
Total	$911.42	$15.48	$895.94	

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
6	5.9	895.94	Answer: -151.903441 Re-keyed as: -151.90	0	12	12

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
1	1	1	748.4450383	147.494961	4.4050383
2	2	2	600.224893	148.22014	3.679854
3	3	3	451.275998	148.948894	2.951105
4	4	4	301.594772	149.681226	2.218773
5	5	5	151.177613	150.417159	1.482840
6	6	6	0.020903	151.156710 *add BAL=151.1777613	0.743289
Total	1	6	0.020903	895.919096 * add BAL=895.94	15.480903

The complete amortization schedule is shown in the table above. The total interest Tamara is to pay on her dishwasher is $15.48.

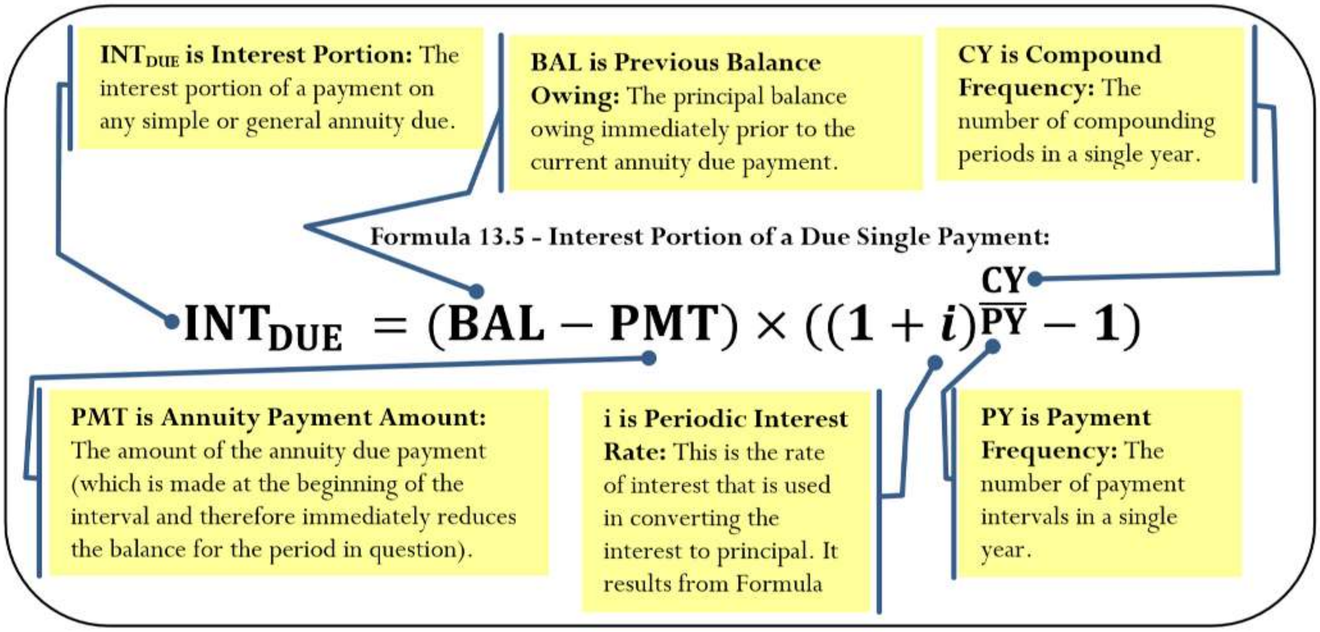
Amortization Schedule Due

In the instance of an annuity due, you require a small modification to the amortization schedule, as illustrated in the next table. Notice that the headers of the second and fifth columns have been modified to clarify the timing of the payment and point in time when the balance is achieved. Each line of the table still represents one payment interval.

Payment Number	Payment Amount at Beginning of Interval ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining at End of Interval($) (PRN)
0–Start	N/A	N/A	N/A	(3)
1	(4)	(5)	(6)	(7)
etc.				
Last payment	(11)	$0.00 (10)	(9)	$0.00
Total	*if needed	(13)	*equals original loan balance	N/A

The Formula

Be sure to determine any calculated amounts such as PMT through the appropriate annuity due formulas and not the ordinary annuity formulas. Recall that in an annuity the annuity payment is made at the beginning of the interval, which immediately reduces the balance eligible for interest. Calculating interest requires Formula 13.1 to be adapted and reintroduced as Formula 13.5.



INT_{DUE} is Interest Portion: The interest portion of a payment on any simple or general annuity due.

BAL is Previous Balance Owing: The principal balance owing immediately prior to the current annuity due payment.

CY is Compound Frequency: The number of compounding periods in a single year.

Formula 13.5 - Interest Portion of a Due Single Payment:

$$\text{INT}_{\text{DUE}} = (\text{BAL} - \text{PMT}) \times \left((1+i)^{\frac{\text{CY}}{\text{PY}}} - 1\right)$$

PMT is Annuity Payment Amount: The amount of the annuity due payment (which is made at the beginning of the interval and therefore immediately reduces the balance for the period in question).

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal. It results from Formula

PY is Payment Frequency: The number of payment intervals in a single year.

Formula 13.5

How It Works

When you work with the amortization of annuities due, you must make the following modifications to the 13-step process:

Step 1: Ensure that you use appropriate annuity due formulas when solving for any unknown variable.

Step 5: Recall that the payment is made at the beginning of the interval, which immediately reduces the balance eligible for interest. Calculating interest requires Formula 13.5, not Formula 13.1.

Step 10: The last annuity due payment reduces the balance to zero. Therefore, no interest is accrued for the last payment interval.

Important Notes

The BAII Plus Calculator

On the calculator, the AMORT function is designed only for ordinary amortization; however, a little adaptation allows you to generate amortization due outputs quickly:

- **P1 and P2**: In an annuity due, since the first payment occurs today (time period 0), the second payment is at time period 1, and so on, the payment number of an annuity due is always one higher than the payment number of an ordinary annuity. To adapt on your calculator, always add 1 to the payment number being calculated. For example, if you are interested in payment seven only, set both P1 and P2 to 8. Or, if you are interested in the payment range 14 through 24, set P1 = 15 and P2 = 25.
- **BAL**: To generate the output, the payment numbers are one too high. This results in the balance being decreased by one extra payment. To adapt the procedure, manually increase the balance by adding one payment (or with BAL on your display, use a shortcut key sequence of − RCL PMT =).

- **INT and PRN**: Both of these numbers are correct except for the final payment. If working with the final payment, you need to adjust the PRN by the BAL remaining.

✓ Example $13.3.2$: An Investment Annuity Due

Maisy just moved to Toronto to attend the University of Toronto. Shortly after she settled into her new apartment, her parents sent her an email on September 1 confirming that they had set up an investment annuity for $25,000 at 4.75% compounded semi-annually to assist her with tuition fees and buying textbooks. The annuity will deposit the funds to her bank account annually starting today for four years. Construct a complete amortization schedule and calculate the total interest earned.

Solution

Construct a complete amortization schedule for the investment fund along with the total interest earned. Note that this situation presents an investment annuity due.

What You Already Know

Step 1:

The timeline for the education fund appears in the timeline.

PV_{DUE} = $25,000, IY = 4.75%, CY = 2, PY = 1, Years = 4, FV = $0

How You Will Get There

Step 1 (continued):

Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.5.

Step 2:

Set up the amortization table for an annuity due.

Steps 3 and 4:

Fill in the original principal and payment column. Put the first payment on the first line and deduct it immediately with no interest from the principal.

Steps 5 to 8:

For each line use Formula 13.5 and Formula 13.2 and the rearranged Formula 13.3.

Steps 9 to 11:

Fill in the principal and balance remaining, set the interest portion to zero, and determine the final payment by adding the interest and principal component together.

Step 12:

Check for the "missing penny."

Step 13:

Total up the interest portion.

Perform

Step 1 (continued):

$$i = 4.75\%/2 = 2.375\%; N = 1 \times 4 = 4 \text{ payments}$$

$$\$25,000 = PMT\left[\frac{1-\left[\frac{1}{(1+0.02375)^{\frac{2}{1}}}\right]^{4}}{(1+0.02375)^{\frac{2}{1}}-1}\right]\times(1+0.02375)^{\frac{2}{1}}$$

$$PMT = \frac{\$25,000}{\left[1-\left[\frac{1}{1.048064}\right]^{4}\right.} = \$6,696.74$$

$$\left[\frac{\left[1.048064\right]}{0.048064}\right] \times 1.048064$$

Steps 2 to 11 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount at Beginning of Interval ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining at End of Interval($) (PRN)
0–Start				$25,000.00
1	$6,696.74	(1) $879.73	(2) $5,817.01	(3) $19,182.99
2	$6,696.74	$600.14	$6,096.60	$13,086.39
3	$6,696.74	$307.11	$6,389.63	$6,696.76
4	$6,696.74	$0.00	$6,696.76	$0.00
Total	$26,786.98	$1,786.98	$25,000.00	

(1) $INT_{DUE} = (\$25,000 - \$6,696.74) \times \left((1+0.02375)^{\frac{2}{1}} - 1\right) = \$879.729032$

(2) $PRN = \$6,696.74 - \$879.729032 = \$5,817.010967$

(3) $BAL_{P2} = \$25,000 - \$5,817.010967 = \$19,182.99$

Step 12:

There are no "missing pennies."

Step 13:

Interest is totaled in the table above.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	4	4.75	-25000	Answer: 6,696.744972 Re-keyed as: 6,696.74	0	1	2

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
1	2	2	-12,486.24903 * add PMT=19,192.99	5,817.010967	879.729032
2	3	3	-6,389.648886 * add PMT=13,086.39	6,096.600146	600.139853
3	4	4	-0.021369 * add PMT=6,696.76	6,389.627517 * add BAL=6,389.648886	307.112483
4	5	5	6,696.717603 * add PMT=-0.021369	6,696.738973 * add BAL=6,696.76	0.001027

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
Total	2	5	6,696.717603 * add PMT=-0.021369	24,999.97863 * add BAL=25,000	1,786.98137

The complete amortization schedule is shown in the table. The total interest earned on the education fund is $1,786.98.

The Partial Amortization Schedule

Sometimes, businesses are interested only in creating partial amortization schedules, which are amortization schedules that show only a specified range of payments and not the entire annuity. This may occur for a variety of reasons. For instance, the complete amortization schedule may be too long (imagine weekly payments on a 25-year loan), or maybe you are solely interested in the principal and interest portions during a specific period of time for accounting and tax purposes.

How It Works

The procedure for partial amortization schedules remains almost the same as a complete schedule with the following notable differences:

Step 2: Set up a partial amortization schedule template as per this next table. The corresponding step numbers are identified in the table.

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
Preceding Payment #	N/A	N/A	N/A	(3)
First Payment Number of Partial Schedule	(4)	(5)	(6)	(7)
etc.				
Last payment Number of Partial Schedule	(11)*	(10)*	(9)*	$0.00
Total	*if needed	(13)	(13)	N/A

*Perform these steps only if the final payment in the annuity is involved in the schedule

Step 3: Calculate the balance remaining in the account immediately before the first payment of the partial schedule. Recall that this requires you to calculate the future value of the principal less the future value of all payments made.

Steps 9 to 11: These steps are required only if the final payment is involved in the partial schedule.

Step 13: Sum the principal portion as well as the interest portion.

✓ Example $13.3.3$: A Partial Loan Amortization Schedule on a Loan

Molson Coors Brewing Company just acquired $1.2 million worth of new brewing equipment for its Canadian operations. The terms of the loan require end-of-quarter payments for eight years at 8.3% compounded quarterly. For accounting purposes, the company is interested in knowing the principal and interest portions of each payment for the fourth year and also wants to know the total interest and principal paid during the year. Construct the partial amortization schedule.

Solution

Construct a partial amortization schedule for the fourth year of the loan along with the total interest and principal paid during the year.

What You Already Know

Step 1:

The timeline for the equipment loan appears in the timeline.

PV_{ORD} = \$1,200,000, IY = 8.3%, CY = 4, PY = 4, Years = 8, FV = \$0

How You Will Get There

Step 1 (continued):

Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.4.

Step 2:

Set up the partial amortization table for the ordinary annuity.

Step 3:

Calculate the future value of loan principal for the end of the third year (the 12th payment) using Formulas 9.2 and 9.3. Then calculate the future value of the first 12 payments using Formulas 11.1 and 11.2. Calculate the principal balance after 12 payments through $BAL = FV - FV_{ORD}$.

Step 4:

Fill in the payment column for the four payments made in the fourth year.

Steps 5 to 8:

For each line of the table, use Formula 13.1 and Formula 13.2 and the rearranged Formula 13.3.

Steps 9 to 11:

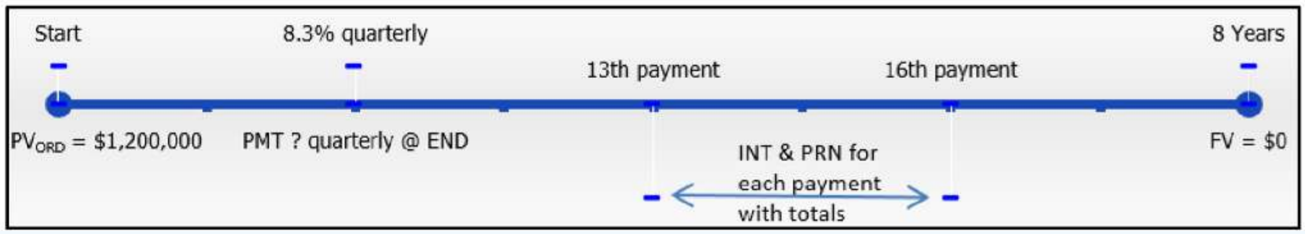
Not needed, since the final payment is not involved.

Step 12:

Check for the "missing penny."

Step 13:

Sum the interest and principal portions.



Perform

Step 1 (continued):

$$i = 8.3\%/4 = 2.075\%; N = 4 \times 8 = 32 \text{ payments}$$

$$\$1,200,000 = PMT\left[\frac{1-\left[\frac{1}{(1+0.02075)^{\frac{4}{4}}}\right]^{32}}{(1+0.02075)^{\frac{4}{4}}-1}\right]$$

$$PMT = \frac{\$1,200,000}{\left[\frac{0.481701}{0.02075}\right]} = \$51,691.71$$

Steps 2 to 11 (with some calculations, including Step 3) are detailed in the table below:

Payment Number	Payment Amount (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining (\$) (PRN)

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
12				(1) $ 839,147.91
13	$51,691.71	(2) $17,412.32	(3) $34,279.39	(4) $804,868.52
14	$51,691.71	$16,701.02	$34,990.69	$769,877.83
15	$51,691.71	$15,974.96	$35,716.75	$734,161.08
16	$51,691.71	$15,233.84	$36,457.87	$697,703.22
Total				

(1) **Step 3**:

Principal: $N = 4 \times 3 = 12$ compounds; $FV = \$1,200,000(1+0.02075)^{12} = \$1,535,373.036$

Payments: $N = 4 \times 3 = 12$ payments;

$$FV_{ORD} = \$51,691.71\left[\frac{\left[(1+0.02075)^{\frac{4}{4}}\right]^{12}-1}{(1+0.02075)^{\frac{4}{4}}-1}\right] = \$51,691.71\left[\frac{0.279477}{0.02075}\right] = \$696,225.1283$$

Balance: $\$1,535,373.036 - \$696,225.1283 = \$839,147.9072$

(2) $INT = \$839,147.9072 \times \left((1+0.02075)^{\frac{4}{4}}-1\right) = \$17,412.31908$

(3) $PRN = \$51,691.71 - \$17,412.31908 = \$34,279.39092$

(4) $BAL_{P2} = \$839,147.9072 - \$34,279.39092 = \$804,868.5163$

Steps 12 to 13:

Adjust for the "missing pennies" (noted in red) and total the interest.

Payment Number	Payment Amount ($) (PMT)	Interest Portion ($) (INT)	Principal Portion ($) (PRN)	Principal Balance Remaining ($) (PRN)
12				(1) $ 839,147.91
13	$51,691.71	(2) $17,412.32	(3) $34,279.39	(4) $804,868.52
14	$51,691.71	$16,701.02	$34,990.69	$769,877.83
15	$51,691.71	$15,974.96	$35,716.75	$734,161.08
16	$51,691.71	$15,233.85	$36,457.86	$697,703.22
Total	$206,766.84	$65,322.15	$141,444.69	

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
32	8.3	1200000	Answer: -51,691.71391 Re-keyed as: -51691.71	0	4	4

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
12	12	12	839,147.9072	N/A	N/A
13	13	13	804,868.5163	34,279.39092	17,412.31908
14	14	14	769,877.828	34,990.68829	16,701.02171

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
15	15	15	734,161.083	35,716.74507	15,974.96493
16	16	16	697,703.2154	36,457.86753	15,233.84247
Total	13	16	697,703.2154	141,444.6918	65,322.14819

The partial amortization schedule for the fourth year is shown in the table above. The total interest paid in the year is \$65,322.15, and the principal portion is \$141,444.69

Contributors and Attributions

- Template:ContribBusiMathOlivier

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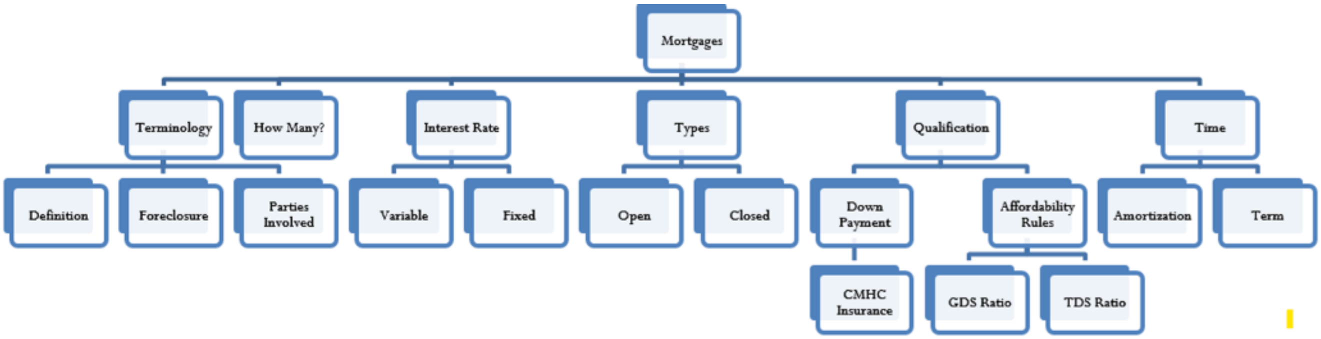
13.4: Special Application - Mortgages

If you buy a home in Canada, you will almost certainly need to take out a mortgage. After all, the expense is huge: on average a house costs $784,567 in Vancouver, $432,576 in Calgary, $261,666 in Winnipeg, $503,094 in Toronto, $272,069 in Halifax, and $318,363 in the Northwest Territories!1 As mentioned earlier in this book, the average Canadian home costs approximately $378,369.2 Compare that with Canadian median family gross income of $69,860,3 and once you consider the income left after all deductions and expenses it becomes obvious that you will have to pay for your home gradually over many years. That means you must make use of a financial institution's money over time, all the while paying for this privilege in the form of interest.

Your home mortgage may well be the biggest personal financial transaction you will ever make, so understanding the factors that determine your mortgage payments can save you a lot of money. This section explains mortgage fundamentals and shows you how to amortize and calculate payments on various mortgages.

Mortgages Explained

This figure illustrates the various concepts related to mortgages. Each of these concepts is discussed below.



Terminology

A **mortgage** is a special type of loan that is collaterally secured by real estate. In essence, the loan has a lien against the property, that is, the right to seize the property for the debt to be satisfied. An individual or business taking out a mortgage is obliged to pay back the amount of the loan with interest based on a predetermined contract. The financial institution, though, has a claim on the real estate property in the event that the mortgage goes into default, meaning that it is not paid as per the agreement. In these instances, financial institutions will pursue **foreclosure** of the property, which allows for the tenants to be evicted and the property to be sold. The proceeds of the sale are then used to pay off the mortgage. A mortgage always involves two parties. The individual or business that borrows the money is referred to as the **mortgagor**, and the financial institution that lends the money is referred to as the **mortgagee**.

How Many Mortgages

A real estate property can have more than one mortgage. The main mortgage based on the amount of money borrowed to purchase the home is called the **first mortgage**. But property owners can choose to have other mortgages as well. For example, most home equity lines of credit (HELOCs) are secured by a **second mortgage** against the property. Sometimes homeowners borrow money to make home improvements or renovations and have these amounts placed as an additional mortgage against the property. Any number of mortgages against a property is possible, though it is uncommon to have more than three. The order of the mortgages is important. If the mortgage goes into default and foreclosure occurs, the mortgagee with the first mortgage gets access to the proceeds first. If any money is left over after the first mortgage is paid off, the mortgagee of the second mortgage gets the remainder. If anything is left, the third mortgagee gets the balance and so on until all proceeds have been expended. Ultimately, any money left over after all mortgages and costs have been paid belongs to the mortgagor.

Interest Rate

In the mortgage contract, the two parties can agree to either a fixed interest rate or a variable interest rate. Either way, the mortgage always forms an ordinary annuity since interest is not payable in advance.

- Under a fixed interest rate, the principal is repaid through a number of equal payments that cover both the interest and principal components of the loan. The interest portion is highest at the beginning and gradually declines over the amortization period of the mortgage. In Canada, fixed interest rates are either annually or semi-annually compounded; the latter is the prevailing choice.
- In a variable interest rate mortgage, the principal is repaid through an agreed-upon number of unequal payments that fluctuate with changes in borrowing rates. The principal and interest portions of the payment vary as interest rates fluctuate, meaning that the interest portion can rise at any point with any increase in rates. When rates change, a common practice in many financial institutions is to change the variable interest rate as of the first day of the next month. If the rate change does not coincide with a mortgage payment date, then the interest portion is calculated in a way similar to the procedure for a demand loan (see Section 8.5), where the exact number of days at the different rates must be determined. In Canada, variable interest rates are usually compounded monthly.

Types of Mortgages

The mortgage agreement can be open or closed. An **open mortgage** has very few rules and it allows the mortgagor to pay off the debt in full or make additional prepayments at any given point in any amount without penalty. A **closed mortgage** has many rules that determine how the mortgage is to be paid. It does not allow the mortgager to pay off the debt in full until the loan matures. As a marketing tool, most closed mortgages have “top-up” options that allow the mortgagor to make additional payments (such as an additional 20% per year) against the mortgage without penalty. Any payments exceeding the maximums or early payment of the mortgage are penalized heavily, with a three-month minimum interest charge that could be increased up to a measure called the interest rate differential, which effectively assesses the bank's loss and charges the mortgagor this full amount.

Qualification

Purchasing a home and obtaining a mortgage requires a down payment. You can get a mortgage with a 25% down payment or more; this is known as a loan-to-value ratio of 75% or less. The loan-to-value ratio divides the principal borrowed by the value of the house. If you do not have enough down payment to meet this criterion, you can make the real estate purchase with as little as 5% down (a 95% loan-to-value ratio); however, your mortgage must then be insured by the Canada Mortgage and Housing Corporation (CMHC). The premium charged for the insurance can range from 0.5% to 3.3% of the principal borrowed, depending on the mortgage particulars.

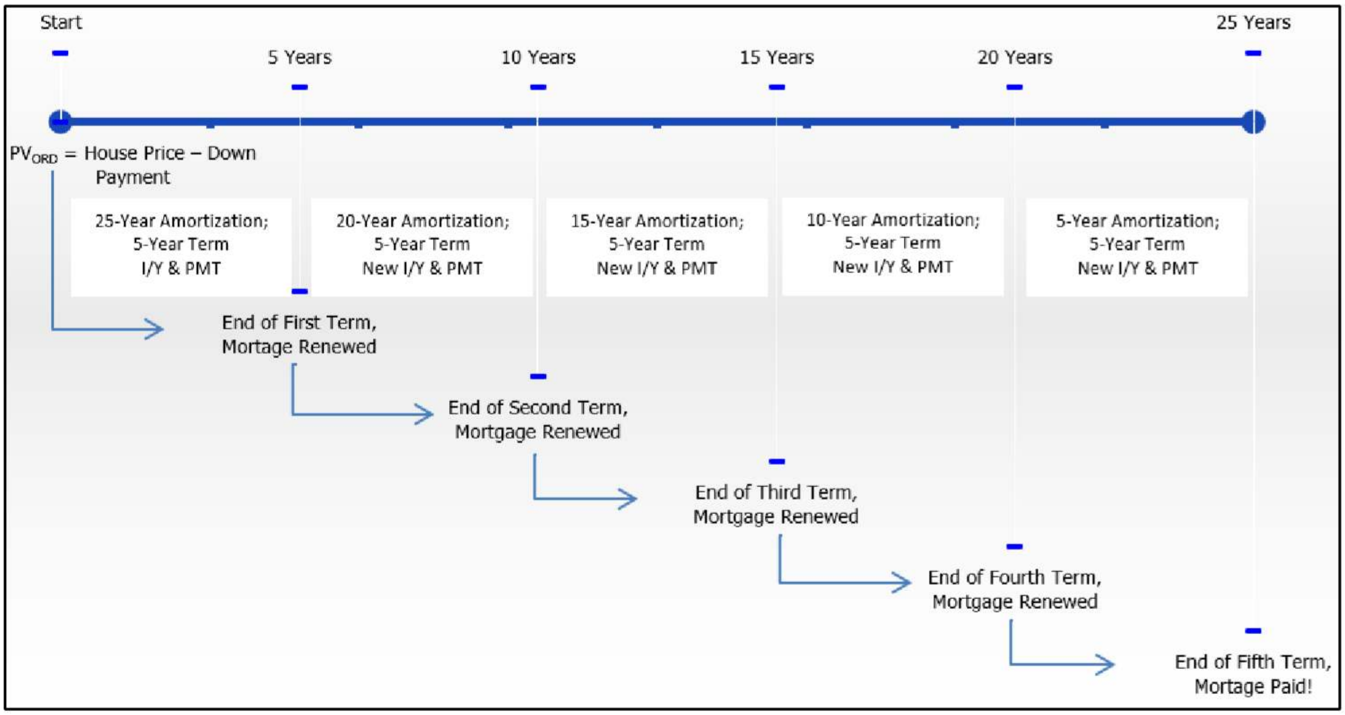
To qualify for a mortgage, you must also meet two affordability rules:

1. The first affordability rule states that your principal, interest, taxes, and heating expenses, or PITH, must not exceed 32% of your gross monthly household income. If the real estate involves a condominium, then 50% of the condo fees are also included in the PITH. This is known as the gross debt service (GDS) ratio.
2. The second affordability rule states that the PITH plus all other debt requirements must not exceed 40% of gross monthly household income. Other debt can include any other debt payments such as car loans, credit cards, or student loans. This is known as the total debt service (TDS) ratio.

It is important to note that these ratios have some degree of flexibility depending on the financial institution issuing the mortgage. Each of the percentages for the ratios commonly varies by up to ±3% depending on the particular institution’s policies.

Time

Finally, due to the large amount of principal borrowed in a real estate purchase, the mortgage may be amortized over an extended time period. The most common amortization is 25 years with a maximum of 30 years. That is a long time frame for any contract. Consequently, mortgages are taken out in shorter terms, usually up to no more than five years, although some institutions do offer seven- or ten-year terms. This allows the financial institution regular opportunities to update the interest rate to reflect current mortgage rates. What this means to the mortgagor is that the mortgage becomes due in full at the end of each term. Most people then renew their mortgage for another term, though the amortization period becomes shorter in accordance with the number of years elapsed since the initial principal was borrowed. The figure below illustrates the typical mortgage amortization process, where the amortization is initially established at 25 years and the mortgagor uses five consecutive five-year terms to pay off the debt.


Start
5 Years
10 Years
15 Years
20 Years
25 Years
PV_ORD = House Price – Down Payment
25-Year Amortization; 5-Year Term I/Y & PMT
20-Year Amortization; 5-Year Term New I/Y & PMT
15-Year Amortization; 5-Year Term New I/Y & PMT
10-Year Amortization; 5-Year Term New I/Y & PMT
5-Year Amortization; 5-Year Term New I/Y & PMT
End of First Term, Mortage Renewed
End of Second Term, Mortgage Renewed
End of Third Term, Mortgage Renewed
End of Fourth Term, Mortgage Renewed
End of Fifth Term, Mortage Paid!

How Mortgage Interest Rates Are Determined

The actual interest rate you can obtain on a mortgage is determined by five factors in combination:

1. **The Bank of Canada Rate**. This rate sets the basis from which all variable rates are determined. The prime rate (the Bank of Canada rate +2%) is usually the lowest obtainable interest rate. Some banks, however, discount this rate slightly by a margin between 0.1% and 0.4%.
2. **Bond Market Rates**. Bond market rates set the basis for which all fixed interest rates are determined. Fluctuations in the prime rate do not necessarily affect bond market rates.
3. **The Mortgage Type**. Open mortgages have substantially higher rates than equivalent closed mortgages because of the uncertainty of when the mortgage will be paid off. Banks protect their interests and minimize the risks of open mortgages through the higher rate.
4. **The Term**. As the length of the term increases, the interest rate increases as well. This is due to the uncertainty of future interest rates. The financial institution must protect itself against rising rates in the future.
5. **The Type of Rate**. Variable rates are lower than fixed rates since the financial institution can adjust the rate at any time to match prevailing conditions. Fixed rates do not share this benefit of adjustability and therefore are higher to protect the financial institution.

The table on the next page illustrates some actual posted semi-annually compounded interest rates at the time of writing.

Term	Closed Mortgage		Open Mortgage	
	Variable Rate	Fixed Rate	Variable Rate	Fixed Rate
1 year	2.10%	3.09%	3.14%	6.30%

Term	Closed Mortgage		Open Mortgage	
5 year	3.00%	5.34%	4.00%	8.25%

Calculating the Mortgage Payment

The following mortgage variables are always known:

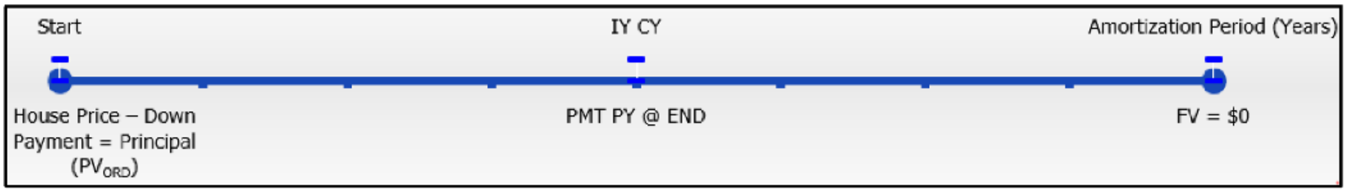
- Mortgage interest rates are posted by every financial institution. The posted rate is almost always negotiable, and a smart mortgagor may be able to negotiate up to a 1% deduction from the posted rate.
- The mortgagor chooses the amortization period, the term, and the payment frequency and also negotiates these variables with the financial institution.
- The principal is determined by the value of the home purchased less any down payment plus any fees or premiums.
- All mortgages are ordinary annuities.

What is left? The unknown variable is the mortgage payment amount that matches the time value of money variables. When calculating this amount, the most important variable is the amortization period, which determines the length of time over which the loan is repaid. It forms the basis for calculating the payment. Note that the term has no effect on the payment calculation. It dictates only the time frame during which the current mortgage arrangement (interest rate, payment frequency, type, and so on) remains in effect.

How It Works

Follow these steps to calculate a mortgage payment:

Step 1: Visualize the mortgage by drawing a timeline as illustrated below. Identify all other time value of money variables, including PV_{ORD}, IY, CY, PY, and Years. The future value, FV, is always zero (the mortgage is repaid).


Start
IY CY
Amortization Period (Years)
House Price – Down Payment = Principal (PV_{ORD})
PMT PY @ END
FV = $0

Step 2: Calculate the periodic interest rate (i) using Formula 9.1.

Step 3: Calculate the number of annuity payments (N) using Formula 11.1. Remember to use the amortization period and not the term for the Years variable in this calculation.

Step 4: Calculate the ordinary annuity payment amount using Formula 11.4 and rearranging for PMT. You must round this calculated amount to two decimals since it represents a physical payment amount.

? Exercise 13.4.1: Give It Some Thought

In each of the following situations determine which mortgage results in a higher mortgage payment. Assume all other time value of money variables remain constant.

a. Open or closed
b. Fixed or variable
c. Two-year fixed rate or five-year fixed rate
d. Ten-year amortization or 20-year amortization

Answer

a. Open (open mortgages always have a higher interest rate than closed mortgages).
b. Fixed rates are always higher to account for future uncertainties.
c. Five-year fixed rate (the longer the term, the higher the rate).
d. Ten-year amortization (there is less time to pay off the debt, meaning higher payments are required).

Example 13.4.1: What Are the Payments?

The Olivers are looking to purchase a new home from Pacesetter Homes in a northeastern Calgary suburb. An Appaloosa 3 model show home can be purchased for \$408,726.15. They are planning on putting \$50,000 as a down payment with a 25 year amortization and weekly payments. If current mortgage rates are fixed at 5.29% compounded semi-annually for a five year closed term, determine the mortgage payment required.

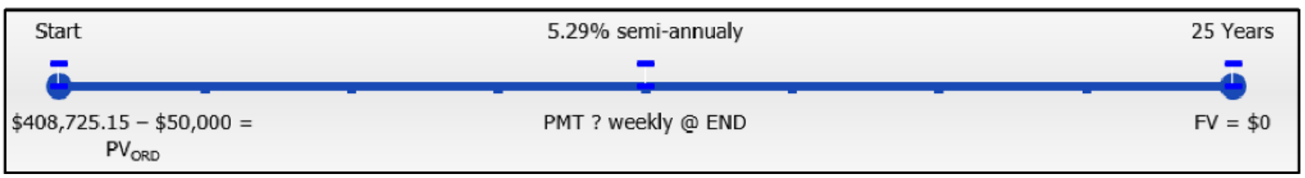
Solution

Calculate the mortgage payment amount required (PMT).

What You Already Know

Step 1:

The mortgage timeline appears below.



PV_{ORD} = \$408,726.15 − \$50,000 = \$358,726.15, IY = 5.29%, CY = 2, PY = 52, Years = 25, FV = \$0

How You Will Get There

Step 2:

Apply Formula 9.1.

Step 3:

Apply Formula 11.1.

Step 4:

Calculate the mortgage payment using Formula 11.4, rearranging for PMT.

Perform

Step 2:

$$i = 5.29\%/2 = 2.645\%$$

Step 3:

$$N = 52 \times 25 = 1,300 \text{ payments}$$

Step 4:

$$\$358,726.15 = PMT\left[\frac{1-\left[\frac{1}{(1+0.02645)^{\frac{2}{52}}}\right]^{1300}}{(1+0.02645)^{\frac{2}{52}}-1}\right]$$

$$PMT = \frac{\$358,726.15}{\left[\frac{1-\left[\frac{1}{1.001004}\right]^{1,300}}{0.001004}\right]} = \$494.40$$

Calculator Instructions

Mode	N	I/Y	PV

Mode	N	I/Y	PV
END	1300	5.29	358726.15

PMT	FV	P/Y	C/Y
Answer: -494.398328	0	52	2

If the Olivers purchase this home as planned, they are mortgaging $358,726.15 for 25 years. During the first five-year term of their mortgage, they make weekly payments of $494.40. After the five years, they must renew their mortgage.

Renewing the Mortgage

When the term of a mortgage expires, the balance remaining becomes due in full. Typically the balance owing is still quite substantial, so the mortgage must be renewed. As discussed earlier, this means that the mortgagor assumes another mortgage, not necessarily with the same financial institution, and the amortization term is typically reduced by the length of the first term. The length of the second term of the mortgage then depends on the choice of the mortgagor. Other variables such as payment frequency and the interest rate may or may not change.

For example, assume a mortgage is initially taken out with a 25-year amortization and a five-year term. After five years, the mortgage becomes due in full. Unable to pay it, the mortgagor renews the mortgage for the remaining 20-year amortization, and also opts for a three-year term in assuming the new mortgage. When those three years are over, the mortgagor renews the mortgage for the remaining 17-year amortization and again makes another term decision. This process repeats until the debt is ultimately paid off.

How It Works

Follow these steps to renew a mortgage:

Steps 1 to 4: The steps for calculating the mortgage payment amount remain unchanged.

Step 5: Determine the balance remaining at the end of the mortgage term. This involves the following:

1. Calculate the future value of the mortgage principal (FV) at the end of the term using Formulas 9.2 (Number of Compounding Periods for Single Payments) and 9.3 (Compound Interest for Single Payments).
2. Calculate the future value of the mortgage payments (FV_{ORD}) made throughout the term using Formulas 11.1 (Number of Annuity Payments) and 11.2 (Future Value of an Ordinary Annuity).
3. Calculate the remaining balance by taking $FV - FV_{ORD} = BAL$.

Step 6: Depending on the information being sought, repeat the above steps as needed for each mortgage renewal using the new amortization remaining, the new interest rate, any changes in payment frequency, and the new term. For example, if you are looking for the mortgage payment in the second term, repeat just steps 2 through 4. If looking for the balance remaining at the end of the second term, repeat step 5 as well.

Important Notes

Most commonly, mortgagors progress along the original path of amortization with each mortgage term renewal. The mortgaging process does not require this, though. With any renewal, the mortgagor may choose either to shorten or to lengthen the amortization period. If shortening the amortization period causes no TDS or GDS ratio concerns, the mortgagor can pay off the mortgage faster. If the mortgagor wishes to lengthen the amortization period, the financial institution may look at the overall time to pay the debt and put an upper cap on how long the amortization period may be increased.

Things To Watch Out For

The most common mistake with mortgages is to confuse the term and the amortization period. Remember:

- When you determine the mortgage payment amount, you use the amortization period to determine the N.
- When you calculate the balance remaining at the end of a mortgage period, you use the term to determine the N.

Paths To Success

When you use your BAII Plus calculator to calculate the remaining balance at the end of the term, you can arrive at this number in one of two ways. Once you have computed the mortgage payment amount and re-entered it into the calculator with only two decimals, you determine the last payment number for the mortgage term and then either

1. Input this value into the N and solve for FV, or
2. Open up the AMORT function and input the last payment number into both P1 and P2. Scroll down to BAL for the solution.

Exercise $13.4.2$: Give It Some Thought

A mortgage was taken out in its first term with a semi-annually compounded interest rate of 6%. If the mortgage remains on the same amortization schedule, what happens to the mortgage payment if the new semi-annually compounded interest rate upon renewal is

a. 6.5%
b. 5.5%
c. 6%

Answer

a. PMT rises, since more interest is charged to the mortgage.
b. PMT falls, since less interest is charged to the mortgage.
c. PMT remains the same, since there is no change in the interest charged to the mortgage.

Example $13.4.2$: A Mortgage Renewal

The Chans purchased their home three years ago for $389,000 less a $38,900 down payment at a fixed semi-annually compounded rate of 4.9% with monthly payments. They amortized the mortgage over 20 years. The Chans will renew the mortgage on the same amortization schedule at a new rate of 5.85% compounded semi-annually. How much will their monthly payments increase in the second term?

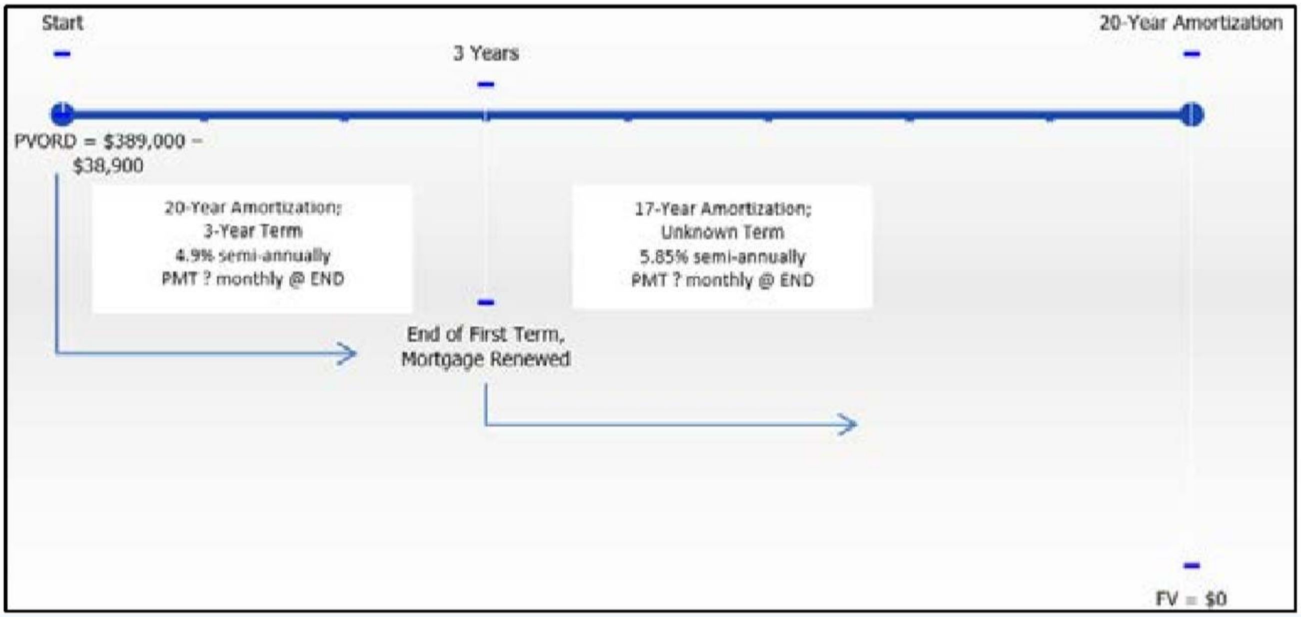
Solution

Calculate the original mortgage payment in the first term, or PMT_1. Then renew the mortgage and recalculate the mortgage payment in the second term, or PMT_2. The amount by which PMT_2 is higher is their monthly payment increase.

What You Already Know

Step 1:

The Chans's mortgage timeline appears below.


Start
20-Year Amortization
3 Years
PVORD = $389,000 − $38,900
20-Year Amortization;
3-Year Term
4.9% semi-annually
PMT ? monthly @ END
End of First Term,
Mortgage Renewed
17-Year Amortization;
Unknown Term
5.85% semi-annually
PMT ? monthly @ END
FV = $0

First Term: PV_{ORD} = $389,000 − $38,900 = $350,100, IY = 4.9%, CY = 2, PY = 12, Years = 20, FV = $0

Second Term: $PV_{ORD} = BAL$ after first term, IY = 5.85%, CY = 2, PY = 12, Years = 17, FV = $0

How You Will Get There

Step 2:

For the first term, apply Formula 9.1.

Step 3:

For the first term, apply Formula 11.1.

Step 4:

For the first term, calculate the mortgage payment using Formula 11.4, rearranging for PMT.

Step 5:

Calculate the future value of the principal at the end of the first term using Formulas 9.2 and 9.3 for the principal and Formulas 11.1 and 11.2 for the mortgage payments. Determine the remaining balance through $BAL = FV - FV_{ORD}$.

Step 6:

Calculate the new mortgage payment through Formulas 9.1, 11.1, and 11.4, rearranging for PMT.

Step 7:

Calculate the increase from the first to the second payment.

Perform

Step 2:

$$i = 4.9\%/2 = 2.45\%$$

Step 3:

$$N = 12 \times 20 = 240 \text{ payments}$$

Step 4:

$$\$350,100 = PMT\left[\frac{1-\left[\frac{1}{(1+0.0245)^{\frac{2}{12}}}\right]^{240}}{(1+0.0245)^{\frac{2}{12}}-1}\right]$$

$$PMT = \frac{\$350,100}{\left[\frac{1-\left[\frac{1}{1.004042}\right]^{240}}{0.004042}\right]} = \$2,281.73$$

Step 5:

Principal:

$$N = 2 \times 3 = 6 \text{ compounds; } FV = \$350,100(1+0.0245)^{6} = \$404,821.7959$$

Payments:

$$N = 12 \times 3 = 36 \text{ payments}$$

$$FV_{ORD} = \$2,281.73\left[\frac{\left[(1+0.0245)^{\frac{2}{12}}\right]^{36}-1}{(1+0.0245)^{\frac{2}{12}}-1}\right] = \$88,228.30609$$

Balance:

$$\$404,821.7959 - \$88,228.30609 = \$316,593.49$$

Step 6:

$$i = 5.85\%/2 = 2.925\%; N = 12 \times 17 = 204 \text{ payments}$$

$$\$316,593.49 = PMT\left[\frac{1-\left[\frac{1}{(1+0,02925)^{\frac{2}{12}}}\right]^{204}}{(1+0.02925)^{\frac{2}{12}}-1}\right]$$

$$PMT = \frac{\$316,593.49}{\left[\frac{1-\left[\frac{1}{(1+0.02925)^{\frac{2}{12}}}\right]^{204}}{(1+0.02925)^{\frac{2}{12}}-1}\right]} = \$2,440.73$$

Step 7:

$$\$2,440.73 - \$2,281.73 = \$159.00$$

Action	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
First Term Payment	END	240	4.9	350100	Answer: -2,281.730755	0	12	2
Balance at end of First Term	✓	36	c	✓	-2,281.73	Answer: -316,593.4898	✓	✓
Second Term Payment	✓	204	5.85	316593.49	Answer: -2,440.733592	0	✓	✓

The initial mortgage payment for the three-year term was $2,281.73. Upon renewal at the higher interest rate, the monthly payment increased by $159.00 to $2,440.73.

References

1 Canadian Real Estate Association (CREA), “Housing Market Stats”, https://www.crea.ca/housing-market-stats/ (accessed July 9, 2011).

2 Living In Canada, “Canadian House Prices,” www.livingin-canada.com/house-prices-canada.html (accessed July 9, 2011).

3 Statistics Canada, “Median Total Income, by Family Type, by Province and Territory, CANSIM table 111-0009, www40.statcan.ca/l01/cst01/famil108a-eng.htm (accessed October 19, 2010).

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13.S: Summary

Key Concepts

13.1: Calculating Interest and Principal Components (Where Did My Money Go?)

- The concept of amortization
- How to calculate interest and principal components for a single payment
- How to calculate interest and principal components for a series of payments

13.2: Calculating the Final Payment (The Bank Wants Every Penny)

- Understanding why the final payment is different than all other payments
- How to calculate the exact amount of the final payment including both interest and principal components
- How to calculate interest and principal components involving the final payment

13.3: Amortization Schedules (How Much Is Mine, How Much Is Theirs?)

- The development of a complete amortization schedule
- The development of a complete amortization schedule due
- The development of a partial amortization schedule

13.4: Special Application: Mortgages (Your Biggest Purchase)

- The language and concepts involved in mortgages
- What determines the mortgage interest rate
- How to calculate the mortgage payment
- The procedure involved in renewing a mortgage

The Language of Business Mathematics

amortization

A process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. amortization period The length of time it will take for the principal of a loan to be reduced to zero.

amortization schedule

A table that shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. amortization term The length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged.

closed mortgage

A type of mortgage that has many strict rules and does not allow the mortgager to pay off the debt in full until the loan matures. Early payment incurs substantial penalties.

first mortgage

A mortgage with first claim to the proceeds of a foreclosure process on a real estate property.

foreclosure

A process that allows a financial institution to evict tenants from a mortgaged property and put the property up for sale. The proceeds of the sale are then used to pay off the mortgage.

gross debt service (GDS) ratio

On a mortgage, your principal, interest, taxes, and heating expenses (PITH), plus 50% of condo fees if applicable, must not exceed 32% of your gross monthly household income.

mortgage

A special type of loan that is collaterally secured by real estate property.

mortgagee

The financial institution that lends the money for a mortgage.

mortgagor

The individual or business that borrows the money for a mortgage.

open mortgage

A type of mortgage that has very few rules and allows the mortgagor to pay off the debt in full or with additional prepayments at any given point without penalty.

partial amortization schedule

Amortization schedules that show only a specified range of payments and not the entire annuity.

second mortgage

A mortgage with second claim to the proceeds of a foreclosure process on a real estate property. Only the remaining funds after the first mortgage has been paid off are available for the second mortgage to be paid off.

Atotal debt service (TDS) ratio

On a mortgage, your principal, interest, taxes, and heating expenses (PITH) plus all other debt requirements must not exceed 40% of your gross monthly household income.

The Formulas You Need to Know

Symbols Used

BAL = principal balance immediately after an annuity payment

BAL_{P1} = principal balance immediately prior to the first payment in a series of annuity payments

BAL_{P2} = principal balance immediately after the last payment in a series of annuity payments

CY = compounding frequency

i = periodic interest rate

INT = interest portion of an ordinary single annuity payment or a series of annuity payments

INT_{DUE} = interest portion of a due single annuity payment

N = number of annuity payments (for annuities) or number of compounding periods (for lump sums)

PMT = annuity payment amount

PRN = principal portion of a single annuity payment or a series of annuity payments

PY = payment frequency

Formulas Introduced

Formula 13.1 Interest Portion of an Ordinary Single Payment: $INT = BAL \times \left((1+i)^{\frac{CY}{PY}} - 1\right)$

Formula 13.2 Principal Portion of a Single Payment: $PRN = PMT - INT$

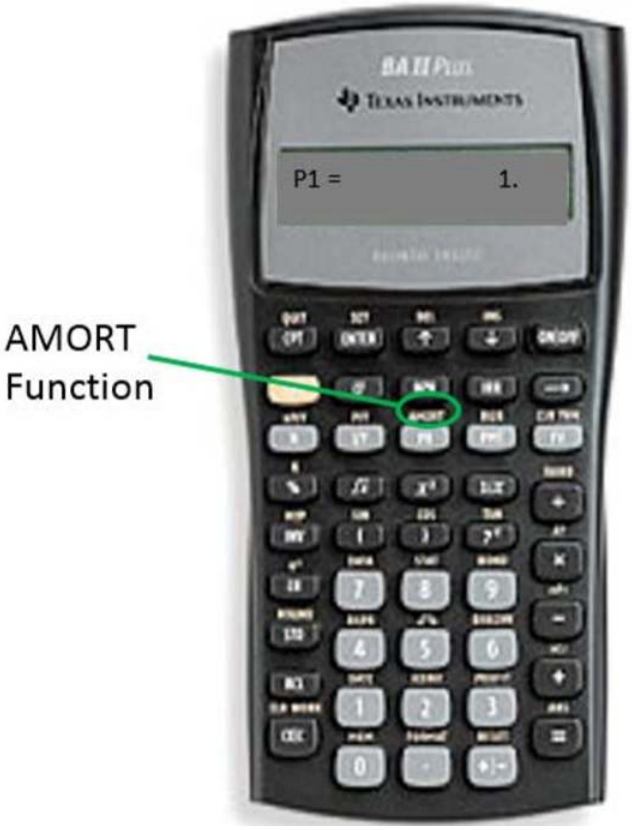
Formula 13.3 Principal Portion for a Series of Payments: $PRN = BAL_{P1} - BAL_{P2}$

Formula 13.4 Interest Portion for a Series of Payments: $INT = N \times PMT - PRN$

Formula 13.5: Interest Portion of a Due Single Payment: $INT_{DUE} = (BAL - PMT) \times \left((1+i)^{\frac{CY}{PY}} - 1\right)$

Technology

Calculator



Amortization (AMORT) Function

1. AMORT is located on the 2nd shelf above the PV button, as illustrated in the photo.
2. There are five variables (use ↓ or ↑ to scroll).
   - P1 is the starting payment number. The calculator can work with a single payment or a series of payments.
   - P2 is the ending payment number. This number is the same as P1 when you are concerned with just a single payment. When working with a series of payments, you can set it to a higher number.
   - BAL is the principal balance remaining after the payment number entered into the P2 variable. The cash flow sign is correct as indicated on the calculator display.
   - PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
   - INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
3. To use the Amortization function, the commands are as follows:
   a. Enter all seven of the time value of money variables accurately ($N, I/Y, PV, PMT, FV, P/Y$, and C/Y). If PMT was computed, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
   b. Press 2nd AMORT.
   c. Enter a value for P1 and press Enter followed by ↓.
   d. Enter a value for P2 and press Enter followed by ↓. Note that the higher the numbers entered in P1 or P2, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.
   e. Using ↓ and ↑, scroll through BAL, PRN, and INT to read the output.

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13.CS: Case Study - Cash Or Credit?

The Situation

Pay cash or take out a loan? That is the question the purchasing manager for Lightning Wholesale faces today. Lightning Wholesale needs to replace its fleet of 10 forklifts for its warehouse operations. After shopping around and putting out a request for quotes, it has chosen Combilift as its supplier. The purchasing manager for Lightning Wholesale has been informed that the company does have the cash available to pay off the debt immediately. Alternatively, Lightning Wholesale could take out a loan and finance the purchase. The purchasing manager wonders what recommendation she should make— pay cash or take the loan? She knows she will need to support this decision with facts to her superiors.

The Data

- The quote from Combilift for the fleet of forklifts was for $225,000 inclusive of all taxes, delivery, and other costs.
- The purchase could be alternatively financed by taking out a loan with month-end payments over three years at 4.79% compounded monthly.
- Prevailing interest rates on three-year investments are sitting at 5.95% compounded annually.

Important Information

- Businesses have other considerations in making these decisions, such as interest expense deductions on loans or interest taxes payable on investments. For purposes of this analysis, though, focus solely on the financial decision being made, and do not factor these other components into the decision.

Your Tasks

1. Many people would not even consider taking the loan since they do not want to be paying interest on purchases. From a financial perspective, explain why this decision is not as simple as choosing not to pay interest. (Hint: Think about the interest rate.)
2. Calculate the loan payment amount if Lightning Wholesale pursued the loan alternative. What would be the amount of the final loan payment?
3. If Lightning Wholesale wanted to invest a single amount of money today to make the payments on the loan, what amount today must be invested? (Hint: Calculate the present value of the loan payments using the investment rate of return.)
4. Based on your calculations, what should the purchasing manager recommend? How much better is the chosen alternative in today's dollars?
5. Can you develop a general rule (exclusive of other considerations) to help decide more easily between paying cash or financing? (Hint: Try the above calculations using a loan rate of 6.79% instead of 4.79% and see what decision is made.)

Contributors and Attributions

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13.E: Understanding Amortization and its Applications (Exercises)

13.1: Calculating Interest and Principal Components

Mechanics

For each of the following ordinary annuities, calculate the interest and principal portion of the payment indicated.

Principal	Interest	Payment Frequency	Loan Term	Payment Number to Find
1. $5,000	8% compounded quarterly	Monthly	3 years	16
2. $45,000	7.65% compounded monthly	Monthly	5 years	23
3. $68,000	6.5% compounded semi-annually	Quarterly	7 years	19
4. $250,000	4.9% compounded annually	Semi-annually	10 years	17

For each of the following ordinary annuities, calculate the total interest and principal portions for the series of payments indicated.

Principal	Interest	Payment Frequency	Loan Term	Payment Series to Find (inclusive)
5. $20,000	5% compounded quarterly	Quarterly	4 years	Year 1
6. $15,000	9% compounded monthly	Monthly	4 years	Year 2
7. $39,000	6% compounded semi-annually	Quarterly	5 years	Year 3
8. $50,000	7.5% compounded annually	Biweekly	6 years	Year 4
9. $750,000	4% compounded monthly	Monthly	10 years	67 to 78
10. $500,000	6.25% compounded annually	Semi-annually	8 years	7 to 11

Applications

11. A $14,000 loan at 6% compounded monthly is repaid by monthly payments over four years.
    a. What is the size of the monthly payment?
    b. Calculate the principal portion of the 25th payment.
    c. Calculate the interest portion of the 33rd payment.
    d. Calculate the total interest paid in the second year.
    e. Calculate the principal portion of the payments in the third year.
12. Quarterly payments are to be made against a $47,500 loan at 5.95% compounded annually with a six-year amortization.
    a. What is the size of the quarterly payment?

   b. Calculate the principal portion of the sixth payment.
   c. Calculate the interest portion of the 17th payment.
   d. Calculate how much the principal will be reduced in the fourth year.
   e. Calculate the total interest paid in the first year.
13. A lump sum of $100,000 is placed into an investment annuity to make end-of-month payments for 20 years at 4% compounded semi-annually.
   a. What is the size of the monthly payment?
   b. Calculate the principal portion of the 203rd payment.
   c. Calculate the interest portion of the 76th payment.
   d. Calculate the total interest received in the fifth year.
   e. Calculate the principal portion of the payments made in the seventh year.
14. For his son's college education, Pat deposited $25,000 into an annuity earning 4.2% compounded quarterly. His son is to receive payments at the end of every quarter for five years.
   a. How much will his son receive each quarter?
   b. How much of the third payment is interest?
   c. How much of the payments made in the third year will come from the account's principal?
   d. If his son finishes his education in four years instead of five and closes the account upon graduation, what total interest will he have received?
15. Cathy and Bill just acquired a new Honda Odyssey Touring Edition minivan for $60,531.56 under the dealership's purchase financing of 5.65% compounded annually for eight years.
   a. What are their monthly car payments?
   b. In the first year, what total amount of interest will they pay?
   c. In the fourth year, by how much will the principal be reduced?

Challenge, Critical Thinking, & Other Applications

16. Yangjing deposits $30,000 into an investment annuity for her daughter, who is currently living far away. The annuity is to earn 6.3% compounded semi-annually and make monthly payments starting today for the next five years. Calculate the interest portion of the payments made in the second year.
17. At the age of 54, Hillary just finished all the arrangements on her parents' estate. She is going to invest her $75,000 inheritance at 6.25% compounded annually until she retires at age 65, and then she wants to receive month-end payments for the following 20 years. The income annuity is expected to earn 3.85% compounded annually.
   a. What are the principal and interest portions for the first payment of the income annuity?
   b. What is the portion of interest earned on the payments made in the second year of the income annuity?
   c. By what amount is the principal of the income annuity reduced in the fifth year?
18. Art Industries just financed a $10,000 purchase at 5.9% compounded annually. It fixes the loan payment at $300 per month.
   a. How long will it take to pay the loan off?
   b. What are the interest and principal components of the 16th payment?
   c. For tax purposes, Art Industries needs to know the total interest paid for payments 7 through 18. Calculate the amount.
19. Explore the impact of the term on the interest component of a loan. For a $200,000 loan at 5% compounded semiannually with monthly payments, calculate the following:
   a. Interest component for the entire loan for each term of 10, 15, 20, and 25 years.
   b. Between each increment of term in part (a), by what amount and what percentage did the annuity payment decrease?
   c. Between each increment of term in part (a), by what amount and percentage did the interest portion increase?
   d. Comment on your findings.

20. Explore the impact of the interest rate on the interest component of a loan. For a $200,000 loan for 25 years with monthly payments, calculate the following:
    a. Interest component for the entire loan for each semi-annually compounded interest rate of 4%, 5%, 6%, 7%, and 8%.
    b. Between each increment of rate in part (a), by what amount and what percentage did the annuity payment decrease?
    c. Between each increment of rate in part (a), by what amount and what percentage did the interest portion increase?
    d. Comment on your findings.

13.2: Calculating the Final Payment

Mechanics

For each of the following ordinary annuities, calculate the final payment amount.

Principal	Interest	Payment Frequency	Loan Term
1. $15,000	10% compounded quarterly	Quarterly	3 years
2. $85,000	6.75% compounded monthly	Monthly	7 years
3. $32,000	8.25% compounded annually	Annually	10 years
4. $250,000	5.9% compounded semi-annually	Monthly	20 years

For each of the following ordinary annuities, calculate the final payment amount along with the total interest and principal portions for the series of payments indicated.

Principal	Interest	Payment Frequency	Loan Term	Payment Series to Find (inclusive)
5. $21,000	8% compounded quarterly	Quarterly	4 years	Year 4
6. $115,000	7% compounded semi-annually	Monthly	10 years	Years 9 and 10
7. $13,750	9.5% compounded monthly	Semi-annually	6 years	Years 5 and 6
8. $500,000	7.25% compounded semi-annually	Biweekly	25 years	Year 25
9. $71,000	3.85% compounded quarterly	Annually	15 years	Years 13 to 15
10. $47,500	10.25% compounded annually	Quarterly	9 years	Year 9

Applications

11. A $28,250 loan at 9% compounded quarterly is repaid by monthly payments over five years. a. What is the amount of the final payment? b. Calculate the principal and interest portions of the payments in the final year.
12. Semi-annual payments are to be made against a $97,500 loan at 7.5% compounded semi-annually with a 10-year amortization.
    a. What is the amount of the final payment?
    b. Calculate the principal and interest portions of the payments in the final two years.

13. A $250,000 lump sum placed into an investment annuity is to make end-of-month payments for 17 years at 5% compounded annually.
    a. Calculate the principal and interest portions of the payments in the first five years.
    b. What is the amount of the final payment?
    c. Calculate the principal and interest portions of the payments in the last five years.
14. A $65,000 trust fund is set up to make end-of-year payments for 15 years while earning 3.5% compounded quarterly.
    a. What is the amount of the final payment?
    b. Calculate the principal and interest portion of the payments in the final three years.
15. Stuart and Shelley just purchased a new $65,871.88 Nissan Titan Crew Cab SL at 8.99% compounded monthly for a seven-year term.
    a. Calculate the principal and interest portions of the monthly payments in the first two years.
    b. What is the amount of the final monthly payment?
    c. Calculate the principal and interest portions of the payments in the last two years.

Challenge, Critical Thinking, & Other Applications

16. Mirabel Wholesale has a retail client that is struggling and wants to make instalments against its most recent invoice for $133,465.32. Mirabel works out a plan at 12.5% compounded monthly with beginning-of-month payments for two years.
    a. What will be the amount of the final payment?
    b. Calculate the principal and interest portions of the payments for the entire agreement.
17. A new hotel built in Banff cost $36 million to build. The owner's financing arrangements allow for quarterly payments at 6% compounded semi-annually over the next 30 years. The first payment is to be made today.
    a. What is the amount of the final payment?
    b. Calculate the principal and interest portions of the payments in the final five years.
18. Through a government arrangement, a new $110 million state-of-the-art baseball stadium will be constructed. Under terms of this arrangement, the owner of the baseball team will be charged 8.8% compounded annually and will be allowed to defer the payments for five years before making beginning-of-year payments for 40 years.
    a. What is the amount of the final payment?
    b. Calculate the principal and interest portion of the payments in the final five years.
19. Wile E. Coyote owes the ACME Corporation $75,000 for various purchased goods. Wile agrees to make $1,000 payments at the end of every month at 10% compounded quarterly until the debt is repaid in full.
    a. What is the amount of the final payment?
    b. Calculate the principal and interest portion of the final six payments.
20. Explore how the term affects the adjustment that needs to be made to a final payment. Consider a $450,000 loan at 7.5% compounded semi-annually with month-end payments.
    a. Calculate the final payment for each term of 10, 15, 20, 25, 30, and 35 years.
    b. Comment on the adjustments you made to the final payment based on your results.

13.3: Amortization Schedules

Mechanics

For each of the following ordinary annuities, create the complete amortization table and calculate the total interest.

Principal	Interest	Payment Frequency	Loan Term
1. $27,500	6.8% compounded quarterly	Quarterly	2 years

Principal	Interest	Payment Frequency	Loan Term
1. $27,500	6.8% compounded quarterly	Quarterly	2 years
2. $192,000	4.75% compounded semi-annually	Semi-annually	4 years
3. $1,854.25	8.99% compounded annually	Monthly	6 years
4. $425,000	5.9% compounded monthly	Annually	7 years

For each of the following ordinary annuities, calculate the partial amortization schedule for the payment series indicated along with the total interest and principal portions.

Principal	Interest	Payment Frequency	Loan Term	Partial Schedule Payments (inclusive)
5. $221,000	7.14% compounded quarterly	Quarterly	15 years	Year 11
6. $109,900	3.8% compounded monthly	Monthly	5 years	Payments 38 to 43
7. $450,000	7.5% compounded semi-annually	Monthly	25 years	Payments 26 to 29
8. $500,000	9.5% compounded annually	Semi-annually	12 years	Years 11 and 12

Applications

For questions 9 through 11, construct a complete amortization schedule and calculate the total interest.

9. A farmer purchased a John Deere combine for $369,930. The equipment dealership sets up a financing plan to allow for end-of-quarter payments for the next two years at 7.8% compounded monthly.
10. Jennifer purchased an Aqua Shield Sunroom for $19,097. Terms of purchase require her to put $2,000 down, with the balance financed through six equal end-of-month payments at 11.2% compounded semi-annually.
11. Jerry's Concrete installed a $13,544 concrete driveway for a client with no money down and four equal end-of-month installments at 4.9% compounded monthly.

For questions 12 through 14, construct the partial amortization schedule indicated and calculate the total principal and interest portions represented by the partial schedule.

12. Some real estate is purchased for $850,000 with a 30-year amortization at 6.8% compounded semi-annually. Create a schedule for the end-of-month payments in the first half of the 14th year.
13. Marcel had a new cedar fence installed around his oversized house lot for $22,900. The fence company allows him to finance the purchase with end-of-month instalments for one year at 7.39% compounded annually. Construct a schedule for the first two payments and the last four payments.
14. Ron and Natasha had Oasis Leisure and Spa install an in-ground swimming pool for $51,000. The financing plan through the company allows for end-of-month payments for two years at 6.9% compounded quarterly. Ron and Natasha instruct Oasis to round their monthly payment upward to the next dollar amount evenly divisible by $500. Create a schedule for the first three payments, payments seven through nine, and the last three payments.

Challenge, Critical Thinking, & Other Applications

15. David leased an Acura CSX for $30,185 with no down payment on a 48-month lease at 2.9% compounded annually. The residual value is $11,516. Set up a partial amortization schedule for monthly payments 7 through 13. What is the total interest paid?
16. Merryweather's $40,000 trust fund is set to mature and will make its first semi-annual payment to her today. The fund can earn 4.4% compounded annually during its five-year term. Construct a complete amortization schedule including the total interest earned.

17. Construct a new complete amortization schedule for the farmer in question 9 if he has a prosperous third quarter and is able to put $35,000 toward his debt in addition to his regular third-quarter payment. How much interest in total would he save?
18. Hillary acquired an antique bedroom set recovered from a European castle for $118,000. She will finance the purchase at 7.95% compounded annually through a plan allowing for payments of $18,000 at the end of every quarter.
    a. Create a complete amortization schedule and indicate her total interest paid.
    b. Recreate the complete amortization schedule if Hillary pays two additional top-up payments consisting of 10% of the principal remaining after her third payment as well as her fifth payment. What amount of interest does she save?
19. Sinbad is an agent for RE/MAX. He purchased a Cadillac STS for his realty business so that he could drive clients to various homes. Through the bank, he financed the $85,595 vehicle at 5.95% compounded annually on a 24-month term. To file his taxes, he needs a complete amortization schedule with the total interest paid for the first 12 months and then for the next 12 months.
20. Astrid just had a $450,000 custom home built by Hallmark Homes. She took out a 25-year amortization on her mortgage at 6.4% compounded semi-annually and locked the rate in for the first three years. Construct a partial amortization schedule for the first three years and calculate the total interest and principal portions.

13.4: Special Application - Mortgages

Mechanics

Determine the mortgage payment amount for each of the following mortgages. Assume all interest rates are fixed and compounded semi-annually.

Principal	Interest Rate	Payment Frequency	Amortization Period	Mortgage Term (Years)
1. $693,482	5.49%	Monthly	30	5
2. $405,551	6.5%	Bi-weekly	25	3
3. $227,167	2.85 %	Weekly	20	4

Determine the mortgage payment amount upon renewal in the second term for each of the following mortgages. In all cases, assume the amortization period is reduced appropriately upon renewal and that all interest rates are fixed and compounded semi-annually.

Original Principal	Amortization Period (Years)	First-Term Information	Second-Term Information
4. $434,693	30	4.5%; Monthly payments; 3-year term	5.25%; Monthly payments; 2-year term
5. $287,420	15	3.29%; Weekly payments; 1-year term	6.15%; Monthly payments; 5-year term
6. $318,222	25	3%; Biweekly payments 2-year term	6.8%; Biweekly payments; 4-year term

Applications

7. Jean-Claude wants to take out a mortgage in Montreal for $287,420 with a 20-year amortization. Prevailing interest rates are 6.2% compounded semi-annually. Calculate his monthly payment amount.
8. Luc has noticed that mortgage interest rates are low and that he can take out a mortgage for 3.7% compounded semiannually. If he mortgages a home in Halifax for $255,818 amortized over 25 years, determine his biweekly payment amount.

9. Five years ago, Asia purchased her $322,000 home in Edmonton with a 25-year amortization. In her first five-year term, she made monthly payments and was charged 4.89% compounded semi-annually. She will renew the mortgage on the same amortization timeline for another five-year term at 5.49% compounded semi-annually with monthly payments. Calculate the balance remaining after the first term and the new mortgage payment amount for the second term.

For exercises 10 to 12, calculate the following:

a. **Balance remaining after the first term**
b. **Interest and principal portions in the first term**
c. **New mortgage payment amount in the second term**
d. **Balance remaining after the second term**

10. Three years ago, Phalatda took out a mortgage on her new home in Kelowna for $628,200 less a $100,000 down payment at 6.49% compounded semi-annually. She is making monthly payments over her three-year term based on a 30-year amortization. At renewal, she is able to obtain a new mortgage on a four-year term at 6.19% compounded semi-annually while continuing with monthly payments and the original amortization timeline.
11. Dagny signed a 30-year amortization mortgage seven years ago on her home, which she purchased for $984,000 less a $150,000 down payment at 6.9% compounded monthly with monthly payments. She renews her mortgage on the same amortization schedule for another seven-year term at 7.15% compounded semi-annually with weekly payments.
12. Two years ago, Eleonora started a $380,000 mortgage with a 25-year amortization at 4.15% compounded semi-annually with bimonthly payments. Upon renewal, she has decided to shorten the amortization period by five years and takes out a five-year term at 4.98% compounded annually with weekly payments.

For exercises 13 and 14, calculate the following:

a. **Balance remaining after all three terms**
b. **Total interest and total principal portion across all three terms**

13. Ten years ago, Travis purchased his starter home with a mortgage principal of $180,700. He amortized the mortgage over 25 years and through his first five-year term he made monthly payments at 3.85% compounded semi-annually. Upon renewal, he took out another five-year term at 4.65% compounded semi-annually with monthly payments. Today, he renews his mortgage for another five-year term at 5.39% compounded semi-annually with monthly payments.
14. Seven years ago, Aisha's initial principal on her mortgage was $295,900. She set up a 20-year amortization and in her first four-year term of monthly payments her mortgage rate was 5.7% compounded semi-annually. Upon renewal, she took a further three-year term with monthly payments at a mortgage rate of 4.35% compounded semi-annually. Today, she renews the mortgage but decreases the amortization period by three years. She takes a five-year closed fixed rate mortgage of 6.39% compounded semi-annually with bimonthly payments.

Challenge, Critical Thinking, & Other Applications

15. Consider the following information about Jim and Carol, who are considering the purchase of a new home: Gross monthly total income $7,000 Estimated property taxes per month $400.00 Home heating costs per month $100.00 Monthly car payment $735.50 Monthly credit card debt payments $250.00 Monthly student debt payments $175.00 Planned Down Payment $15,000.00 Based on the affordability rules and a 25-year amortization, what is the highest price they could pay for their new home if interest rates are 5.19% compounded semi-annually?
16. The Muswagons have signed a five-year closed variable rate $265,000 mortgage with a 25-year amortization and monthly payments. The initial interest rate was set at 4.5% compounded monthly. It increased by 0.75% after 14 months. Five months before the term expired, it then decreased by 0.25%. Calculate the balance remaining after the five-year term along with the total interest and principal portions paid.
17. The Verhaeghes have signed a three-year closed fixed rate mortgage with a 20-year amortization and monthly payments. They negotiated an interest rate of 4.84% compounded semi-annually. The terms of the mortgage allow

for the Verhaeghes to make a single top-up payment at any one point throughout the term. The mortgage principal was $323,000 and 18 months into the term they made one top-up payment of $20,000.

  a. What is the balance remaining at the end of the term?
  b. By what amount was the interest portion reduced by making the top-up payment?

18. Assume a 15-year amortization on a $200,000 mortgage paid off through three consecutive five-year terms with monthly payments. Also assume that the same interest rate of 5.5% compounded semi-annually is maintained throughout the entire amortization. Construct a table showing the remaining balance, the principal portion, and the interest portion at the end of every term. (Hint: Don't forget in the final term to make the appropriate final payment adjustment.)
19. Fifteen years ago Clarissa's initial principal on her mortgage was $408,650. She set up a 30-year amortization, and in her first 10-year term of monthly payments her mortgage rate was 7.7% compounded semi-annually. Upon renewal, she took a further five-year term with monthly payments at a mortgage rate of 5.69% compounded semi-annually. Today, she renews the mortgage but shortens the amortization period by five years when she sets up a three-year closed fixed rate mortgage of 3.45% compounded semi-annually with monthly payments. What principal will she borrow in her third term and what is the remaining balance at the end of the term? What total interest portion and principal portion will she have paid across all 18 years?
20. Many people fail to understand how a mortgage rate increase would affect their mortgage renewal. The first decade of the new millennium has seen unprecedented low mortgage rates. As a result, many people purchased homes at the maximum mortgage that they could afford. At some point rates are going to rise again. When they do, many people may find themselves no longer able to afford their current homes since they were already maximized at the lower rate.
    a. Take a $350,000 mortgage with a five-year term and 25-year amortization using monthly payments. In the initial term, a rate of 3% compounded semi-annually was obtained. Upon renewal five years later on the same amortization schedule, explore the percent change in the monthly payment required if the semi-annually compounded interest rates had risen to each of 4%, 5%, and 6%. Comment on your findings.
    b. Assume that when the mortgage was started the homeowners had monthly taxes of $400 and home heating costs of $100 per month. If the mortgage was the maximum they could afford, according to the GDS ratio what was their annual gross income? Now assume that upon renewal the taxes and heating had increased to $450 and $120, respectively. At each of the new interest rates from part (a), by what percentage would the homeowners' gross income need to rise over the same time frame for them to be able to continue to afford the mortgage? Comment on your findings.

Review Exercises

Mechanics

1. Quinn placed $33,000 into a five-year ordinary investment annuity earning 7.75% compounded quarterly. He will be receiving quarterly payments. Calculate the principal and interest components of the sixth payment.
2. Annanya took out a $42,500 ordinary loan at 6.6% compounded monthly with monthly payments over the six-year amortization period. Calculate the total principal and interest portions for the third year.
3. Two years ago, Sumandeep invested $20,000 at 9.45% compounded monthly. She has been receiving end-of-month payments since, and the last payment will be today. Calculate the amount of the final payment.
4. Hogwild Industries borrowed $75,000 to purchase some new equipment. The terms of the ordinary loan require quarterly payments for three years with an interest rate of 7.1% compounded semi-annually. Calculate the total interest and principal portions for the third year.
5. Dr. Strong of Island Lakes Dental Centre acquired a new Panoramic X-ray machine for his practice. The $7,400 for the machine, borrowed at 8.8% compounded annually, is to be repaid in four end-of-quarter instalments. Develop a complete amortization schedule and total the interest paid.
6. Dr. Miller acquired a new centrifuge machine from Liaoyang Longda Pharmaceutical Machinery Company (LLPMC) for his medical practice. He is to pay off the $60,341 through 20 month-end payments. LLPMC has set the interest

rate on the loan at 9.5% compounded quarterly. Develop a partial amortization schedule for the third to sixth payments.

7. Kerry, who is a pharmacist, just became a new franchisee for Shoppers Drug Mart. As part of her franchising agreement, her operation is to assume a $1.2 million mortgage to be financed over the next 15 years. She is to make payments after every six months. Head office will charge her a rate of 14.25% compounded annually. Determine the amount of her mortgage payment.
8. Alibaba took out a 25-year amortization $273,875 mortgage five years ago at 4.85% compounded semi-annually and has been making monthly payments. He will renew the mortgage for a three-year term today at an interest rate of 6.1% compounded semi-annually on the same amortization schedule. What are his new monthly mortgage payments?

Applications

9. Monthly payments are to be made against an $850,000 loan at 7.15% compounded annually with a 15-year amortization.
   a. What is the size of the monthly payment?
   b. Calculate the principal portion of the 100th payment.
   c. Calculate the interest portion of the 50th payment.
   d. Calculate how much the principal will be reduced in the second year.
   e. Calculate the total interest paid in the fifth year.
10. An investment annuity of $100,000 earning 4.5% compounded quarterly is to make payments at the end of every three months with a 10-year amortization.
    a. What is the size of the quarterly payment?
    b. Calculate the principal portion of the 20th payment.
    c. Calculate the interest portion of the 33rd payment.
    d. Calculate how much the principal will be reduced in the second year.
    e. Calculate the total interest paid in the seventh year.
11. Presto Pizza just purchased a new $18,810 Chevrolet Aveo (including all taxes) as a delivery vehicle. The loan rate is 5.4% compounded annually for a seven-year amortization.
    a. What is the amount of the final monthly payment?
    b. Calculate the principal and interest portions of the payments in the first two years.
    c. Calculate the principal and interest portions of the payments in the last two years.
12. A retail credit card allows its users to make purchases and pay off the debts through month-end payments equally spread over four months. The interest rate is 28.8% compounded annually on such purchases. A customer just placed $1,000 on the card and intends to use this plan.
    a. What is the amount of the final payment?
    b. Calculate the total interest incurred by using the payment plan.
13. The marketing department at Allsports Industries has been assigned a lump sum of $5 million to use toward various research projects. The marketing research manager wants to invest the money at 6.4% compounded semi-annually and will withdraw $1 million at the end of every six months to fund ongoing research. Construct a complete amortization schedule and determine the total interest received.
14. Horizon Insurance just leased $21,200 worth of new computer equipment for the next four years with no residual value expected. The lease rate is 7.9% compounded quarterly, and the company will make payments quarterly. Construct a partial amortization schedule and total the interest for the second year.
15. Shelley Shearer Dance School took out a mortgage in Winnipeg for $500,000 on a five-year term with a 20-year amortization. The mortgage rate is 4.89% compounded semi-annually. Calculate the weekly mortgage payment amount.
16. Four years ago, Katrina became a landlord and opened her new four-unit apartment housing unit with an initial mortgage at 6.83% compounded semi-annually in the amount of $971,000 less a $100,000 down payment. She

amortized over 30 years and opted for monthly payments. Upon renewing her mortgage today, she is taking a two-year term at 5.1% compounded semi-annually while continuing with monthly payments and the original amortization timeline.

a. Calculate the interest and principal portions in her first term.
b. What is the balance remaining after the first term?
c. What is the new mortgage payment amount in the second term?
d. What is the balance remaining after the second term?

Challenge, Critical Thinking, & Other Applications

17. Five years ago, the Staples signed a closed fixed rate mortgage with a 25-year amortization and monthly payments. They negotiated an interest rate of 4.49% compounded semi-annually. The terms of the mortgage allow for the Staples to make a single top-up payment at any one point throughout the term. The mortgage principal was $179,000 and they made one top-up payment of $10,000 three years into the term. They are renewing the mortgage today for another five-year term but have reduced the amortization period by five years.
    1. What is the balance remaining at the end of the first term?
    2. By what amount was the interest portion of the first term reduced by making the top-up payment?
    3. Calculate the mortgage payment amount for the second term if the interest rate remains unchanged.
18. The human resources department just received a $1 million gift from a former employee. The employee's instructions are that the funds are to be invested with an amortization of 10 years and the employees of the company are to receive annual end-of-year Christmas bonuses from the fund. The company employs four managers, eight supervisors, and 20 workers. The annual bonuses are to be distributed such that a supervisor would receive twice as much as a worker, and a manager would receive twice as much as a supervisor.
    a. The fund is invested at 6.2% compounded semi-annually. Construct a complete amortization schedule for the investment.
    b. Except for the last year, how much can each worker, supervisor, and manager expect to receive as the annual Christmas bonus?
19. Speedy Courier purchased a new Toyota Prius for $27,800 to add to its fleet of courier vehicles. It is being financed at 7.39% compounded monthly with monthly payments for six years. For each year of the vehicle loan, calculate the total interest and principal portions paid.
20. B.O.S. Designer Candles Inc. has acquired some manufacturing space for $750,000 in an industrial section of the city. The owners intend to mortgage the property over a 15-year amortization period through successive five-year terms with monthly payments. The semi-annually compounded interest rate for the first term is 5%, and the owners foresee rates of 6% and 6.75% for the future terms. For each term, determine the mortgage payment amount, the total payments made, the total interest paid, and the total principal reduction.

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CHAPTER OVERVIEW

14: Bonds and Sinking Funds

- 14.0: Introduction
- 14.1: Determining the Value of a Bond
- 14.2: Calculating a Bond’s Yield
- 14.3: Sinking Fund Schedules
- 14.4: Debt Retirement and Amortization
- 14.S: Summary
- 14.CS: Case Study - Financing An Acquisition Through Bonds
- 14.E: Bonds and Sinking Funds (Exercises)

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14.0: Introduction

One of the most powerful ways for an institution to raise large amounts of capital is by issuing marketable bonds. A company or a government will essentially borrow the required financing from investors such as you and me. Individually we cannot provide the needed funds, but the bond issuer can raise a large sum of money from us collectively.

For example, Manitoba Hydro is currently building new hydro lines extending from the Keewatinoow Converter Station north of Lakes Winnipeg and Manitoba to the southern distribution network. The project, called Bipole III, is awaiting environmental approvals and licensing. When approved, it is projected to take almost five years to complete the 1,364 kilometre project involving approximately 2,800 45-metre transmission towers. The cost of the project is around $3.28 billion. Manitoba Hydro's 2011–2012 net income was $61 million, while its 2012–2013 net income was $92 million.1 It obviously cannot just ask a bank for a $3.28 billion loan! And even in situations when a bank loan is available, financing through bonds is generally more flexible.

One of the largest bond issuers in Canada, with $468 billion in outstanding marketable bonds as of August 2012, is the Government of Canada.2 This money is primarily used to fund Canada's national debt. On the consumer side, marketable bonds represent another investment opportunity you can include in your RRSP portfolio. They feature regular interest payments and a chance to grow your savings through trading on the bond market.

This chapter begins with the basics of how bonds operate. Then you will learn how to calculate the purchase price of a bond and work with bond interest rates. Bond issuance usually comes with a sinking fund provision, which is explained in Section 14.3. Finally, we will explore the financial and accounting responsibilities associated with amortization of bond premiums and accrual of bond discounts.

References

1. Manitoba Hydro-Electric Board, 62nd Annual Report for the Year Ended March 31, 2013, www.hydro.mb.ca/corporate/ar/2012/publish/index.html#66-67 (accessed October 2, 2013).
2. Statistics Canada, "Canada: Economic and Financial Data," www40.statcan.gc.ca/l01/cst01/indi01f-eng.htm (accessed November 27, 2012).

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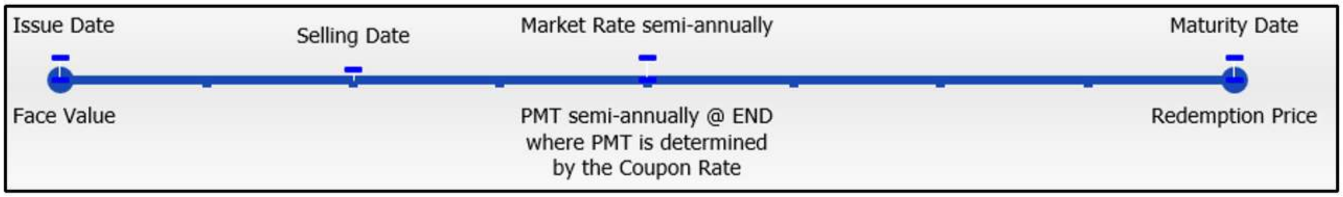
14.1: Determining the Value of a Bond

Exactly what is your marketable bond worth? When you inspect the financial section of your local newspaper, you pay particular attention to the quotes on bonds, into which you invested a portion of your RRSP portfolio. Ten years ago, when prevailing bond rates were 5.5%, you purchased 10 Government of Canada $5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. At the time you paid $4,699.02 each for them. Today's prevailing bond rates have dropped to 3.35% but are expected to rise in the near future. Should you hold on to those bonds? Should you cash them in by selling them in the bond market?

To help answer those questions, it is critical to know the value, or selling price, of your bonds today. This section explains the concept of a marketable bond along with its important characteristics and terminology. You calculate the bond's selling price first on interest payment dates and then on all other dates, which present a more complex situation.

Bond Basics

A marketable **bond**, as illustrated in the photo, is a debt that is secured by a specific corporate asset and that establishes the issuer’s responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date. A **debenture** is the same as a marketable bond, except that the debt is not secured by any specific corporate asset. Mathematically, the calculations are identical for these two financial tools, which this textbook refers to as bonds for simplicity. A typical bond timeline looks like the one below. Key terms are discussed as well.



Bond Issue Date

The **bond issue date** is the date that the bond is issued and available for purchase by creditors. Interest accrues from this date.

Bond Face Value

Also called the par value or denomination of the bond, the **bond face value** is the principal amount of the debt. It is what the investor lent to the bond-issuing corporation. The amount, usually a multiple of $100, is found in small denominations up to $10,000 for individual investors and larger denominations up to $50,000 or more for corporate investors.

Bond Coupon Rate

Also known as the bond rate or nominal rate, the **bond coupon rate** is the nominal interest rate paid on the face value of the bond. The coupon rate is fixed for the life of the bond. Most commonly the interest is calculated semi-annually and payable at the end of every six-month period over the entire life of the bond, starting from the issue date. All coupon rates used in this textbook can be assumed to be semi-annually compounded unless stated otherwise.

Bond Market Rate

The **bond market rate** is the prevailing nominal rate of interest in the open bond market. Since bonds are actively traded, this rate fluctuates based on economic and financial conditions. On the issue date, the market rate determines the coupon rate that is tied to the bond. Market rates are usually compounded semi-annually, as will be assumed in this textbook unless otherwise stated. Therefore, marketable bonds form ordinary simple annuities, since the interest payments and the market rate are both compounded semi-annually, and the payments occur at the end of the interval.

Bond Redemption Price

Also called the redemption value or maturity value, the **bond redemption price** is the amount the bond issuer will pay to the bondholder upon maturity of the bond. The redemption price normally equals the face value of the bond, in which case the bond is said to be “redeemable at par” because interest on the bond has already been paid in full periodically throughout the term, leaving only the principal in the account. In some instances a bond issuer may in fact redeem the bond at a premium, which is a price greater than the face value. The redemption price is then stated as a percentage of the face value, such as 103%. For introductory purposes, this text sticks to the most common situation, where the redemption price equals the face value.

Bond Maturity Date

Also known as the redemption date or due date, the **bond maturity date** is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.

Bond Selling Date

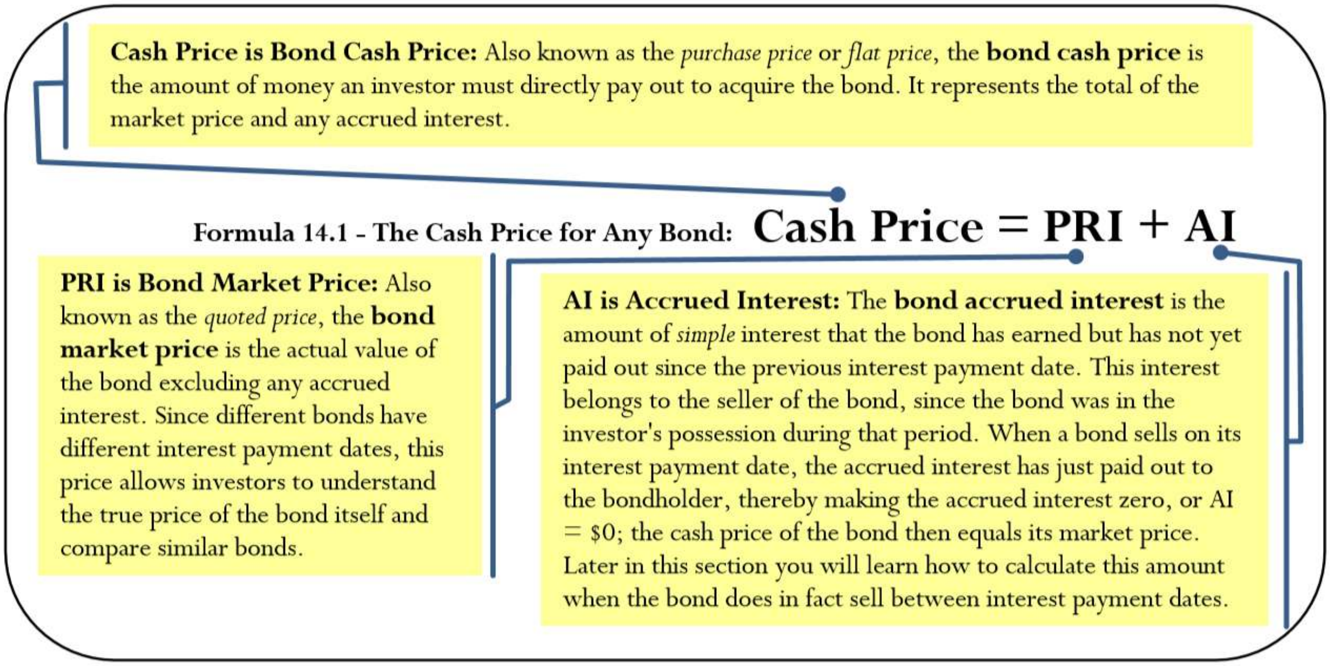
The date that a bond is actively traded and sold to another investor through the bond market is known as the **bond selling date**. In the timeline, the selling date can appear anywhere on the timeline between the issue date and maturity date, and it may occur more than once as the bond is sold by one investor after another.

Calculating the Bond Price on an Interest Payment Date

Marketable bonds and debentures are nonredeemable, which means the only way to cash these bonds in before the maturity date is to sell them to another investor. Therefore, the key mathematical calculation is what to pay for the bond. The selling date, maturity date, coupon rate, redemption price, and market rate together determine the bond price. On the bond's issue date, the market rate determines the coupon rate, so these two rates are identical. As a result, the price of the bond equals its face value. After the bond is issued, interest starts to accrue on it, and the market rate begins to fluctuate based on market conditions. This changes the price of the bond.

The Formula

Because the bond pays interest semi-annually, two days of the year are defined as the interest payment dates. To determine a bond's selling price on these two days, you must use the formulas for present value of an ordinary annuity. Once you understand how to perform these basic calculations we will move on to the more complex formulas and techniques required to determine the selling price on the other 363 days of the year. Regardless of the selling date, Formula 14.1 expresses how to determine the price of any bond.



Cash Price is Bond Cash Price: Also known as the *purchase price* or *flat price*, the **bond cash price** is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and any accrued interest.

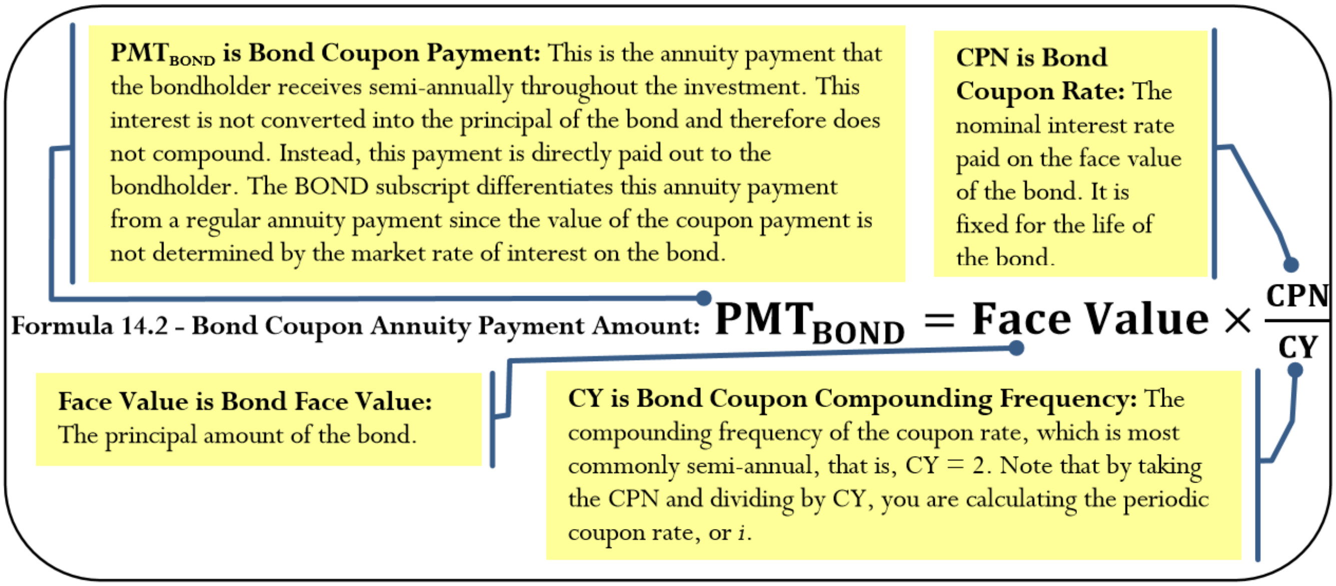
Formula 14.1 - The Cash Price for Any Bond: $\text{Cash Price} = PRI + AI$

PRI is Bond Market Price: Also known as the *quoted price*, the **bond market price** is the actual value of the bond excluding any accrued interest. Since different bonds have different interest payment dates, this price allows investors to understand the true price of the bond itself and compare similar bonds.

AI is Accrued Interest: The **bond accrued interest** is the amount of *simple* interest that the bond has earned but has not yet paid out since the previous interest payment date. This interest belongs to the seller of the bond, since the bond was in the investor's possession during that period. When a bond sells on its interest payment date, the accrued interest has just paid out to the bondholder, thereby making the accrued interest zero, or AI = \$0; the cash price of the bond then equals its market price. Later in this section you will learn how to calculate this amount when the bond does in fact sell between interest payment dates.

Formula 14.1

To determine the selling price of the bond, you must know the amount of the semi-annual interest payment to the bondholder. You use Formula 14.2 to calculate this amount.



PMT_{BOND} is Bond Coupon Payment: This is the annuity payment that the bondholder receives semi-annually throughout the investment. This interest is not converted into the principal of the bond and therefore does not compound. Instead, this payment is directly paid out to the bondholder. The BOND subscript differentiates this annuity payment from a regular annuity payment since the value of the coupon payment is not determined by the market rate of interest on the bond.

CPN is Bond Coupon Rate: The nominal interest rate paid on the face value of the bond. It is fixed for the life of the bond.

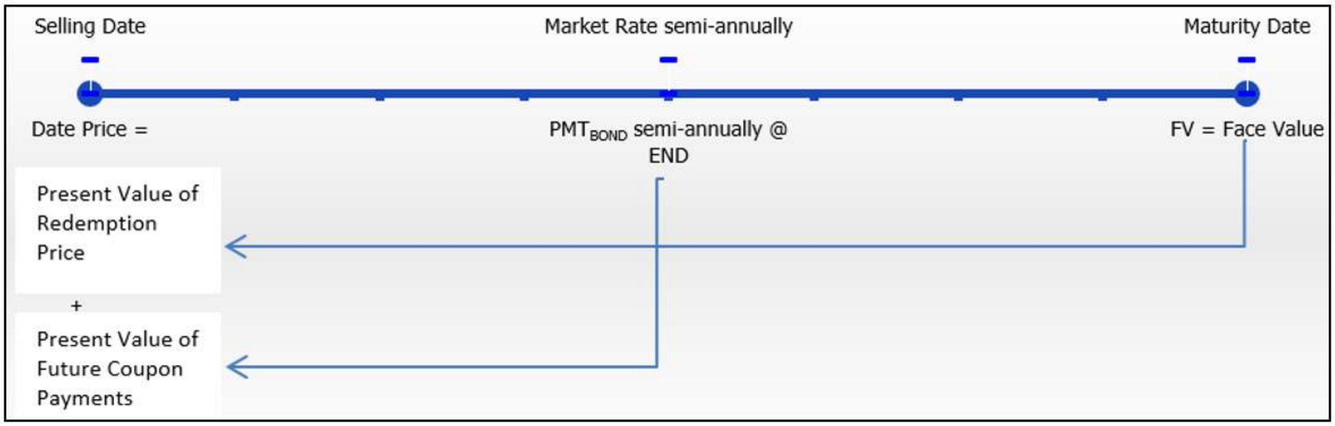
Formula 14.2 - Bond Coupon Annuity Payment Amount: $$PMT_{BOND} = \text{Face Value} \times \frac{CPN}{CY}$$

Face Value is Bond Face Value: The principal amount of the bond.

CY is Bond Coupon Compounding Frequency: The compounding frequency of the coupon rate, which is most commonly semi-annual, that is, CY = 2. Note that by taking the CPN and dividing by CY, you are calculating the periodic coupon rate, or *i*.

Formula 14.2

The market price of a bond on its selling date is the present value of all the future cash flows, as illustrated in the figure below. For the bond purchaser, this is a combination of the remaining coupon annuity payments plus the redemption price at maturity (which in this textbook always equals the face value). Formula 14.3 summarizes this calculation, which combines Formulas 9.3 and 11.4 together and simplifies the resulting expression.

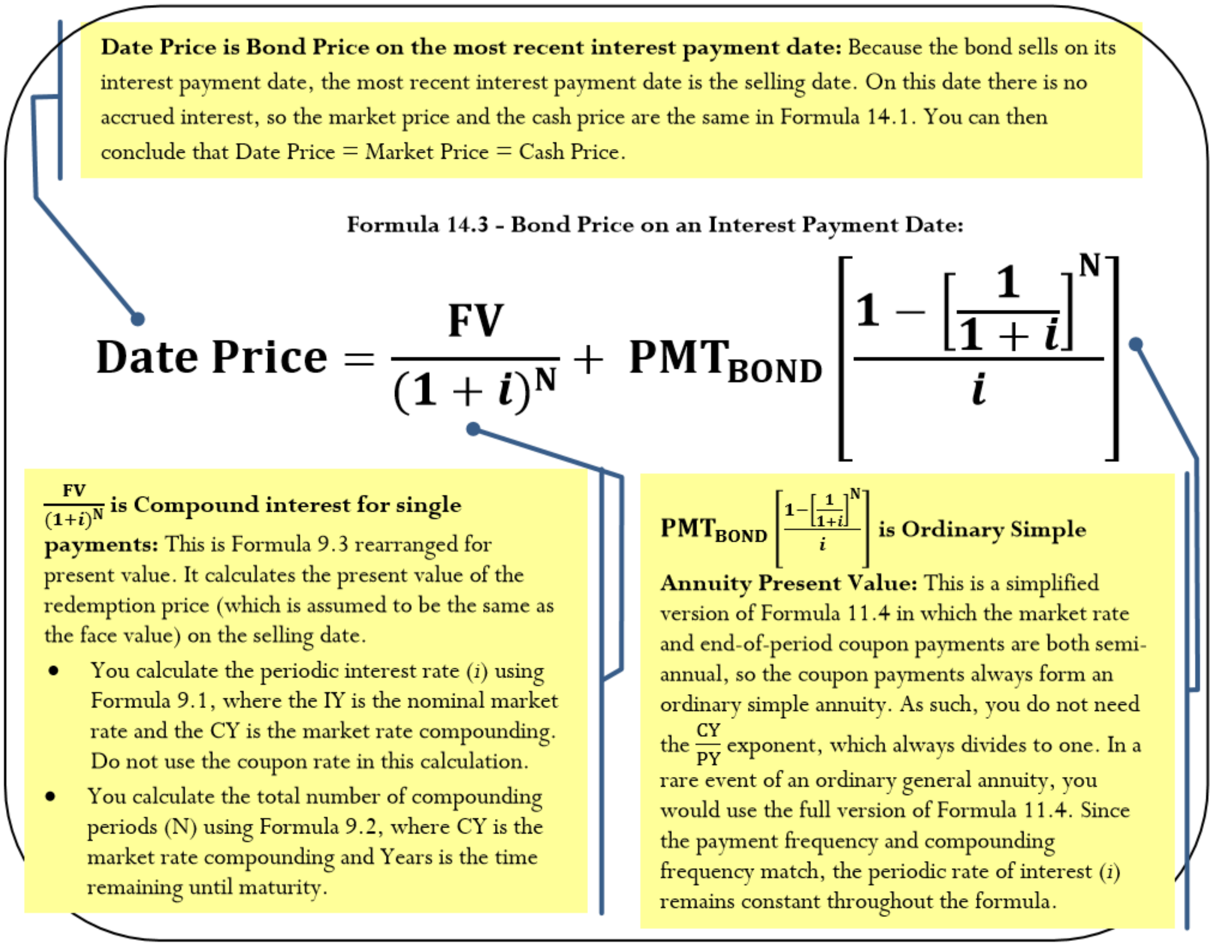

Selling Date
Market Rate semi-annually
Maturity Date
Date Price =
PMT_{BOND} semi-annually @ END
FV = Face Value
Present Value of Redemption Price
+
Present Value of Future Coupon Payments

The price of a bond fluctuates with the market rate over time. If the bond sells for a price higher than its face value, the difference is known as a **bond premium**. If the bond sells for a price lower than its face value, the difference is known as a **bond discount**. The amount of the premium or discount excludes any accrued interest on the bond. Why does the selling price change like this? Remember that the interest paid by the bond is a fixed rate (the coupon rate) determined at the time of issue.

- Assume a coupon rate of 5%. If the market rate has increased to 6%, it means that investors can buy bonds paying 6%. If you are trying to sell your 5% bond, no one wants to buy it unless you "put it on sale" in an amount that compensates for the 1% difference. Hence, you discount your bond.

- Alternatively, if the market rate decreases to 4%, it means that investors can buy bonds paying 4%. If you are trying to sell your 5% bond, it is very attractive to investors, so you add some extra margin, raising the price by an amount not exceeding the 1% difference. Hence, you sell at a premium price.

The figure after Formula 14.3 illustrates the relationship between the market rate, coupon rate, and the selling price of the bond. Notice that when the coupon rate is higher than the market rate, the selling price rises above its face value. Alternatively, when the coupon rate is lower than the market rate, the selling price falls below its face value. Apply Formula 14.4 to calculate the amount of the premium or discount on a bond.



Date Price is Bond Price on the most recent interest payment date: Because the bond sells on its interest payment date, the most recent interest payment date is the selling date. On this date there is no accrued interest, so the market price and the cash price are the same in Formula 14.1. You can then conclude that Date Price = Market Price = Cash Price.

Formula 14.3 - Bond Price on an Interest Payment Date:

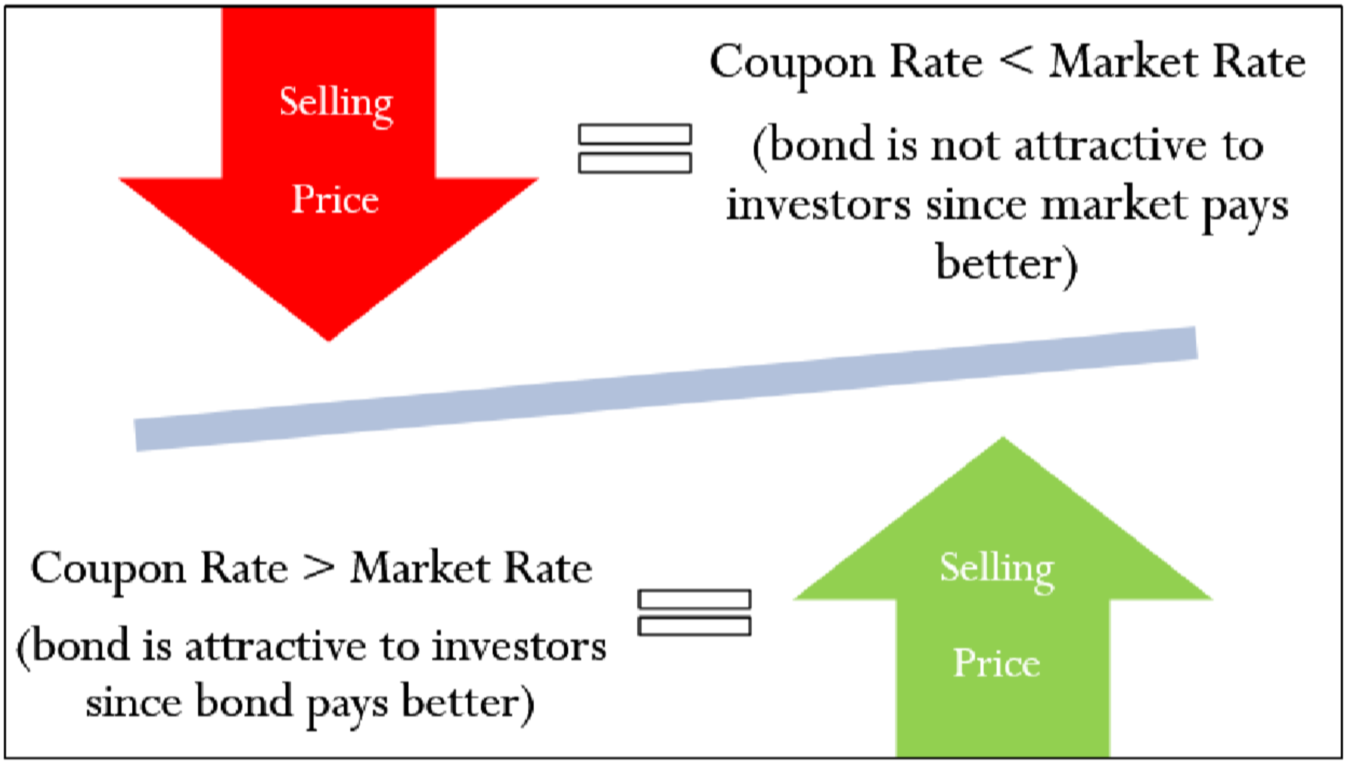
$$\text{Date Price} = \frac{\text{FV}}{(1+i)^{\text{N}}} + \text{PMT}_{\text{BOND}}\left[\frac{1-\left[\frac{1}{1+i}\right]^{\text{N}}}{i}\right]$$

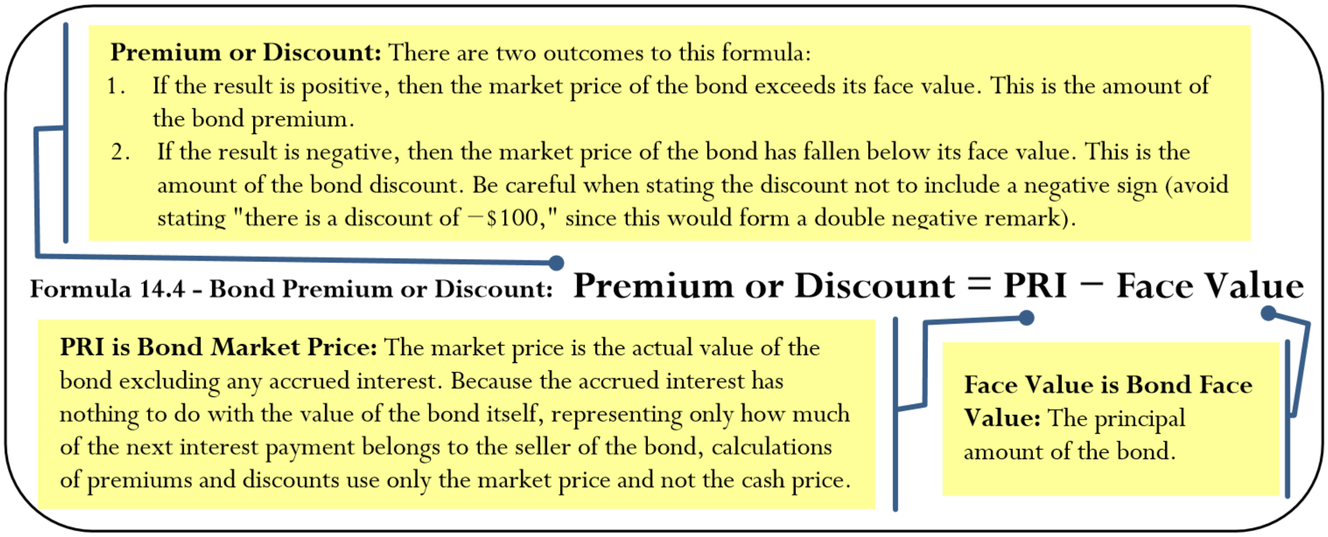
$\frac{\text{FV}}{(1+i)^{\text{N}}}$ **is Compound interest for single payments:** This is Formula 9.3 rearranged for present value. It calculates the present value of the redemption price (which is assumed to be the same as the face value) on the selling date.

- You calculate the periodic interest rate (i) using Formula 9.1, where the IY is the nominal market rate and the CY is the market rate compounding. Do not use the coupon rate in this calculation.
- You calculate the total number of compounding periods (N) using Formula 9.2, where CY is the market rate compounding and Years is the time remaining until maturity.

$\text{PMT}_{\text{BOND}}\left[\frac{1-\left[\frac{1}{1+i}\right]^{\text{N}}}{i}\right]$ **is Ordinary Simple Annuity Present Value:** This is a simplified version of Formula 11.4 in which the market rate and end-of-period coupon payments are both semi-annual, so the coupon payments always form an ordinary simple annuity. As such, you do not need the $\frac{\text{CY}}{\text{PY}}$ exponent, which always divides to one. In a rare event of an ordinary general annuity, you would use the full version of Formula 11.4. Since the payment frequency and compounding frequency match, the periodic rate of interest (i) remains constant throughout the formula.

Formula 14.3


Selling
Price
Coupon Rate < Market Rate
(bond is not attractive to investors since market pays better)
Coupon Rate > Market Rate
(bond is attractive to investors since bond pays better)
Selling
Price



Premium or Discount: There are two outcomes to this formula:

1. If the result is positive, then the market price of the bond exceeds its face value. This is the amount of the bond premium.
2. If the result is negative, then the market price of the bond has fallen below its face value. This is the amount of the bond discount. Be careful when stating the discount not to include a negative sign (avoid stating "there is a discount of −$100," since this would form a double negative remark).

Formula 14.4 - Bond Premium or Discount: $\text{Premium or Discount} = PRI - \text{Face Value}$

PRI is Bond Market Price: The market price is the actual value of the bond excluding any accrued interest. Because the accrued interest has nothing to do with the value of the bond itself, representing only how much of the next interest payment belongs to the seller of the bond, calculations of premiums and discounts use only the market price and not the cash price.

Face Value is Bond Face Value: The principal amount of the bond.

Formula 14.4

How It Works

Follow these steps to calculate the price of a bond on its interest payment date:

Step 1: Draw a timeline extending from the selling date to the maturity date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the regular bond interest payment. For future calculations do not round this number.

Step 3: Using Formula 14.3, calculate the date price of the bond. On an interest payment date, the date price is equal to both the market price and cash price.

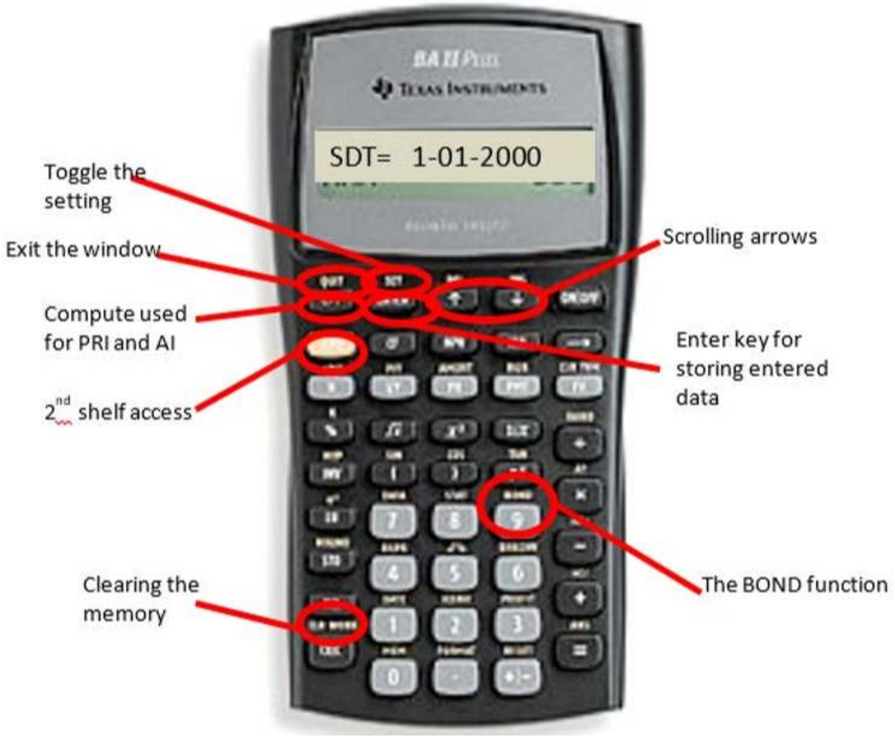
- Use the market rate for Formula 9.1 (Periodic Interest Rate).

- Typically, to calculate N you need both Formula 9.2 (Number of Compound Periods for Single Payments) for the redemption value and Formula 11.1 (Number of Annuity Payments) for the annuity. However, since compound periods and annuity payments are both semi-annual, then each formula would produce the same value of N. Therefore, you can use just Formula 11.1, recognizing that it represents both the number of compound periods as well as the number of annuity payments.
- The date price from Formula 14.3 equals the market price. Since there is no accrued interest, the cash price is the same as the date price.

Step 4: If required, use Formula 14.4 to calculate any bond premium or discount.

Important Notes

The BOND Function on the BAII Plus Calculator. On an interest payment date, you can solve any bond problem using the regular time value of money buttons on your calculator since bonds represent ordinary simple annuities. However, a builtin function for bonds on the BAII Plus calculator greatly simplifies bond price calculations, particularly when the bond is being sold on a date other than an interest payment date. The BOND function is located on the second shelf above the number 9 key and is accessed by pressing 2nd BOND.



This spreadsheet has nine lines, which you scroll through using the ↓ and ↑ arrows. The first seven lines are considered the data entry lines, while the last two are the output lines. Upon opening the window, you should use the 2nd CLR Work function to erase previously loaded data. The data entry lines are as follows:

- SDT is the selling date. It is entered in the standard date format of MM.DDYY where MM is the month number (one or two digits), DD is the two-digit day number, and YY is the last two digits of the year. You must press the ENTER key to store this information. If the question does not involve specific dates, use January 1, 2000, or 1.0100 so as to determine the redemption date more easily.
- CPN is the nominal coupon rate. It is formatted as a percentage but without the percent sign; thus 5.5% is keyed in as 5.5. You must press the ENTER key to store this information.
- RDT is the redemption date or maturity date. It must be entered in the standard date format, and you must press ENTER to store this information. If the question does not involve specific dates, key in the appropriate date based on a selling date of January 1, 2000.
- RV is the redemption value or redemption price expressed as a percentage of the face value. Since the redemption price equals the face value, use the default setting of 100. This textbook never requires you to alter this number.

- ACT/360 is a toggle that you change by pressing 2nd SET. ACT counts the actual number of days in the transaction, while 360 treats every month as having 30 days. In Canada, ACT is the standard, so you should leave the calculator on this setting.
- 2/Y OR 1/Y is a toggle that you change by pressing 2nd SET. 2/Y indicates a semi-annual compound for both the market rate and coupon rate, while 1/Y indicates an annual compound. This textbook always uses the 2/Y setting.
- YLD is the nominal market rate for bonds at the time of sale. It follows the same format as the CPN, that is, a percentage but without the percent sign. You must press the ENTER key to store the information. The output lines are as follows:
- PRI is the market price of the bond. After scrolling to this line, press the CPT button to calculate this output, which is not computed automatically. The output is a percentage of the redemption price (which is the same as the face value). Thus, if you have a $1,000 face value bond, you need to take the output divided by 100 multiplied by the face value to arrive at the market price.
- AI is the accrued interest of the bond. This output is automatically calculated when you press the CPT button on the PRI line. If you are on an interest payment date, it has a value of zero. If you are in between payments, the output is, just like the PRI, a percentage of the redemption price, so you need to convert it to a value in the same manner as the PRI.

If you are interested in the cash price of the bond, you must add together the values associated with the PRI and AI outputs, as in Formula 14.1. When you are finished with the BOND function, press 2nd QUIT to leave the window.

Things To Watch Out For

When you calculate the price of a bond on the interest payment date, the date price is in fact calculating the market price. Recall that the cash price of the bond is always determined by Formula 14.1, where the market price and accrued interest must be totalled to arrive at the cash price. On interest payment dates, there is no accrued interest, so it always has a value of zero. When working with bonds, get in the habit now of thinking in the manner of Formula 14.1. Later on, when the bond is sold on a noninterest payment date and accrued interest is involved, this habit is handy for figuring out bond prices. With respect to the BAII Plus calculator, always add together the outputs of the PRI and AI windows to arrive at the selling price (cash price) of the bond.

Paths To Success

Earlier in this textbook Canada Savings Bonds were discussed. Note that these bonds are fully redeemable at any point, in that you can cash them in at any point with any financial institution before maturity. Therefore, Canada Saving Bonds are not considered marketable bonds and do not operate according to the principles discussed in this section.

? Give It Some Thought

a. Explain the relationship between changes in the bond market rate and the price of the bond.
b. What five variables are used in determining the price of a bond?
c. True or false: If a $1,000 face value bond has a cash price of $1,125 and a market price of $1,100, it is selling at a bond premium of $100.
d. What three variables determine the coupon payment amount?

Answer

a. As the market rate rises the bond price decreases, and vice versa.
b. Selling date, maturity date, coupon rate, redemption price, and market rate
c. True; the bond premium is the difference between the face value and the market price.
d. Face value, nominal coupon rate, and coupon rate compounding frequency; see Formula 14.2

✓ Pricing a Bond with Exact Dates on an Interest Payment Date

A Government of Canada $50,000 bond was issued on January 15, 1991, with a 25-year maturity. The coupon rate was 10.15% compounded semi-annually. What cash price did the bond have on July 15, 2005, when prevailing market rates were 4.31% compounded semi-annually? What was the amount of the bond premium or discount?

Solution

This bond makes interest payments six months apart, on January 15 and July 15 each year. The bond is being sold on July 15, which is an interest payment date. On an interest payment date, solve for the date price, which is the same as the cash price. Also calculate the premium or discount.

What You Already Know

Step 1:

The timeline for the bond sale appears below.

Coupon Interest Payment: CPN = 10.15%, CY = 2, Face Value = $50,000

Bond: FV = $50,000, IY = 4.31%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 10.5

How You Will Get There

Step 2:

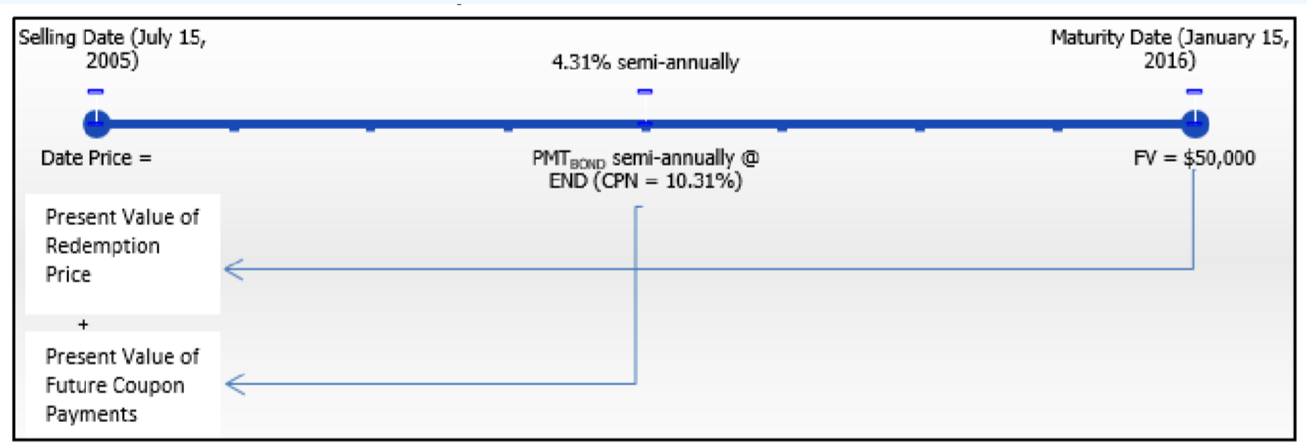
Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date. The cash price in Formula 14.1 equals the date price.

Step 4:

Apply Formula 14.4 to determine the bond premium or discount.


Selling Date (July 15, 2005)
4.31% semi-annually
Maturity Date (January 15, 2016)
Date Price =
PMT_BOND semi-annually @ END (CPN = 10.31%)
FV = $50,000
Present Value of Redemption Price
+
Present Value of Future Coupon Payments

Perform

Step 2:

$$PMT_{BOND} = \$50,000 \times \frac{0.1015}{2} = \$50,000 \times 0.05075 = \$2,537.50$$

Step 3:

$$i = 4.31\%/2 = 2.155\%; N = 2 \times 10.5 = 21 \text{ (compounds and payments)}$$

$$\text{Date Price } = \frac{\$50,000}{(1+0.02155)^{21}} + \$2,537.50\left[\frac{1-\left[\frac{1}{1+0.02155}\right]^{21}}{0.02155}\right] = \$74,452.86$$

Step 4:

$$\text{Premium or Discount } = \$74,452.86 - \$50,000.00 = \$24,452.86$$

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
7.1505	10.15	1.1516	100	ACT	2/Y	4.31	Answer: 148.905720	Answer: 0

Transforming the output:

$$\frac{148.905720+0}{100}\times \$50,000=\$74,452.86012$$

The bond has a coupon rate that is substantially higher than the market rate, so it is sold at a premium of $24,452.86 for a cash price of $74,452.86. Note that because the sale occurs on the interest payment date, there is no accrued interest, so the market price and cash price are the same.

✓ Example 14.1.2: Pricing a Bond with General Dates on an Interest Payment Date

A $25,000 Government of Canada bond was issued with a 25-year maturity and a coupon rate of 8.92% compounded semiannually. Two-and-a-half years later the bond is being sold when market rates have increased to 9.46% compounded semiannually. Determine the selling price of the bond along with the amount of premium or discount.

Solution

Although there are no specific dates, the coupon is semi-annual, making interest payments every six months. If the bond is being sold 2½ years after issue, this makes the sale date an interest payment date. On an interest payment date, solve for the date price, which is the same as the cash price. Also calculate the premium or discount.

What You Already Know

Step 1:

The timeline for the bond sale appears below.

Coupon Interest Payment: CPN = 8.92%, CY = 2, Face Value = $25,000

Bond: FV = $25,000, IY = 9.46%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 22.5

How You Will Get There

Step 2:

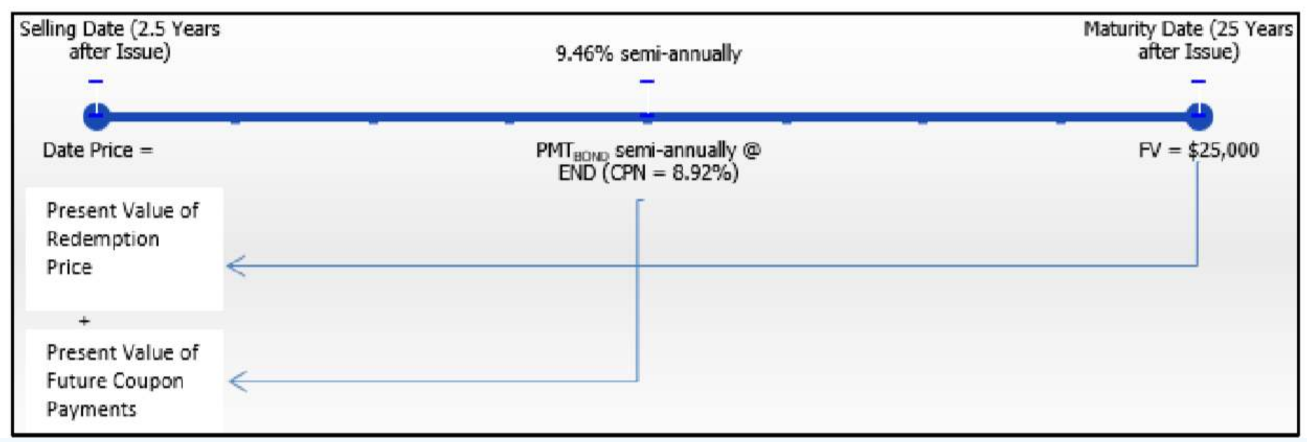
Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date. The cash price in Formula 14.1 equals the date price.

Step 4:

Apply Formula 14.4 to determine the bond premium or discount.


Selling Date (2.5 Years after Issue)
9.46% semi-annually
Maturity Date (25 Years after Issue)
Date Price =
PMT_BOND semi-annually @ END (CPN = 8.92%)
FV = $25,000
Present Value of Redemption Price
+
Present Value of Future Coupon Payments

Perform

Step 2:

$$PMT_{BOND}=\$25,000\times\frac{0.0892}{2}=\$25,000\times 0.0446=\$1,115$$

Step 3:

$$i = 9.46\%/2 = 4.73\%; N = 2 \times 22.5 = 45 \text{ (compounds and payments)}$$

$$\text{Date Price} = \frac{\$25,000}{(1+0.0473)^{45}} + \$1,115\left[\frac{1-\left[\frac{1}{1+0.0473}\right]^{45}}{0.0473}\right] = \$23,751.28$$

Step 4:

$$\text{Premium or Discount} = \$23,751.28 - \$25,000.00 = -\$1,248.72$$

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
1.0100	8.92	7.0122	100	ACT	2/Y	9.46	Answer: 95.005105	Answer: 0

Transforming the output:

$$\frac{95.005105+0}{100} \times \$25,000 = \$23,751.27631$$

The bond has a coupon rate that is slightly lower than the market rate, so it is sold at a discount of $1,248.72 for a cash price of $23,751.28. Note that because the sale occurs on the interest payment date, there is no accrued interest, so the market price and cash price are the same.

Calculating the Bond Price on a Non-interest Payment Date

There are only two days of the year upon which the cash price and the market price of the bond are the same value. Those days are the interest payment dates, when you determine the bond's value using Formula 14.3. However, what happens if the bond is sold on one of the other 363 days of the year?

On these other dates, the cash price and the market price are not equal. For each day that elapses after an interest payment date, interest for the next payment starts to accrue such that over the next six months, enough interest is available to make the next interest payment.

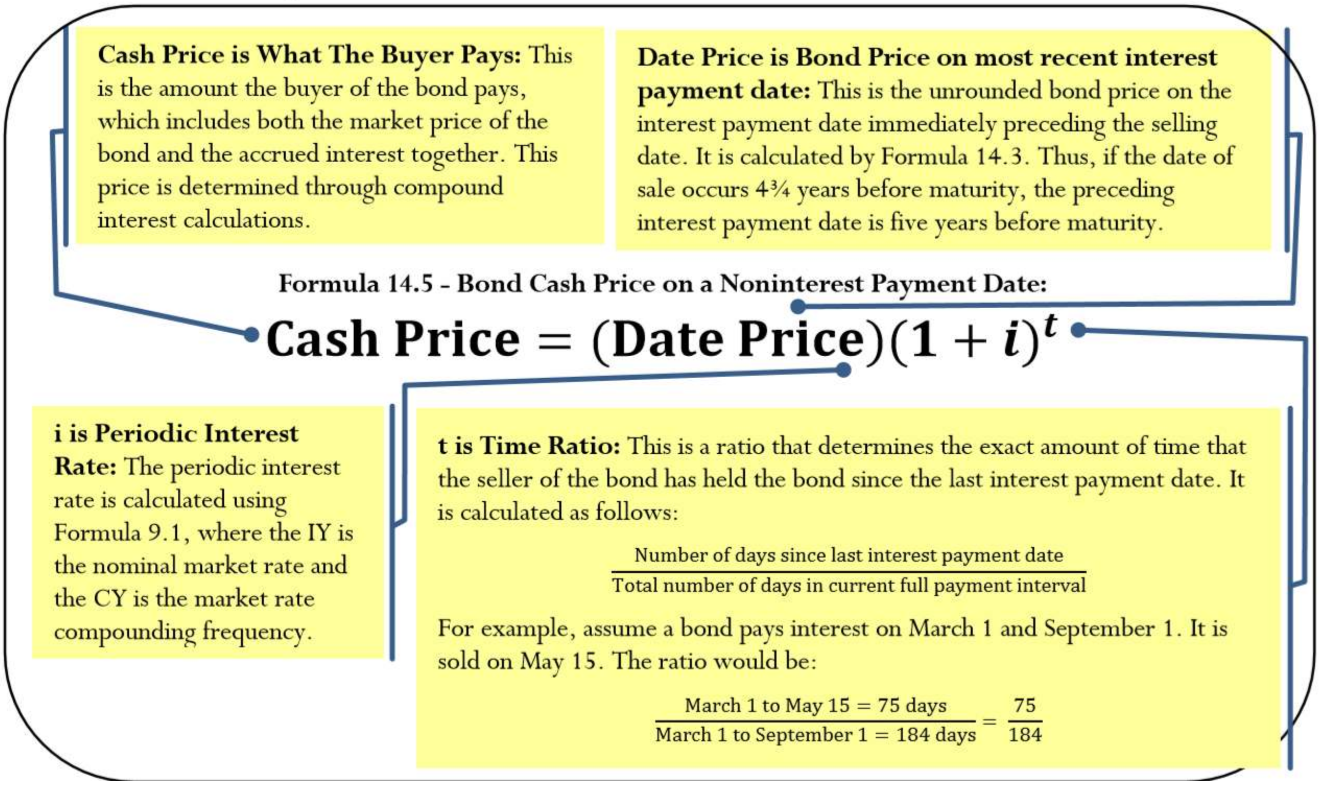
According to Formula 14.1, when an investor wants to purchase a bond in between interest payment dates, the buyer has to pay the seller a cash price equalling the market price of the bond plus the accrued interest. Why? Assume a bond makes semi-annual interest payments of $50. When the buyer acquires the bond from the seller, two months have elapsed since the last interest payment date. Since the seller held the bond for two months of the six-month payment interval, it is fair and reasonable for the seller to receive the interest earned during that time frame. However, the bond will not make its next interest payment until four months later, at which time the buyer, who now owns the bond, will receive the full $50 interest payment for the full six months. Thus, at the time of buying the bond, the buyer has to pay the seller the bond's market price plus the portion of the next interest payment that legally belongs to the seller. In this example, an interest amount representing two of the six months needs to be paid.

The Formula

Arriving at the bond's price in between interest payment dates is a little complex because the cash price is increasing according to a compound interest formula, while in practice the accrued interest on the bond is increasing according to a simple interest formula. If this seems peculiar to you, you would be right, but that is just how bonds happen to work!

You require three important pieces of information: what to pay (the cash price), how much simple interest was included in the cash price (the accrued interest), and what the actual value of the bond was (the market price). To arrive at these three numbers, follow these steps:

1. First calculate the cash price of the bond as shown in Formula 14.5.
2. Calculate the accrued interest included in that price as shown in Formula 14.6.
3. Determine the market price from Formula 14.1.



Cash Price is What The Buyer Pays: This is the amount the buyer of the bond pays, which includes both the market price of the bond and the accrued interest together. This price is determined through compound interest calculations.

Date Price is Bond Price on most recent interest payment date: This is the unrounded bond price on the interest payment date immediately preceding the selling date. It is calculated by Formula 14.3. Thus, if the date of sale occurs 4¾ years before maturity, the preceding interest payment date is five years before maturity.

Formula 14.5 - Bond Cash Price on a Noninterest Payment Date:

$$\text{Cash Price} = (\text{Date Price})(1+i)^t$$

i is Periodic Interest Rate: The periodic interest rate is calculated using Formula 9.1, where the IY is the nominal market rate and the CY is the market rate compounding frequency.

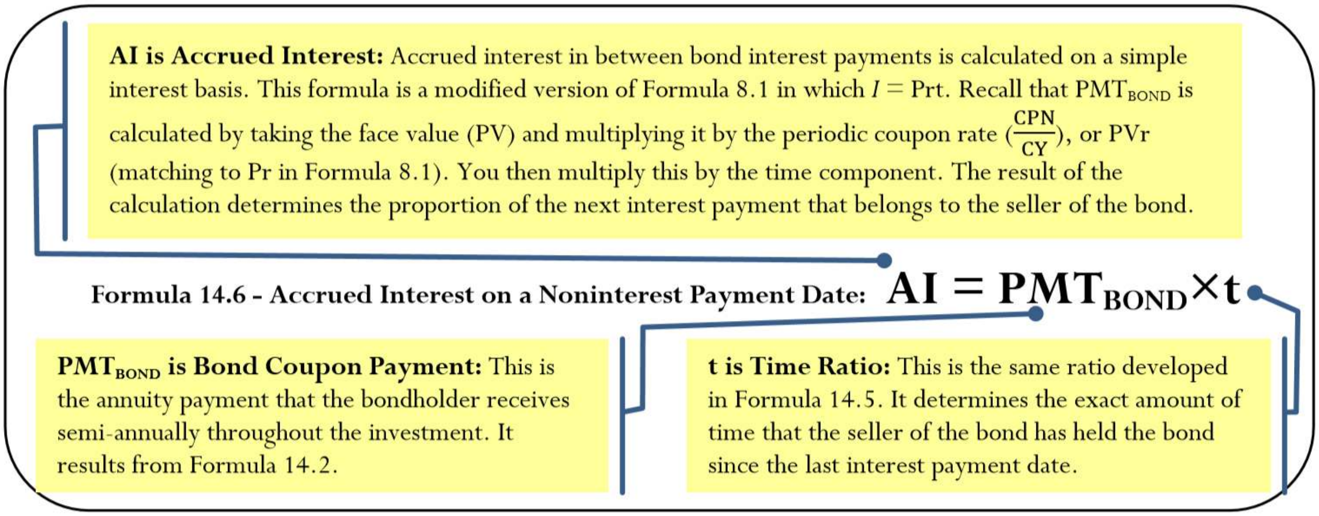
t is Time Ratio: This is a ratio that determines the exact amount of time that the seller of the bond has held the bond since the last interest payment date. It is calculated as follows:

$$\frac{\text{Number of days since last interest payment date}}{\text{Total number of days in current full payment interval}}$$

For example, assume a bond pays interest on March 1 and September 1. It is sold on May 15. The ratio would be:

$$\frac{\text{March 1 to May 15} = 75\text{ days}}{\text{March 1 to September 1} = 184\text{ days}} = \frac{75}{184}$$

Formula 14.5



AI is Accrued Interest: Accrued interest in between bond interest payments is calculated on a simple interest basis. This formula is a modified version of Formula 8.1 in which $I = Prt$. Recall that PMT_{BOND} is calculated by taking the face value (PV) and multiplying it by the periodic coupon rate ($\frac{CPN}{CY}$), or PVr (matching to Pr in Formula 8.1). You then multiply this by the time component. The result of the calculation determines the proportion of the next interest payment that belongs to the seller of the bond.

Formula 14.6 - Accrued Interest on a Noninterest Payment Date: $AI = PMT_{BOND} \times t$

PMT_{BOND} is Bond Coupon Payment: This is the annuity payment that the bondholder receives semi-annually throughout the investment. It results from Formula 14.2.

t is Time Ratio: This is the same ratio developed in Formula 14.5. It determines the exact amount of time that the seller of the bond has held the bond since the last interest payment date.

Formula 14.6

How It Works

Follow these steps to calculate the price of a bond in between interest payment dates:

Step 1: Draw a timeline like the one to the right, extending from the preceding interest payment date to the maturity date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the regular bond interest payment. For future calculations do not round this number.

Step 3: Using Formula 14.3, calculate the date price of the bond on the interest payment date just preceding the selling date.

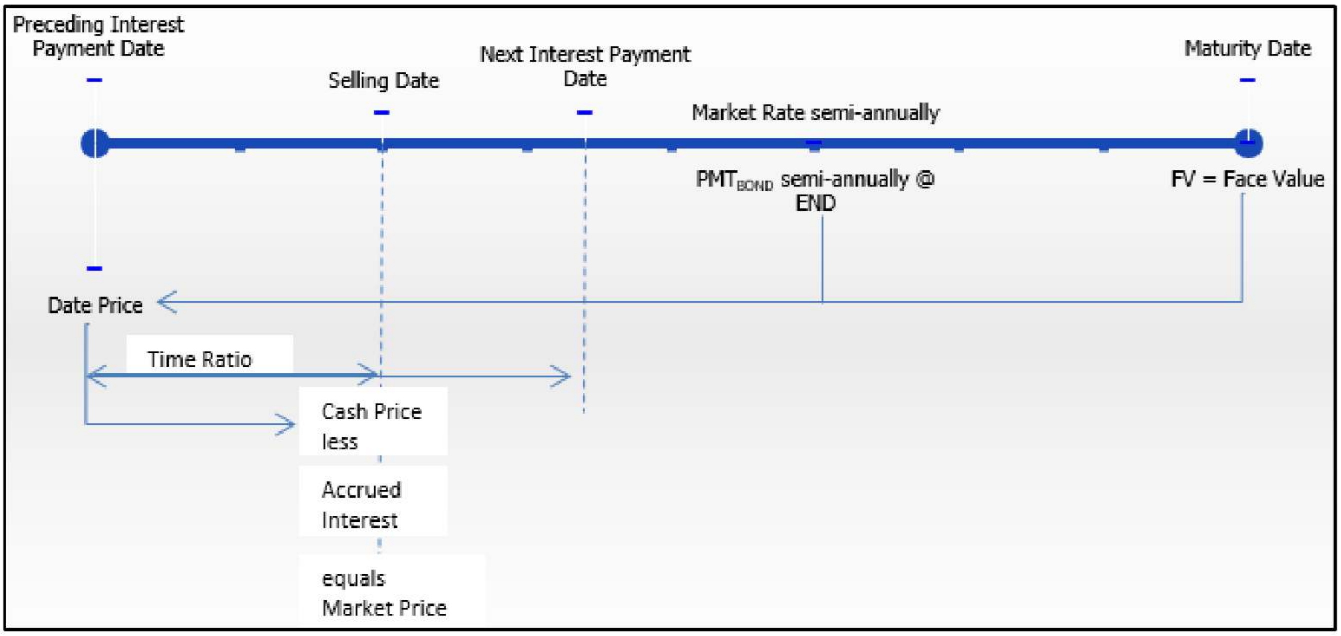
- Use the market rate for Formula 9.1 (Periodic Interest Rate).
- Recall that you use only Formula 11.1 and recognize that it represents both the number of compound periods as well as the number of annuity payments.

Step 4: Calculate the cash price of the bond using Formula 14.5. Calculate the time ratio by determining the exact number of days the seller held the bond as well as the exact number of days involved in the current payment interval.

Step 5: Calculate the accrued interest on the bond using Formula 14.6. Use the time ratio from step 5.

Step 6: Calculate the market price of the bond using Formula 14.1.

Step 7: If required, use Formula 14.4 to calculate the bond premium or discount.


Preceding Interest Payment Date
Selling Date
Next Interest Payment Date
Maturity Date
Market Rate semi-annually
PMT$_{BOND}$ semi-annually @ END
FV = Face Value
Date Price
Time Ratio
Cash Price less
Accrued Interest
equals Market Price

Important Notes

The DATE Function on the BAII Plus Calculator. To create the time ratio, determine the number of days since the last interest payment date as well as the total number of days in the current payment interval. You can compute this through the DATE function. For a full discussion of this function, recall the instructions at the end of Chapter 8. To arrive at the required numbers:

1. Compute the number of days since the last interest payment date by entering the last interest payment date as DT1 and the selling date as DT2. Compute the DBD.
2. Compute the total number of days in the current payment interval by entering the last interest payment date as DT1 and the next interest payment date as DT2. Compute the DBD.

Things To Watch Out For

In calculations of bond premiums and discounts on non-interest-payment dates, the most common mistake is to use the cash price instead of the market price. Remember that the cash price includes both the accrued interest and the market price. The accrued interest does not factor into the value of the bond, since it represents a proportioning of the next interest payment between the seller and the buyer. Therefore, the amount of the bond premium or discount should not include the accrued interest. Use only the market price to determine the premium or discount.

Paths To Success

On the BAII Plus, the PRI and AI outputs of the BOND worksheet must be summed together to arrive at the cash price. Both of these outputs represent a percentage of the face value (the same base) and can be summed before converting the percentage into a dollar amount. For example, if PRI = 98% and AI = 2.5%, you could take the total of 100.5% to figure out the cash price. Using the calculator efficiently, you would store the PRI into a memory cell, scroll down, and then add the recall of the memory cell to the AI to arrive at the total percentage of face value.

Exercise 14.1.2: Give It Some Thought

a. Is the cash price or market price higher when a bond is sold in between interest payment dates?
b. Why can you not use Formula 14.3 directly to obtain the market price of a bond selling in between interest payment dates?

Answer

a. Cash price; it includes the accrued interest.
b. Accrued interest is calculated using simple interest, which results in slightly different numbers than if the bond price used compound interest alone.

Example 14.1.3: Bond Price on a Noninterest Payment Date

A $20 million face value Bell Canada bond was issued on July 19, 1999, with a 30-year maturity and a coupon rate of 6.55% compounded semi-annually. At what cash price did the bond trade on November 10, 2010, when market yields were posted at 5.892% semi-annually? What was the accrued interest? What was the market price? What premium or discount does this represent?

Solution

Notice that this bond makes interest payments six months apart, on January 19 and July 19 of each year. The bond sale date, November 10, is not an interest payment date. Calculate four variables: the cash price, accrued interest (AI), market price (PRI), and the bond premium or discount.

What You Already Know

Step 1:

The timeline of the bond sale appears below.

Coupon Interest Payment: CPN = 6.55%, CY = 2, Face Value = $20 million

Bond: FV = $20 million, IY = 5.892%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 18.5 years to preceding interest payment date

How You Will Get There

Step 2:

Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its preceding interest payment date.

Step 4:

Apply Formula 14.5 to determine the cash price of the bond.

Step 5:

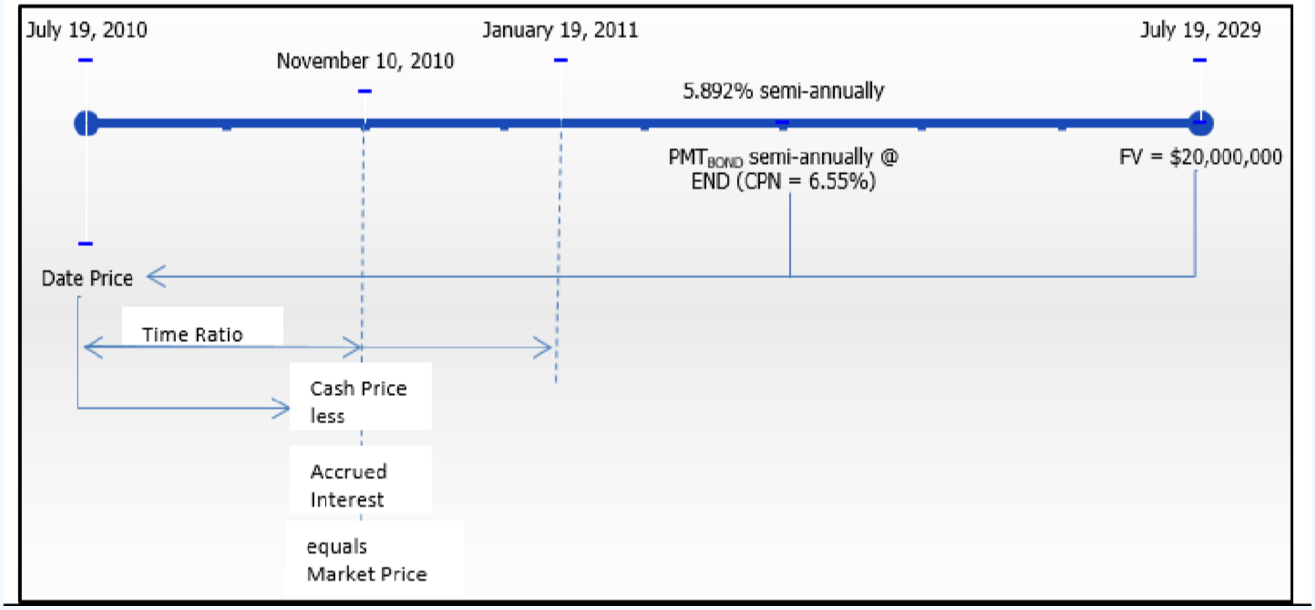
Apply Formula 14.6 to determine the accrued interest.

Step 6:

Apply Formula 14.1 to determine the market price.

Step 7:

Apply Formula 14.4 to determine the bond premium or discount.


July 19, 2010
November 10, 2010
January 19, 2011
July 19, 2029
5.892% semi-annually
PMT_BOND semi-annually @ END (CPN = 6.55%)
FV = $20,000,000
Date Price
Time Ratio
Cash Price less
Accrued Interest
equals Market Price

Perform

Step 2:

$$PMT_{BOND} = \$20,000,000 \times \frac{0.0655}{2} = \$655,000$$

Step 3:

$$i = 5.892\%/2 = 2.946\%; N = 2 \times 19 = 38 \text{ (compounds and payments)}$$

$$\text{Date Price } = \frac{\$20,000,000}{(1+0.02946)^{38}} + \$655,000 \left[\frac{1 - \left[\frac{1}{1+0.02946}\right]^{38}}{0.02946} \right] = \$21,492,511.69$$

Step 4:

$$t = \frac{\text{July 19, 2010 to November 10, 2010}}{\text{July 19, 2010 to January 19, 2011}} = \frac{114}{184}$$

$$\text{Cash Price } = (\$21,492,511.69)(1+0.02946)^{\frac{114}{184}} = \$21,882,632.40$$

Step 5:

$$(\$655,000)\frac{114}{184} = \$405,815.22$$

Step 6:

$$\$21,882,632.40 = PRI + \$405,815.22 \quad \$21,476,817.18 = PRI$$

Step 7:

$$\text{Premium or Discount } = \$21,476,817.18 - \$20,000,000.00 = \$1,476,817.18$$

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
11.1010	6.55	7.1929	100	ACT	2/Y	5.592	Answer: 107.384085	Answer: 2.029076

Transforming the output:

$$PRI = \frac{107.384085}{100} \times \$20,000,000 = \$21,476,817.18$$

$$AI = \frac{2.029076}{100} \times \$20,000,000 = \$405,815.22$$

The bond had a coupon rate that is higher than the market rate, so it sold at a premium of $1,476,817.18 for a market price of $21,476,817.18. The buyer also owed the seller $405,815.22 of accrued interest; therefore, the cash price was $21,882,632.40.

✓ Example 14.1.4: How Did the Investor Do?

A $50,000 Government of Ontario bond was issued on March 1, 1995, with a coupon rate of 9.5% compounded semiannually and 50 years until maturity. Harvey bought the bond on July 17, 1996, when market rates were 8.06% compounded semi-annually, and sold the bond on December 12, 2008, with a semi-annual yield of 3.45%. Based on the market price, what gain or loss did Harvey realize?

Solution

Notice that this bond makes interest payments six months apart, on March 1 and September 1 of each year. Since the bond is being bought on July 17 and sold on December 12, neither date represents an interest payment date. Calculate the market price (PRI) for both dates and then determine the difference.

What You Already Know

Step 1:

The timeline for the bond sale appears below. Coupon Interest Payment: CPN = 9.5%, CY = 2, Face Value = $50,000

Bond Purchase: FV = $50,000, IY = 8.06%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 49 years to preceding interest payment date

Bond Sale: FV = $50,000, IY = 3.45%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 36.5 years to preceding interest payment date

How You Will Get There

Step 2:

Apply Formula 14.2 to determine the periodic bond interest payment.

For each of the purchase and sale, perform steps 3 through 6:

Step 3:

Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its preceding interest payment date.

Step 4:

Apply Formula 14.5 to determine the cash price of the bond.

Step 5:

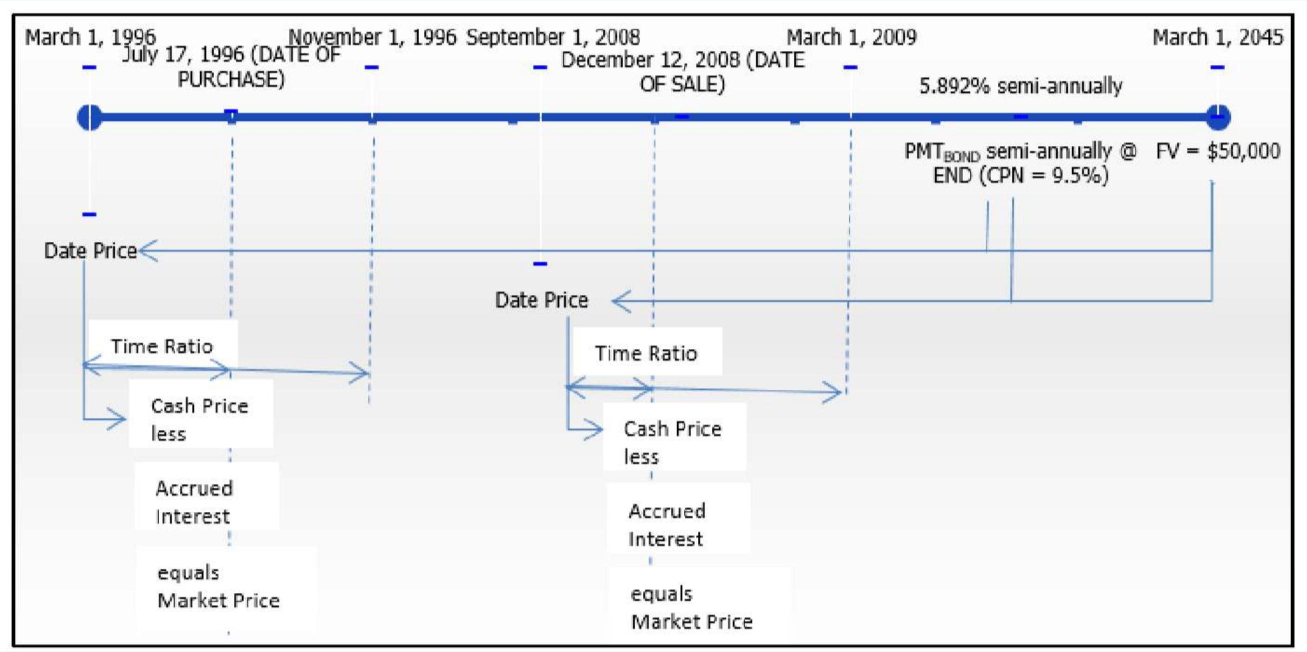
Apply Formula 14.6 to determine the accrued interest.

Step 6:

Apply Formula 14.1 to determine the market price.

Step 7:

Determine the difference between the market prices (PRI) from the purchase to the sale.


March 1, 1996
July 17, 1996 (DATE OF PURCHASE)
November 1, 1996
September 1, 2008
December 12, 2008 (DATE OF SALE)
March 1, 2009
March 1, 2045
5.892% semi-annually
PMT_BOND semi-annually @ END (CPN = 9.5%)
FV = $50,000
Date Price
Date Price
Time Ratio
Cash Price less
Accrued Interest
equals Market Price
Time Ratio
Cash Price less
Accrued Interest
equals Market Price

Perform

Step 2:

$$PMT_{BOND} = \$50,000 \times \frac{0.095}{2} = \$2,375$$

Bond Purchase (July 17, 1996)

Step 3:

$$i = 8.06\%/2 = 4.03\%; N = 2 \times 49 = 98 \text{ (compounds and payments)}$$

$$\begin{aligned}\text{Date Price } &= \frac{\$50,000}{(1+0.0403)^{98}} + \$2,375\left[\frac{1-\left[\frac{1}{1+0.0403}\right]^{98}}{0.0403}\right] \\ &= \$58,747.02738\end{aligned}$$

Step 4:

$$t = \frac{\text{March 1, 1996 to July 17, 1996}}{\text{March 1, 1996 to September 1, 1996}} = \frac{138}{184}$$

$$\begin{aligned}\text{Cash Price } &= (\$58,747.02738)(1+0.0403)^{\frac{138}{184}} \\ &= \$60,513.86\end{aligned}$$

Step 5:

$$AI = (\$2,375)\frac{138}{184} = \$1,781.25$$

Step 6:

$$\begin{aligned}\$60,513.86 &= PRI + \$1,781.25 \\ \$58,732.61 &= PRI\end{aligned}$$

Bond Sale (December 12, 2008)

Step 3:

$$i = 3.45\%/2 = 1.725\%; N = 2 \times 36.5 = 73 \text{ (compounds and payments)}$$

$$\begin{aligned}\text{Date Price } &= \frac{\$50,000}{(1+0.01725)^{73}} + \$2,375\left[\frac{1-\left[\frac{1}{1+0.01725}\right]^{73}}{0.01725}\right]\\ &= \$112,522.6856\end{aligned}$$

Step 4:

$$t = \frac{\text{September 1, 2008 to December 12, 2008}}{\text{September 1, 2008 to March 1, 2009}} = \frac{102}{181}$$

$$\begin{aligned}\text{Cash Price } &= (\$112,522.6856)(1+0.01725)^{\frac{102}{181}}\\ &= \$113,612.43\end{aligned}$$

Step 5:

$$AI = (\$2,375)\frac{102}{181} = \$1,338.40$$

Step 6:

$$\begin{aligned}\$113,612.43 &= PRI + \$1,338.40\\ \$112,274.03 &= PRI\end{aligned}$$

Step 7:

$$\text{Gain } = \$112,274.03 - \$58,732.61 = \$53,541.42$$

Calculator Instructions

Bond	SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
Purchase	7.1796	9.5	3.0145	100	ACT	2/Y	8.06	Answer: 117.465216	Answer: 3.5625
Sale	12.1208	√	√	√	√	√	3.45	Answer: 224.548072	Answer: 2.676795

Bond Purchase (July 17, 1996)

Transforming the output:

$$PRI = \frac{117.465216}{100} \times \$50,000 = \$58,732.60843$$

Bond Sale (December 12, 2008)

Transforming the output:

$$PRI = \frac{224.548072}{100} \times \$50,000 = \$112,274.0362$$

(Difference of $0.01 is due to rounding.)

Harvey acquired the bond for a market price of $58,732.61 and sold the bond approximately 12.5 years later for $112,274.03 because of the very low market rates in the bond market. As a result, the gain on his bond amounts to $53,541.42.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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14.2: Calculating a Bond's Yield

Should you always hold onto a bond until maturity? Is there a best time to sell? In the opener to the last section, you invested in 10 Government of Canada $5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. For these you paid $4,699.02 each when prevailing bond rates were 5.5%. Ten years after you purchased those bonds, prevailing bond market rates have dropped to 3.35% but are expected to rise in the near future. Is now the time to sell those bonds?

This is a complex decision with many variables; however, to help make that decision you must know at least three critical pieces of information: the selling price of the bond, the yield you would realize on your investment if you held onto the bond until maturity, and the yield if you sold the bond today.

This section integrates the calculations of bond prices and bond yields so that you better understand your bond investments. These guidelines apply whether you are investing personally or on behalf of your company.

Yield to Maturity

The **yield to maturity**, also known as basis, is a bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price on the selling date and the redemption price. This yield to maturity is exactly equal to the market rate of return on the date of purchase. Thus, in the example above, if you hang on to those bonds until maturity, you will realize a yield to maturity of 5.5%.

In this section, you are turning around the calculations from Section 14.1, where you answered the question, "Knowing the market rate of return, what will you pay?" Now you are asked, "Knowing what you pay, what is the market rate of return?" Thus, rewording the opening example you would have: “If you paid $4,699.02 for a $5,000 face value bond with 20 years to maturity having a 5% coupon, what yield to maturity would you realize?"

The Formula

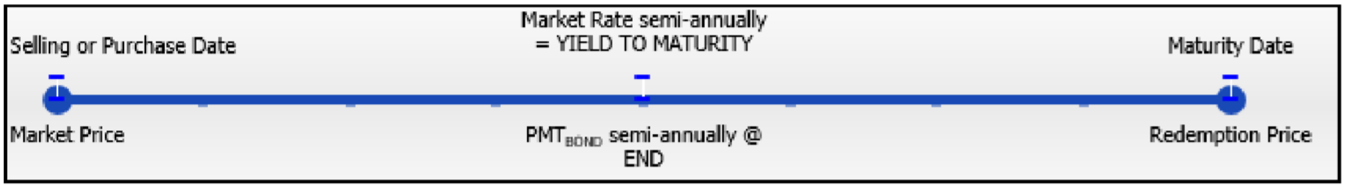
You need no new formulas to calculate a bond's yield to maturity. The goal is to solve for the nominal rate of interest, or IY. You must work with Formulas 14.2, 14.3, and 9.1. Recall that Formula 14.2 determines the semi-annual bond coupon interest payment amount. You substitute this amount into Formula 14.3, which calculates the price of the bond on an interest payment date. However, in this case you must solve Formula 14.3 not for the date price but for the periodic rate of interest, or i. Once you know the periodic interest rate, you can substitute it into Formula 9.1 and solve for the nominal interest rate, or IY.

One problem in using the formula approach is that it is impossible to algebraically isolate the periodic interest rate in Formula 14.3. You must turn to technology such as the BAII Plus calculator or Excel.

How It Works

Follow these steps to calculate a bond's yield to maturity:

Step 1: Draw a timeline like the one presented here, extending from the selling date to the maturity date. Identify all known variables.


Selling or Purchase Date
Market Rate semi-annually = YIELD TO MATURITY
Maturity Date
Market Price
PMT_BOND semi-annually @ END
Redemption Price

Step 2: Using Formula 14.2, calculate the amount of the bond interest payment.

Step 3: As in Section 14.1, use Formula 11.1 to calculate the N. Since the market rate and the coupon rate are both semi-annual, the N is used as both the total number of compounds and the total number of payments.

- If using a manual method, substitute all known numbers into Formula 14.3 and attempt to solve for the periodic interest rate (i). Once you know this, convert it to a nominal interest rate using Formula 9.1.
- Alternatively, if you are using technology such as the calculator, input all known variables and solve directly for the nominal interest rate (IY).

Important Notes

Note two conditions when you calculate yield to maturity:

1. Trading takes place only on interest payment dates.
2. The bondholder reinvests all coupon payments at the same rate of interest. Calculations not meeting these conditions are beyond the introductory scope of this textbook.

? Exercise 14.2.1: Give It Some Thought

If you purchase a bond with a coupon rate of 4% when the market rate is 5%, what yield to maturity will you realize?

Answer

5%; your yield to maturity depends on the market rate, not the coupon rate.

✓ A Basic Yield to Maturity

Suppose a $10,000 face value bond is purchased for $7,688.52 with 20 years until maturity and has a coupon rate of 4% semi-annually. What yield will the bondholder realize if she holds onto it until maturity?

Solution

The yield to maturity represents the nominal rate of interest on the bond, or IY.

What You Already Know

Step 1:

The timeline for the bond sale appears below.

Coupon Interest Payment: CPN = 4%, CY = 2, Face Value = $10,000

Bond: Date Price = $7,688.52, FV = $10,000, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 20

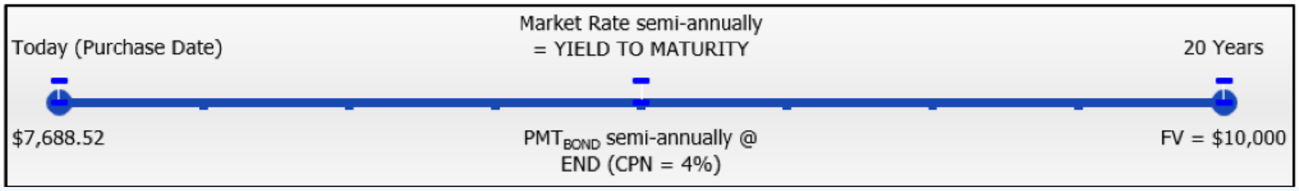
How You Will Get There

Step 2:

Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

Apply Formulas 11.1, 14.3, and 9.1 to determine the yield to maturity (IY). Solve Formulas 14.3 and 9.1 using the calculator.


Today (Purchase Date)
Market Rate semi-annually = YIELD TO MATURITY
20 Years
$7,688.52
PMT$_{BOND}$ semi-annually @ END (CPN = 4%)
FV = $10,000

Calculations

Step 2:

$$PMT_{BOND} = \$10,000 \times \frac{0.04}{2} = \$200$$

Step 3:

$$N = 2 \times 20 = 40 \text{ (compounds and payments)}$$

$$\$7,688.52 = \frac{\$10,000}{(1+i)^{40}} + \$200\left[\frac{1-\left[\frac{1}{1+i}\right]^{40}}{i}\right] \qquad i = \frac{\text{IY}}{2}$$

$$\text{Calculator output} = IY = 6$$

Calculator Instructions

Mode	N	I/Y	PV
END	40	Answer: 6.000002	-7688.52

PMT	FV	P/Y	C/Y
200	10,000	2	2

If she holds onto the bond for the next 20 years, she will realize a yield to maturity of 6% compounded semi-annually.

The Investor’s Yield

The yield-to-maturity calculation requires that the bond be held until its maturity date, at which point the future value redemption price is known and equal to its face value. However, this condition may not hold. The bondholder may sell the bond at any point before the maturity date. If the bond is sold, the future value is based on the prevailing bond rate at that time, and the price is generally not equal to its face value. What then is the bond’s yield for the investor?

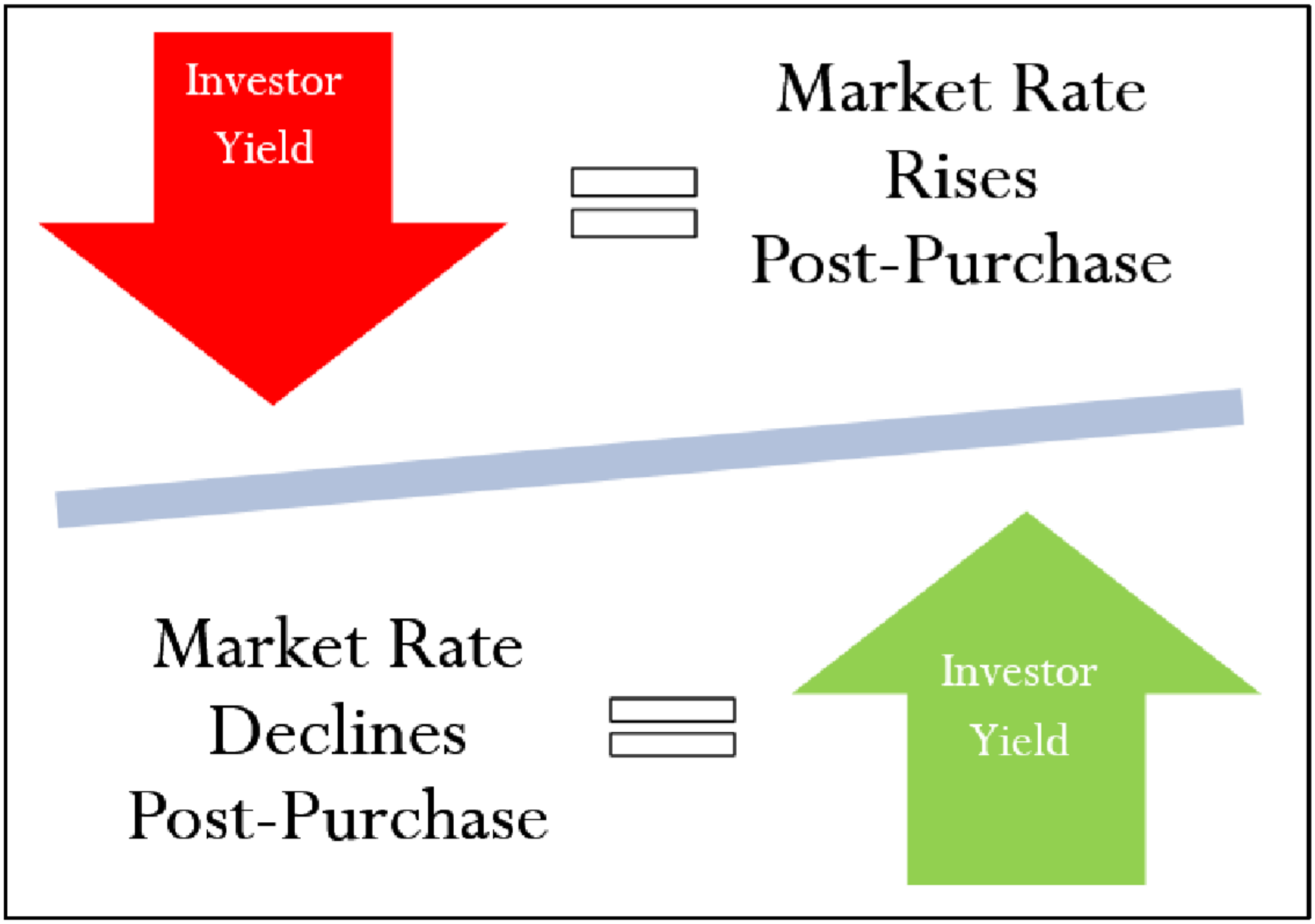

Investor Yield
=
Market Rate Rises Post-Purchase
Market Rate Declines Post-Purchase
=
Investor Yield

Figure 14.2.1: Investor's yield

The investor's yield consists of the coupon payments that have been received while the investor possesses the bond along with the difference achieved between the purchase price and the selling price. The above figure illustrates what happens to the investor’s yield by relating the post-purchase market rate and the investor's yield.

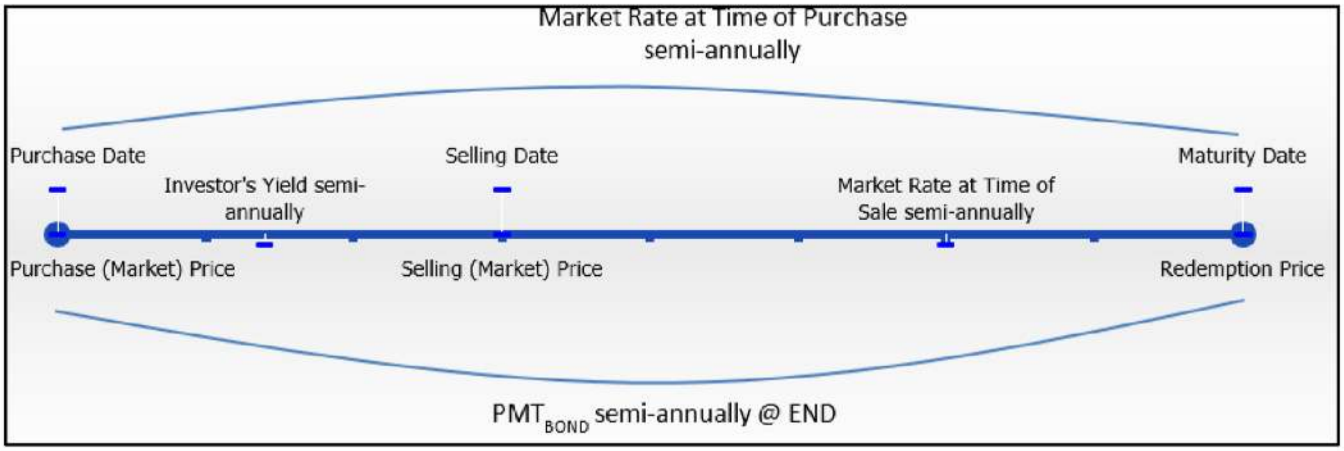
- When the market rate rises after the bond has been purchased, the investor experiences a lower yield than the original yield to maturity since the future bond selling price is reduced. Suppose that investors purchase the bond at a 5% market rate:
  - If they hold it until maturity, they achieve a 5% yield to maturity.
  - If they instead sell the bond before maturity when the market rate is 6%, the investor’s yield is less than 5%.
- When the market rate declines after the bond has been purchased, the investor experiences a higher yield than the original yield to maturity, since the future bond selling price is increased. Suppose again that investors purchase the bond at a 5% market rate:
  - If they hold it until maturity, they of course achieve a 5% yield to maturity.
  - If they sell the bond when the market rate is 4%, the investor’s yield is more than 5%.

The Formula

The existing bond formulas are sufficient for calculating the investor’s yield. You need Formula 14.2 to determine the bond coupon payment amount. You use Formula 14.3 to calculate both the purchase price and the selling price. You then use it a third time to solve for the periodic interest rate. When you solve for i, the future value of the bond that you substitute into the formula must be the selling price of the bond at the future date, not the redemption price.

How It Works

Follow these steps to calculate the investor’s yield:


Market Rate at Time of Purchase semi-annually
Purchase Date
Investor’s Yield semi-annually
Selling Date
Market Rate at Time of Sale semi-annually
Maturity Date
Purchase (Market) Price
Selling (Market) Price
Redemption Price
PMT_BOND semi-annually @ END

Step 1: Draw a timeline like the one depicted here, extending from the purchase date to the maturity date. Clearly indicate the selling date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the bond interest payment.

Step 3: If needed, calculate the purchase price using Formula 14.3. As in previous procedures, use Formula 11.1 to calculate the semi-annually based N representing both the total number of compounds and the total number of payments. Use the market rate at the time of purchase.

Step 4: If needed, calculate the selling price using Formula 14.3 in the same manner. Use the market rate at the time of sale.

Step 5: Solve for the nominal rate of interest (IY) between the purchase date and the selling date. The purchase price is the date price, and the selling price is the future value (FV).

- If using a manual method, substitute all known numbers into Formula 14.3 and solve for the periodic interest rate (i). Convert it into a nominal interest rate using Formula 9.1.

- Alternatively, if using technology such as a calculator, then input all known variables and solve directly for the nominal interest rate (IY).

Important Notes

The same two requirements that applied to calculating yield to maturity persist when you calculate investor's yield:

1. Trading takes place only on interest payment dates.
2. The bondholder reinvests all coupon payments at the same rate of interest.

? Exercise 14.2.2: Give It Some Thought

In each of the following, determine whether the investor's yield increases, decreases, or remains the same.

a. A bond is purchased when the market yield is 5% and then sold when the market rate is 5%.
b. A bond is purchased when the market yield is 5% and then sold when the market rate is 6%.

Answer

a. Increases; a lower market rate causes the selling price to rise.
b. Same; no change in the market rate results in no change to the yield.
c. Decreases; a higher market rate causes the selling price to decline

✓ The Investor's Yield

A $1,000 face value bond with a 7% coupon and 12 years to maturity was purchased for $1,084.68 when market rates were 6%. It sold seven years later for $920.87, when market rates were 9%. What yield did the bondholder realize?

Solution

The investor's yield represents the nominal rate of interest on the bond during the time the investor owned the bond, or IY.

What You Already Know

Step 1:

The timeline for the purchase and sale of the bond appears below.

Coupon Interest Payment: CPN = 7%, CY = 2, Face Value = $1,000 Purchase/Sale of

Bond: PV = $1,084.68, FV = $920.87, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Held = 7

How You Will Get There

Step 2:

Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

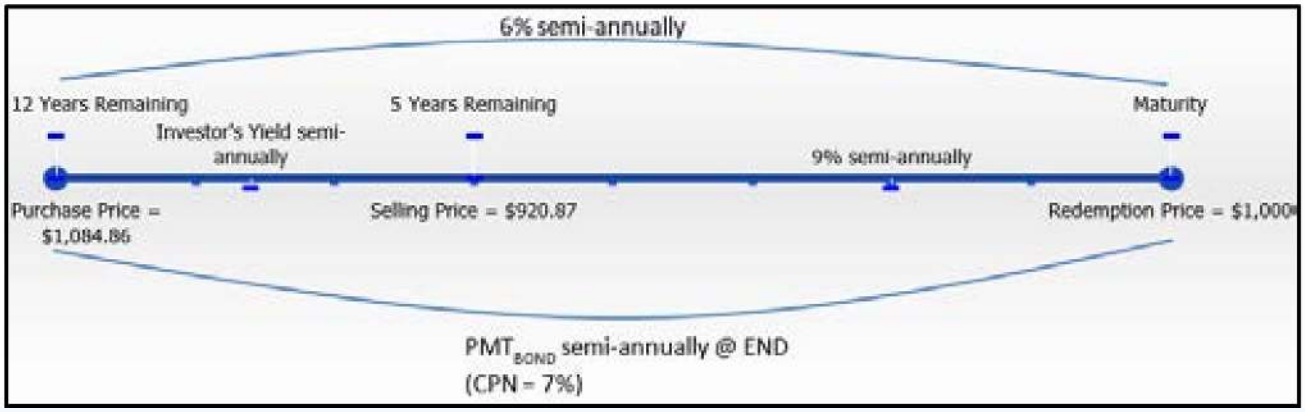
Determine the bond purchase price. It is known at $1,084.68 and does not need to be calculated. This is the date price for step 5.

Step 4:

Determine the bond selling price. It is known at $920.87 and does not need to be calculated. This is the future value (FV) for step 5.

Step 5:

Apply Formulas 11.1, 14.3, and 9.1 to determine the investor's yield (IY). Solve Formulas 14.3 and 9.1 using your calculator.


6% semi-annually
12 Years Remaining
Investor's Yield semi-annually
5 Years Remaining
Maturity
9% semi-annually
Purchase Price = $1,084.86
Selling Price = $920.87
Redemption Price = $1,000
PMT$_{BOND}$ semi-annually @ END
(CPN = 7%)

Perform

Step 2:

$$PMT_{BOND} = \$1,000 \times \frac{0.07}{2} = \$35$$

Step 5:

$$N = 2 \times 7 = 14 \text{ (compounds and payments)}$$

$$\$1,084.68 = \frac{\$920.87}{(1+i)^{14}} + \$35 \left[\frac{1 - \left[\frac{1}{1+i}\right]^{14}}{i} \right] \quad i = \frac{IY}{2}$$

$$\text{Calculator output } = IY = 4.6003\%$$

Calculator Instructions

Mode	N	I/Y	PV
END	14	Answer: 4.600320	-1084.68

PMT	FV	P/Y	C/Y
35	920.87	2	2

The bond market price dropped significantly because of the 3% rise in the market rate over the seven-year time period. Thus, the investor does not realize the original yield to maturity of 6% (the market rate), and in fact the investor's yield has dropped to 4.6003% compounded semi-annually.

✓ What Yield Do You Realize if You Sell Your Bond Now?

In the opening discussions to Sections 14.1 and 14.2, you had invested in 10 Government of Canada $5,000 face value bonds with a 5% coupon and 20 years remaining to maturity, paying $4,699.02 each when prevailing bond rates were 5.5%. You notice today, 10 years later, that prevailing bond rates have dropped to 3.35%. What investor's yield would you realize if you sold your bonds today?

Solution

You need to calculate the yield for just one of the 10 bonds since, of course, the yield is the same across all bonds. The investor's yield represents the nominal rate of interest on the bond during the time you own the bond, or IY.

What You Already Know

Step 1:

The timeline for purchase and sale of the bond appears below.

Coupon Interest Payment: CPN = 5%, CY = 2, Face Value = $5,000

Purchase of Bond: PV = $4,699.02

Sale of Bond: FV = $5,000, IY = 3.35%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 10

Purchase/Sale of Bond: PV = $4,699.02, FV = Sale Price, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Held = 10

How You Will Get There

Step 2:

Apply Formula 14.2 to determine the periodic bond interest payment.

Step 3:

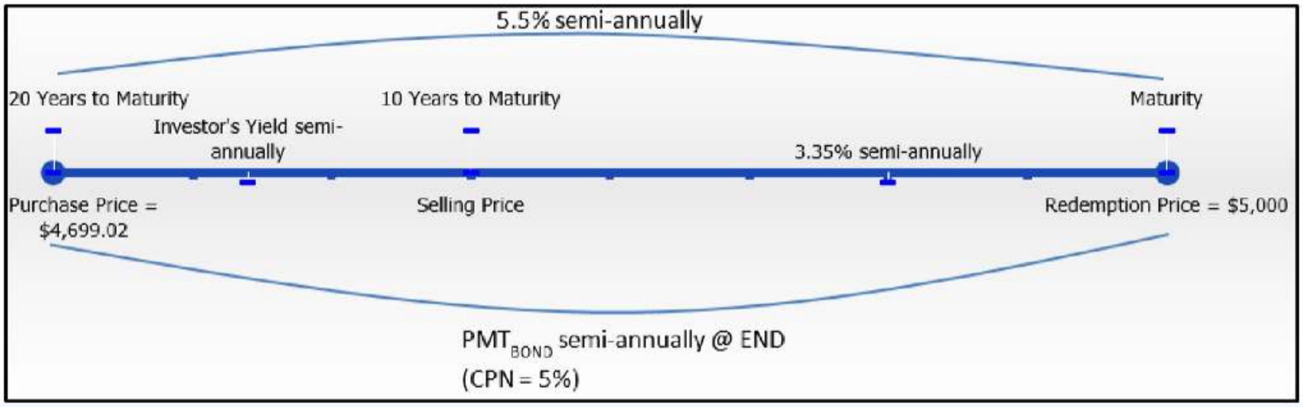
Determine the bond purchase price. It is known at $4,699.02 and does not need to be calculated. This is the date price for step 5.

Step 4:

Determine the bond selling price by applying Formulas 9.1, 11.1, and 14.3. This is the future value (FV) for step 5.

Step 5:

Apply Formulas 11.1, 14.3, and 9.1 to determine the investor's yield (IY). Solve Formulas 14.3 and 9.1 using the calculator.


5.5% semi-annually
20 Years to Maturity
Investor's Yield semi-annually
10 Years to Maturity
Maturity
3.35% semi-annually
Purchase Price = $4,699.02
Selling Price
Redemption Price = $5,000
PMT_{BOND} semi-annually @ END
(CPN = 5%)

Perform

Step 2:

$$PMT_{BOND} = \$5,000 \times \frac{0.05}{2} = \$125$$

Step 4:

$i = 3.35\%/2 = 1.675\%; N = 2 \times 10 = 20$ (compounds and payments)

$$\text{Date Price } = \frac{\$5,000}{(1+0.01675)^{20}} + \$125\left[\frac{1-\left[\frac{1}{1+0.01675}\right]^{20}}{0.01675}\right] = \$5,696.14$$

Step 5:

$$N = 2 \times 10 = 20 \text{ (compounds and payments)}$$

$$\$4,699.02 = \frac{\$5,696.14}{(1+i)^{20}} + \$125\left[\frac{1-\left[\frac{1}{1+i}\right]^{20}}{i}\right] \quad i = \frac{IY}{2}$$

$$\text{Calculator output } = IY = 6.8338\%$$

Calculator Instructions

Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
4	END	20	3.35	Answer: 5,696.138252	125	5000	2	2
5	√	√	Answer: 6.833820	-4,699.02	√	5696.14	√	√

The bond market price increased significantly over the 10-year time period because the market rate dropped from 5.5% to 3.35%. Thus, the investor realizes more than the original yield to maturity of 5.5% (the market rate), and in fact the investor's yield has risen to 6.8338% compounded semi-annually.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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14.3: Sinking Fund Schedules

Individuals and businesses should always plan to save toward their future goals. A sinking fund represents one way of accomplishing this, earning interest while regular contributions build up, all to reach a specified target at the end of the period.

For instance, let's say you are a production manager. In your desk inbox a consultant's report warns about the worrisome state of your production facilities and warehousing operations. Because your current production machinery is really showing its age, you need to replace it within three years. Meanwhile, increasing demand makes a new warehouse in Scarborough necessary within five years. The costs of each project are forecast at $1 million and $3 million, respectively. You see in the financial reports for the company that your company averages annual net profits of about $750,000. Your company needs to start setting aside the required funds now for these urgent projects.

On your way home, you pass by a newly constructed residential neighborhood. You have your eye on a starter home there with a list price of approximately $325,000. The 5% down payment required by the mortgage lender is $16,250. You have been making end-of-month contributions of $239 for the past four years into a savings plan earning 5% compounded monthly. On a whim, you head into the development, where you find a show home currently on the market for $250,000. You are quite smitten with the home and know that it will not take long to sell. Before putting in an offer, you wonder if your savings plan has enough funds to meet the 5% down payment requirement today.

Sinking Funds

A **sinking fund** is a special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds are on hand by a specified date to meet a future savings goal or debt obligation. In its simplest terms, it is a financial savings plan. As the definition indicates, it has either of two main purposes:

1. **Capital Savings**. When your goal is to acquire some form of a capital asset by the end of the fund, you have a capital savings sinking fund. What is a capital asset? It is any tangible property that is not easily converted into cash. Thus, saving up to buy a home, car, warehouse, or even new production machinery qualifies as capital savings.
2. **Debt Retirement**. When your goal is to pay off some form of debt by the end of the fund, then you have a debt retirement sinking fund. Perhaps as a consumer if you were able to get a 0% interest plan with no payments for one year; you might want to make monthly payments into your own savings account such that you would have the needed funds to pay off your purchase when it comes due. Businesses usually set up these funds for the retirement of stocks, bonds, and debentures.

Whether the sinking fund is for capital savings or debt retirement, the mathematical calculations and procedures are identical. Now, why discuss sinking funds in the chapter about bonds? Many bonds carry a sinking fund provision. Once the bond has been issued, the company must start regular contributions to a sinking fund because large sums of money have been borrowed over a long time frame; investors need assurance that the bond issuer will be able to repay its debt upon bond maturity. For example, in the chapter opener the profits needed to repay the financing for the Bipole III project will not just miraculously appear in the company's coffers. Instead, over a period of time the company will accumulate the funds through saved profits.

To provide further assurance to bondholders, the sinking fund is typically managed by a neutral third party rather than the bond-issuing company. This third-party company ensures the integrity of the fund, working toward the debt retirement in a systematic manner according to the provisions of the sinking fund. Investors much prefer bonds or debentures that are backed by sinking funds and third-party management because they are less likely to default.

In the case of bonds or debentures, sinking funds are most commonly set up as ordinary simple annuities that match the timing of the bond interest payments. Thus, when a bond issuer makes an interest payment to its bondholders, it also makes an annuity payment to its sinking fund. In other applications, any type of annuity is possible, whether ordinary or due and general or simple.

Complete Ordinary Sinking Fund Schedules

A complete **sinking fund schedule** is a table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity. It is very similar to an amortization schedule except that (1) the balance increases instead of decreasing, and (2) the interest is being earned instead of being paid.

The Formula

To complete the sinking fund schedule, you must calculate the interest earned and thus apply Formula 13.1, which calculates the interest portion of a single payment:

$$INT = BAL \times \left((1+i)^{\frac{CY}{PY}} - 1\right)$$

Aside from that formula, the table is mathematically simple, requiring only basic addition.

How It Works

Follow these steps to develop a complete sinking fund schedule:

Step 1: Draw a timeline. Identify all of your time value of money variables ($N, IY, FV_{ORD}, PMT, PV, PY, CY$). If either N or PMT is unknown, solve for it using an appropriate formula. Remember to round PMT to two decimals.

Step 2: Set up a sinking fund schedule following the template below, (the cells are marked with the step numbers that follow). The number of payment rows in your table is equal to the number of payments in the annuity (N).

Payment Number	Payment Amount at End ($) ($PMT$)	Interest Earned or Accrued ($) ($INT$)	Principal Balance Accumulated at End of Payment Interval ($) ($BAL$)
0 - Start	N/A	N/A	(3)
1	(4)	(5)	(6)
...			
Last Payment			
Total	(9)	(9)	N/A

Step 3: Fill in the present value of the annuity (PV). You typically start with a zero balance when you save toward a future goal, although sometimes you may make an immediate lump-sum deposit to start the account.

Step 4: Fill in the rounded annuity payment (PMT) all the way down the column, including the final payment row.

Step 5: Calculate the interest portion of the sinking fund's current balance (INT) using Formula 13.1. Round the number to two decimals for the table, but keep track of all decimals throughout.

Step 6: Calculate the new balance by taking the previous un-rounded balance on the line above and adding both the annuity payment and the un-rounded interest on the current row. Round the result to two decimals for the table, but keep track of all decimals for future calculations.

Step 7: Repeat steps 5 and 6 for each annuity payment until the schedule is complete.

Step 8: Since all numbers are rounded to two decimals throughout the table, check the table for the "missing penny." This is the same "missing penny" concept as discussed in the Chapter 13 amortization schedules. Ensure that previous balances plus current payments and interest equal the new balance. If not, adjust the interest earned amount as needed. Do not adjust the balance in the account or the payment. Make as many penny adjustments as needed; they typically appear in pairs such that when you must increase one line by one penny, another subsequent line reduces the interest by one penny.

Step 9: Total up the principal payments made and interest earned for the entire schedule.

Important Notes

When you work with sinking fund schedules, remember three key considerations with respect to the annuity type, final payment adjustment, and the rounding of the sinking fund payment:

1. You can use the ordinary sinking fund schedule for both simple and general annuities.
    - Under a simple sinking fund, the payments and interest always occur together on the same line of the schedule, which raises no complications of realized versus accrued interest.
    - However, for the general annuity, although you could create a table where interest is converted to principal between each annuity payment, it makes no fundamental or mathematical difference in the balance of the fund at any given time. Therefore, this textbook's sinking fund schedules reflect the interest earned regardless of whether it has been converted to principal or remains in accrued format as of each annuity payment date.
2. You are not legally required to be precise when saving up for a future goal. You do not need to adjust the final payment to bring the sinking fund to an exact balance. If the account balance is marginally below or above the goal, you can either top it up as needed or use the extra funds for another purpose.

For the above reason, the sinking fund annuity payment amount is frequently rounded off to a convenient whole number. In this textbook you will be clearly instructed if such rounding is to occur; otherwise, use the exact annuity payment rounded to two decimals.

Your BAII Plus Calculator

Although the calculator has no function called "sinking fund," sinking funds have the same characteristics as amortization schedules. Therefore, use the AMORT function located on the 2nd shelf above the PV key to create the sinking fund schedule. You can find full instructions for the AMORT function in Chapter 13. The key difference in using this function for sinking funds is that the principal grows instead of declines.

With respect to the cash flow sign convention, your PV (if not zero) and PMT are negative, since money is being invested into the account. The FV is a positive number, since it can be withdrawn in the future. As in Chapter 13, you need to input all time value of money buttons accurately before accessing this function.

The BAL and INT outputs are accurate and true to definition. The BAL represents the balance in the fund. The INT represents the interest earned for the specified payments. The PRN output is also accurate, but its definition is changed to represent the total of the annuity payments made and the interest earned. It represents the total increase in the balance of the fund over the course of the specified payments.

? Give It Some Thought

In an amortization schedule, the interest portion for each payment steadily declines with each annuity payment. What do you think happens to the interest earned with each payment in a sinking fund?

Answer

It steadily gets larger since the principal balance is increasing instead of decreasing.

✓ A Sinking Fund for a Bond

The City of Winnipeg issued a $500,000 face value bond with three years until maturity. It will make contributions at the end of every six months to a sinking fund earning 5.8% compounded semi-annually. Construct a complete sinking fund schedule and calculate the total principal contributions and interest components. Add text here.

Solution

You are to construct a complete sinking fund schedule for the bond. The schedule needs to provide the total principal contributions, which is the same as the total payments (PMT) made to the fund, as well as the total interest (INT) earned.

What You Already Know

Step 1:

The timeline for the City of Winnipeg’s sinking fund appears below.

FV_{ORD} = \$500,000, IY = 5.8%, CY = 12, PY = 2, Years = 3, PV = \$0

How You Will Get There

Step 1 (continued):

Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT).

Step 2:

Set up the sinking fund schedule.

Steps 3 and 4:

Fill in the original principal with zero (since this is the opening balance) and payment column with the PMT from step 1.

Steps 5 to 7:

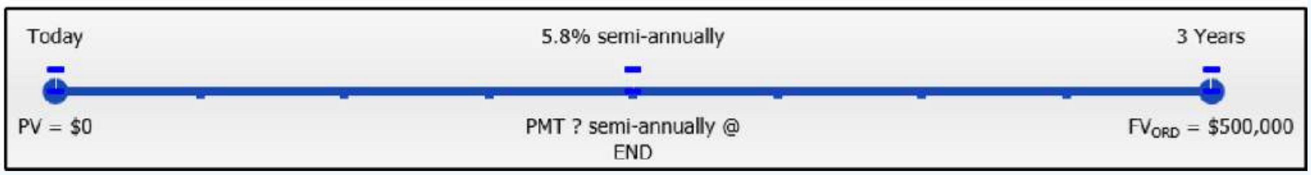
Use Formula 13.1 to calculate the interest and add the row to get the new balance for each line.

Step 8:

Check for the "missing penny."

Step 9:

Sum the interest portion as well as the total payments for the principal contribution.



Perform

Step 1 (continued):

$$i = 5.8\%/2 = 2.9\%; N = 2 \times 3 = 6 \text{ payments}$$

$$\$500,000 = PMT\left[\frac{\left[(1+0.029)^{\frac{2}{2}}\right]^{6}-1}{(1+0.029)^{\frac{2}{2}}-1}\right]$$

$$PMT = \frac{\$500,000}{\left[\frac{\left[(1+0.029)^{\frac{2}{2}}\right]^{6}-1}{(1+0.029)^{\frac{2}{2}}-1}\right]} = \$77,493.07$$

Steps 2 to 7 (with some calculations) are detailed in the table below.

Payment Number	Payment Amount at End (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0 - Start			\$0.00
1	\$77,493.07	(1) \$0.00	(2) \$77,493.07
2	\$77,493.07	(3) \$2,247.30	(4) \$157,233.44
3	\$77,493.07	(5) \$4,559.77	(6) \$239,286.28
4	\$77,493.07	\$6,939.30	\$323,718.65
5	\$77,493.07	\$9,387.84	\$410,599.56

Payment Number	Payment Amount at End ($) ($PMT$)	Interest Earned or Accrued ($) ($INT$)	Principal Balance Accumulated at End of Payment Interval ($) ($BAL$)
6	$77,493.07	$11,907.39	$500,000.02
Total	$464,958.42	$35,041.60	

(1) $\text{INT} = \$0 \times \left((1+0.029)^{\frac{2}{2}} - 1\right) = \$0$

(2) $\text{New Balance} = \$0.00 + \$77,493.07 + \$0.00 = \$77,493.07$

(3) $\text{INT} = \$77,493.07 \times \left((1+0.029)^{\frac{2}{2}} - 1\right) = \$2,247.298945$

(4) $\text{New Balance} = \$77,493.07 + \$77,493.07 + \$2,247.298945 = \$157,233.4389$

(5) $\text{INT} = \$157,233.4389 \times \left((1+0.029)^{\frac{2}{2}} - 1\right) = \$4,559.769729$

(6) $\text{New Balance} = \$157,233.4389 + \$77,493.07 + \$4,559.769729 = \$239,286.28$

Steps 8 to 9:

There are no rounding errors and the table is correct. Total the interest and principal contributions.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	6	5.8	0	Answer: -77,493.06705 Rekeyed as -77493.07	500000	2	2

Payment	P1	P2	BAL (Output)	INT (Output)
1	1	1	-77,493.07	0
2	2	2	-157,233.439	2,247.29903
3	3	3	-239,286.2788	4,559.769732
4	4	4	-323,718.6508	6,939.302084
5	5	5	-410,599.5617	9,387.840875
6	6	6	-500,000.019	11,907.38729
Total	1	6	-500,000.19 Note: Total contributions are PMT × 6	35,041.59901

The complete sinking fund schedule is shown in the table above. The total interest earned by the City of Winnipeg is $35,041.60 in addition to the $464,958.42 of principal contributions made. Note that the fund has an extra $0.02 in it after the three years.

Partial Ordinary Sinking Fund Schedules

For the same reasons as with amortization schedules, sometimes you are interested in creating partial sinking fund schedules for a specified range of payments and not the entire sinking fund. This may be because of any of the following reasons:

- The sinking fund schedule is too long.
- Your sole interest is in the interest portion during a specific period of time for accounting and tax purposes.
- You are assisting in the financial planning of the individual or organization at a particular point in time.

How It Works

For a partial sinking fund schedule, follow almost the same procedure as for a complete sinking fund schedule, with the following notable differences:

Step 2: Set up the partial sinking fund schedule according to the template provided here. The table identifies the corresponding step numbers.

Step 3: Calculate the balance accumulated in the account immediately before the first payment of the partial schedule. Recall the three calculations required to determine this value:

Payment Number	Payment Amount at End ($) (PMT)	Interest Earned or Accrued ($) (INT)	Principal Balance Accumulated at End of Payment Interval ($) (BAL)
Preceding Payment #	N/A	N/A	(3)
First Payment Number of Partial Schedule	(4)	(5)	(6)
...			
Last Payment Number of Partial Schedule			
Total	(9)	(9)	N/A

- Calculate the future value of the original principal. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments).
- Calculate the future value of all annuity payments already made. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).
- Sum the above two calculations to arrive at the principal balance (BAL) immediately before the first payment required in the partial sinking fund schedule.

Things To Watch Out For

When you use Excel to create amortization schedules, recall that your template does not correct for the “missing penny.” The situation is no different with respect to sinking fund schedules. Therefore, a schedule generated using the Excel template may have the occasional penny difference on any given line since each displayed number is rounded to two decimals, but the total interest and total principal are always correct since the un-rounded numbers are used in their calculations.

✓ A Partial Sinking Fund Schedule

The Alberta Capital Finance Authority (ACFA) issued a $200,000 face value bond with five years until maturity. The sinking fund provision requires payments to be made at the end of every quarter into a fund earning 4.4% compounded quarterly. Create a partial sinking fund schedule for the third year. Calculate the total interest earned and total contributions made in the third year.

Solution

You are to construct a partial sinking fund schedule for the third year of the loan along with the total interest earned and total contributions for the year.

What You Already Know

Step 1:

The timeline for the bond appears below.

FV_{ORD} = \$200,000, IY = 4.4%, CY = 4, PY = 4, Years = 5, PV = \$0

How You Will Get There

Step 1 (continued):

Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT).

Step 2:

Set up the partial sinking fund schedule according to the template.

Step 3:

There is no single payment at the beginning (PV = \$0). Calculate just the future value of the first eight annuity payments using Formulas 11.1, and 11.2. The balance in the account prior to the third year is equal to the FV_{ORD}.

Step 4:

Fill in the payment column for the four payments made in the third year.

Steps 5 to 7:

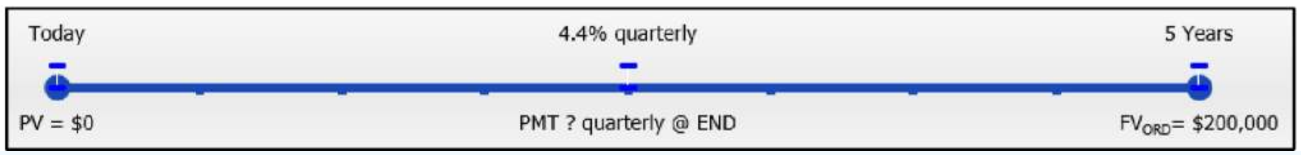
Use Formula 13.1 to calculate interest and add the row to get the new balance for each line.

Step 8:

Check for the "missing penny."

Step 9:

Total up the interest portion as well as the total payments for the principal contribution.



Perform

Step 1 (continued):

$$i=4.4\%/4=1.1\%; N=4\times5=20 \text{ payments}$$

$$\$200,000=PMT\left[\frac{\left[(1+0.011)^{\frac{4}{4}}\right]^{20}-1}{(1+0.011)^{\frac{4}{4}}-1}\right]$$

$$PMT=\frac{\$200,000}{\left[\frac{\bigl[(1+0.011)^{\frac{4}{4}]^{20}-1}-1}{(1+0.011)^{\frac{4}{4}}-1}\right]}=\$8,994.98$$

Steps 2 to 7 (with some calculations, including step 3) are detailed in the table below.

Payment Number	Payment Amount at End (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
8			(1) \$74,792.09
9	\$8,994.98	(2) \$822.71	(3) \$84,609.78

Payment Number	Payment Amount at End ($) ($PMT$)	Interest Earned or Accrued ($) ($INT$)	Principal Balance Accumulated at End of Payment Interval ($) ($BAL$)
10	$8,994.98	$930.71	$94,535.47
11	$8,994.98	$1,039.89	$104,570.34
12	$8,994.98	$1,150.27	$114,715.59
Total	$35,979.92	$3,943.58	

(1) **Step 3:**

$N = 4 \times 2 = 8$ payments;

$$FV_{ORD} = \$8,994.98 \left[\frac{\left[(1+0.011)^{\frac{4}{4}} - 1\right]}{(1+0.011)^{\frac{4}{4}} - 1} \right] = \$8,994.98 \left[\frac{0.091463}{0.011} \right] = \$74,792.0893$$

(2) $INT = \$74,792.0893 \times \left((1+0.011)^{\frac{4}{4}} - 1\right) = \$822.712982$

(3) $\text{New Balance } = \$74,792.0893 + \$8,994.98 + \$822.712982 = \$84,609.78229$

Steps 8 to 9:

In this case there are no rounding errors and the table above is correct. Total the interest and principal contributions.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	20	4.4	0	Answer: -8,994.980862 Re-keyed as -8994.98	200000	4	4

Payment	P1	P2	BAL (Output)	INT (Output)
8	8	8	-74,792.0893	Not needed
9	9	9	-84,609.78228	822.712982
10	10	10	-94,535.46989	930.707605
11	11	11	-104,570.3401	1,039.890169
12	12	12	-114,715.5938	1,150.273741
Total	9	12	-114,715.5938 Note: Total contributions are $PMT \times 4$	3943.584497

The partial sinking fund schedule for the third year is shown in the table above. ACFA contributed $35,979.92 to the fund and earned total interest of $3,943.58.

Sinking Funds Due Schedules

While ordinary sinking funds are typical for bonds, capital savings sinking funds can take any form. Whether saving personally for the down payment on a house or saving at work for the acquisition of a warehouse, the investor determines the timing of the annuity payments. Sinking funds due require a small modification to the headers in the sinking fund due schedule, as illustrated in the table. Recall that the payment occurs at the beginning of the payment interval.

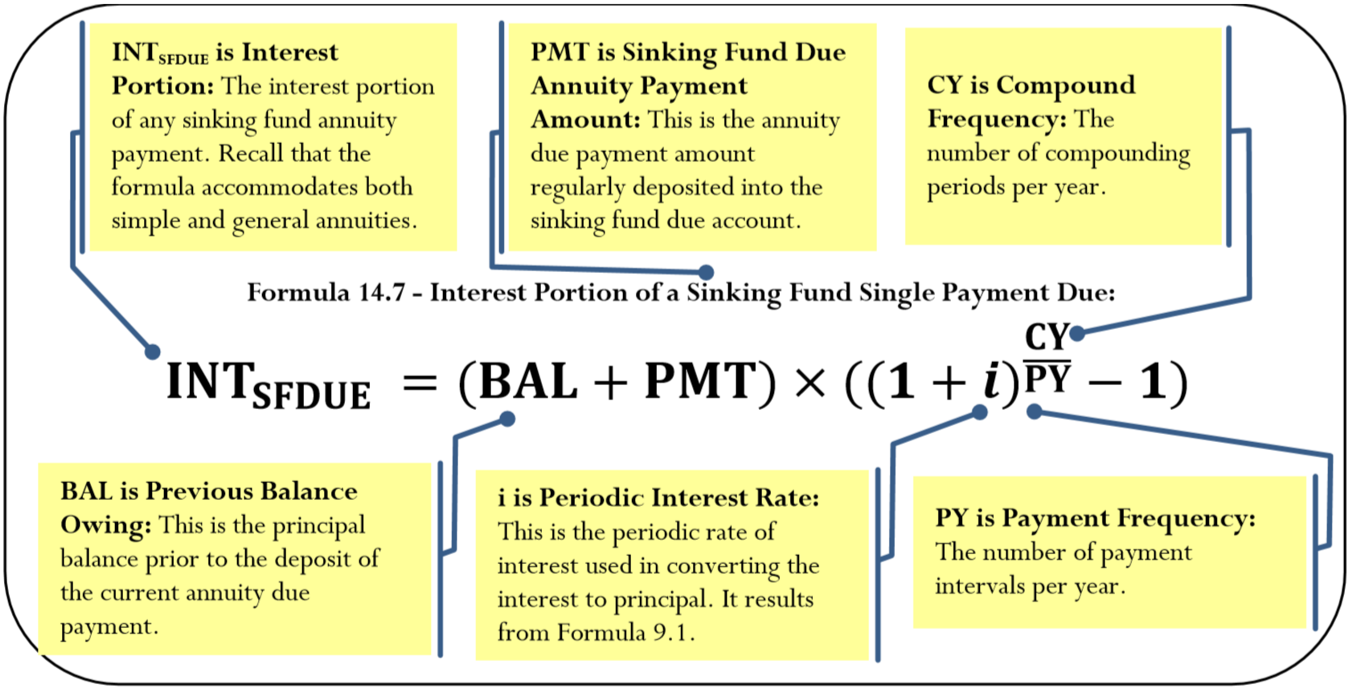
Payment Number	Payment Amount at BGN ($) ($PMT$)	Interest Earned or Accrued ($) ($INT$)	Principal Balance Accumulated at End of Payment Interval ($) ($BAL$)
0 - Start	N/A	N/A	(3)
1	(4)	(5)	(6)
...			
Last Payment			
Total	(9)	(9)	N/A

1. Ensure that any unknown variables like PMT are calculated by an annuity due formula.
2. The first payment occurs at the beginning of the first time period. Hence, this first payment receives interest, unlike the first payment in an ordinary annuity, which receives no interest during the first time period because it is paid at the end.

In accordance with the practice for previous schedules, do not distinguish between earned and accrued interest for general sinking funds due. Both amounts are represented in the third column of the table. If you need a partial sinking fund due schedule, its structure is the same as the table for ordinary sinking funds, although the headers change to match the table above.

The Formula

Calculating the interest portion in a sinking fund due schedule requires a slight modification to Formula 13.1 for the interest portion of a single payment. In an annuity due, the payment is deposited at the beginning of the payment interval and therefore accrues interest during the current payment interval. This means that you must add the annuity payment to the previous balance before calculating interest.



INT_{SFDUE} is Interest Portion: The interest portion of any sinking fund annuity payment. Recall that the formula accommodates both simple and general annuities.

PMT is Sinking Fund Due Annuity Payment Amount: This is the annuity due payment amount regularly deposited into the sinking fund due account.

CY is Compound Frequency: The number of compounding periods per year.

Formula 14.7 - Interest Portion of a Sinking Fund Single Payment Due:

$$INT_{SFDUE}=(BAL+PMT)\times\left((1+i)^{\frac{CY}{PY}}-1\right)$$

BAL is Previous Balance Owing: This is the principal balance prior to the deposit of the current annuity due payment.

i is Periodic Interest Rate: This is the periodic rate of interest used in converting the interest to principal. It results from Formula 9.1.

PY is Payment Frequency: The number of payment intervals per year.

Formula 14.7

How It Works

The steps for creating either a complete or partial sinking fund due schedule remain much the same as for an ordinary sinking fund schedule. You require the following adaptations:

- Step 2: Use the sinking fund due table instead of ordinary sinking fund table.
- Step 5: Use Formula 14.7 instead of Formula 13.1.

Important Notes

Your BAII Plus Calculator.

Probably the most significant change occurs here. The AMORT function is designed only for ordinary amortization, but you can easily adapt it to sinking funds due. These changes are similar to the adaptations required for amortization schedules due.

- **P1 and P2:** In a sinking fund due, since the first payment occurs today (time period 0), the second payment is at time period 1, and so on. So the payment number of a sinking fund due is always one higher than the payment number of an ordinary sinking fund; always add 1 to the payment number being calculated. For example, if interested only in payment seven, set both P1 and P2 to 8. Or if interested in the payment range 14 through 24, set P1 = 15 and P2 = 25.
- **BAL:** To generate the output, the payment numbers are one too high. This results in the balance being increased by one extra payment. To adapt, manually decrease the balance by removing one payment (or with BAL on your display, use the shortcut key sequence of − RCL PMT =).
- **INT and PRN:** Both of these numbers are correct. Recall that PRN reflects the sinking fund payment (PMT) and interest amount (INT) together.

✓ Saving for a Vehicle Down Payment

In an effort to be more environmentally friendly, Bernard is considering leasing a Chevrolet Volt. The lease terms require a down payment of $2,000. To save up, Bernard starts making quarterly contributions today for the next year into a fund earning 5.3% semi-annually. Create a sinking fund due schedule and calculate his total payments and interest.

Solution

You are to construct a complete sinking fund due schedule for car savings along with the total interest (INT) and total principal contributions, or the total payments (PMT) made to the fund.

What You Already Know

Step 1:

The timeline for the car down payment savings plan appears below.

FV_{DUE} = \$2,000, IY = 5.3%, CY = 2, PY = 4, Years = 1, PV = \$0

How You Will Get There

Step 1 (continued):

Solve for the sinking fund due annuity payment (PMT) using Formulas 9.1, 11.1, and 11.3 (rearranging for PMT).

Step 2:

Set up the sinking fund due schedule.

Steps 3 and 4:

Fill in the original principal with zero (since this is the opening balance) and the payment column with the PMT from step 1.

Steps 5 to 7:

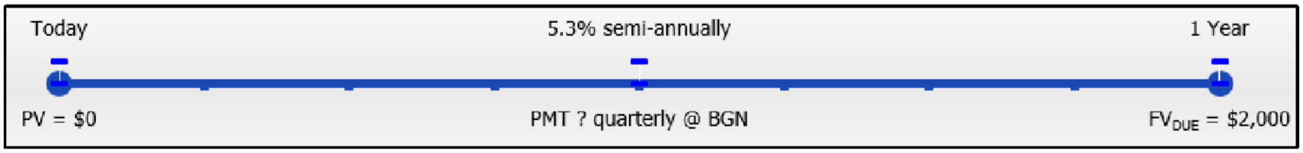
Use Formula 14.7 to calculate interest and add the row to get the new balance for each line.

Step 8:

Check for the "missing penny."

Step 9:

Total up the interest portion as well as the total payments for the principal contribution.



Perform

Step 1 (continued):

$i = 5.3\%/2 = 2.65\%; N = 4 \times 1 = 4$ payments

$$\$2,000 = PMT\left[\frac{\left[(1+0.0265)^{\frac{2}{4}}\right]^{4} - 1}{(1+0.0265)^{\frac{2}{4}} - 1}\right] \times (1+0.0265)^{\frac{2}{4}}$$

$$PMT = \frac{\$2,000}{\left[\frac{\left[(1+0.0265)^{\frac{2}{4}}\right]^{4} - 1}{(1+0.0265)^{\frac{2}{4}} - 1}\right] \times (1+0.0265)^{\frac{2}{4}}} = \$483.87$$

Steps 2 to 7 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount at BGN (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0 - Start			\$0.00
1	\$483.87	(1) \$6.37	(2) \$490.24

2	$483.87	(3) $12.82	(4) $986.93
3	$483.87	$19.36	$1,490.16
4	$483.87	$25.98	$2,000.02
Total			

(1) $INT_{\text{SFDUE}} = (\$0 + \$483.87) \times \left((1+0.0265)^{\frac{2}{4}} - 1\right) = \$6.369356$

(2) $\text{New Balance } = \$0.00 + s483.87 + \$6.369356 = \$490.239356$

(3) $\text{INT}_{SFDUE} = (\$490.239356 + \$483.87) \times \left((1+0.0265)^{\frac{2}{4}} - 1\right) = \$12.822555$

(4) $\text{New Balance } = \$490.239356 + \$483.87 + \$12.822555 = \$986.931911$

Steps 8 to 9:

Adjust for the "missing pennies" (noted in red) and total the payments and interest.

Payment Number	Payment Amount at BGN ($) ($PMT$)	Interest Earned or Accrued ($) ($INT$)	Principal Balance Accumulated at End of Payment Interval ($) ($BAL$)
0 - Start			$0.00
1	$483.87	(1) $6.37	(2) $490.24
2	$483.87	(3) $12.82	(4) $986.93
3	$483.87	$19.36	$1,490.16
4	$483.87	$25.98	$2,000.02
Total	$1,935.48	$64.54	

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	4	5.3	0	Answer: -483.865761 Rekeyed as -483.87	2000	4	2

Payment	P1	P2	BAL (Output)	INT (Output)
1	2	2	-490.239356	6.369
2	3	3	-986.931911	12.822555
3	4	4	-1,490.162611	19.360699
4	5	5	-2,000.017519	25.984907
Total	2	5	-2,000.017519	64.537518

The complete sinking fund due schedule is shown in the table above. The total interest earned by Bernard is $64.54 in addition to the $1,935.48 of principal contributions made. Note that the fund has an extra $0.02 in it because of rounding.

Contributors and Attributions

- Template:ContribBusiMathOlivier

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14.4: Debt Retirement and Amortization

Issuing, buying, and selling bonds results in financial obligations and accounting responsibilities. For instance, your firm is about to issue marketable bonds to finance a major venture in the near future. These bonds require a sinking fund provision to ensure investor confidence. However, the company first needs to foresee its financial obligations if it issues the bonds. How much interest will the company need to pay out to its bondholders annually? What annual sum will it deposit into the sinking fund to satisfy the provision? How does the liability side of the company's balance sheet reflect the fund's provisions? All these questions need to be answered so that you can make an informed decision.

Meanwhile, the finance department reports that your company invested in marketable bonds purchased at a discount. This means your company will benefit from the future bond interest payments and also realize the bond's redemption price upon maturity. Your firm's accounting records must show capital gains being realized over the years, in the form of the difference between the face value and the discounted amount at which the bonds were purchased.

As you can see, the decision to issue or invest in a bond has many financial and tax implications that a company or an individual investor must address. This section explores the financial obligations associated with retiring bond debt and accurate recording of the debt. Also examined is the accounting for premiums and discounts, both of which have tax implications for bondholders and issuers.

Debt Retirement

When an organization issues a bond, the three primary financial implications involve the bond's interest payments, the sinking fund payments, and the balance sheet liability tied to the bond.

1. **Annual Bond Interest Payments**. As discussed earlier in this chapter, most bonds compound semi-annually with semi-annual interest payments, thus $CY = PY$. Because companies manage their books by their fiscal year, they are more interested in the annual total payment commitment than in the amount of each payment, PMT_{BOND}. Therefore, the annual total payment commitment is represented by

   $$\text{Annual Bond Interest Payment} = PMT_{BOND} \times PY$$

   Substituting Formula 14.2 in place of PMT_{BOND}:

   $$\text{Annual Bond Interest Payment} = \text{Face Value } \times \frac{CPN}{CY} \times PY$$

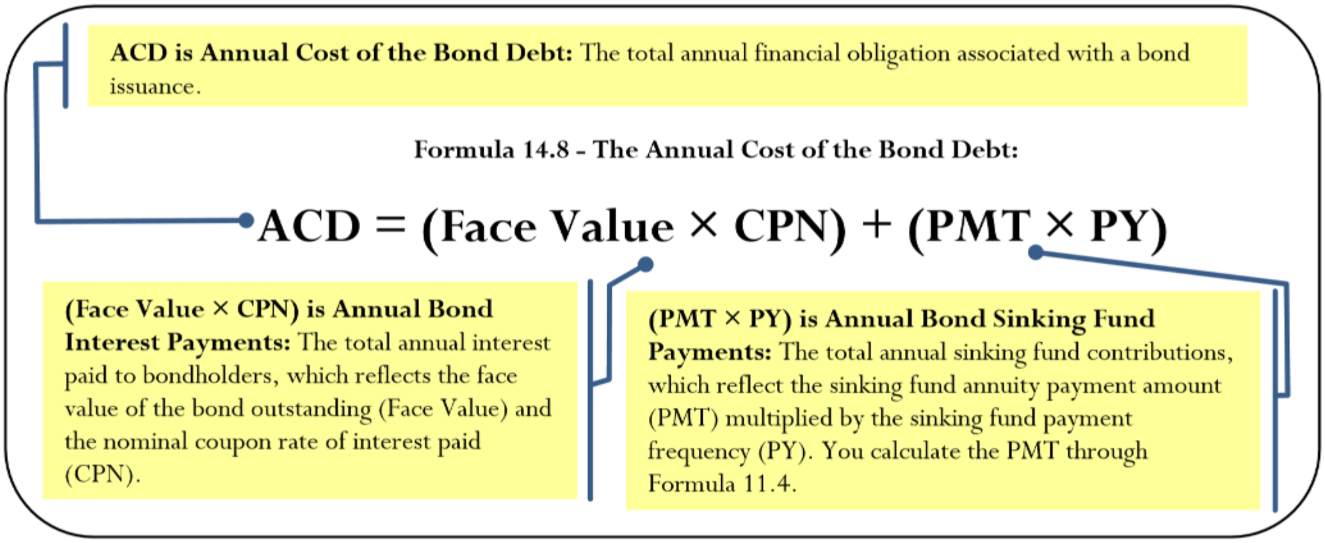
   Since $CY = PY$, simplifying the above formula produces the most direct method of arriving at the annual payment amount:

   $$\text{Annual Bond Interest Payment} = \text{Face Value } \times CPN$$

2. **Annual Bond Sinking Fund Payments**. Sinking fund payments are most commonly set up to match the timing of the bond's interest payments. Hence, these payments are semi-annual. Again, the focus is on the annual sinking fund obligation, which you calculate by multiplying the sinking fund payment by the payment frequency, or $PMT \times PY$. The total of the annual bond interest payments and the annual sinking fund payments is referred to as the **annual cost of the bond debt** or the periodic cost of the bond debt.
3. **Balance Sheet Liability**. The bond represents a debt for the issuing organization. Through the sinking fund, the company saves up money to extinguish that debt. The amount of debt that is outstanding is the difference between the money owing and the money saved at any given point in time. Thus, the **book value of the bond debt** represents the difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time. For example, if the company issued $10 million in bonds and has accumulated $1 million in its sinking fund, the book value of the debt is $9 million. This would be the balance owing if the company used the sinking fund to retire a portion of the bond debt at face value.

The Formula

To calculate the annual cost of the bond debt, you combine both the annual bond interest payments and annual bond sinking fund payments into a single formula.



ACD is Annual Cost of the Bond Debt: The total annual financial obligation associated with a bond issuance.

Formula 14.8 - The Annual Cost of the Bond Debt:

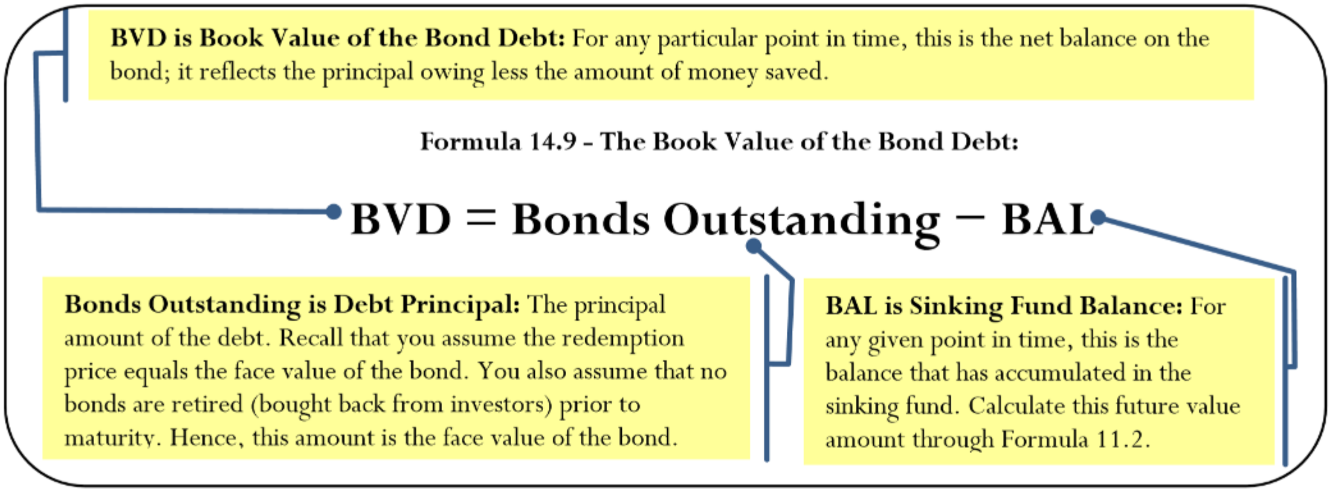
$$\text{ACD} = (\text{Face Value} \times \text{CPN}) + (\text{PMT} \times \text{PY})$$

(Face Value × CPN) is Annual Bond Interest Payments: The total annual interest paid to bondholders, which reflects the face value of the bond outstanding (Face Value) and the nominal coupon rate of interest paid (CPN).

(PMT × PY) is Annual Bond Sinking Fund Payments: The total annual sinking fund contributions, which reflect the sinking fund annuity payment amount (PMT) multiplied by the sinking fund payment frequency (PY). You calculate the PMT through Formula 11.4.

Formula 14.8

Formula 14.9 lets you calculate the book value of the bond debt.



BVD is Book Value of the Bond Debt: For any particular point in time, this is the net balance on the bond; it reflects the principal owing less the amount of money saved.

Formula 14.9 - The Book Value of the Bond Debt:

$$\text{BVD} = \text{Bonds Outstanding} - \text{BAL}$$

Bonds Outstanding is Debt Principal: The principal amount of the debt. Recall that you assume the redemption price equals the face value of the bond. You also assume that no bonds are retired (bought back from investors) prior to maturity. Hence, this amount is the face value of the bond.

BAL is Sinking Fund Balance: For any given point in time, this is the balance that has accumulated in the sinking fund. Calculate this future value amount through Formula 11.2.

Formula 14.9

How It Works

Follow these steps to calculate the annual cost of the bond debt:

Step 1: Identify the face value of the bond and the coupon rate.

Step 2: If the sinking fund payment (PMT) and payment frequency (PY) are known, skip to step 4. Otherwise, draw a timeline for the sinking fund and identify known variables.

Step 3: Calculate the sinking fund payment using Formula 11.4.

Step 4: Calculate the annual cost of the bond debt using Formula 14.8.

Follow these steps to calculate the book value of the bond debt:

Step 1: Identify the face value of the bond.

Step 2: If the balance in the sinking fund (BAL) is known, skip to step 5. Otherwise, draw a timeline for the sinking fund and identify known variables.

Step 3: Calculate the sinking fund payment using Formula 11.4.

Step 4: Calculate the future value of the sinking fund at the point of interest using Formula 11.2. This is the sinking fund balance, or BAL.

Step 5: Calculate the book value of the bond debt using Formula 14.9.

? Give It Some Thought

Suppose a company decides not to wait until maturity and retires some of its bond debt early on an interest payment date. What happens to the BVD if the debt is retired when:

a. the market rate equals the coupon rate?
b. the market rate is higher than the coupon rate?
c. the market rate is lower than the coupon rate?

(Hint: Think about what happens to the market price of the bond.)

Answer

a. The BVD is unchanged. If the two rates are the same, then the bonds sell at par value. Money is transferred dollar for dollar from the sinking fund to directly reduce the bond debt.
b. The BVD decreases. If the market rate is higher, the bonds sell at a discount. The company acquires more bonds at the lower bond price, reducing the number of outstanding bonds.
c. The BVD increases. If the market rate is lower, the bonds sell at a premium. The company acquires fewer bonds at the higher bond price, leaving a larger number of outstanding bonds than by redeeming at par.

✓ The Annual Cost of the Bond Debt

The Bank of Montreal issued a $10,000,000 face value bond carrying a 5.1% coupon with 30 years until maturity. The bond has a matching sinking fund provision for which monies are invested at 4.5%. Calculate the annual cost of the bond debt. Assume all compounding and payments are semi-annual.

Solution

You need to calculate the annual cost of the bond debt (ACD).

What You Already Know

Step 1:

Face Value = $10,000,000, CPN = 5.1%

Step 2:

The timeline for the sinking fund appears below.

FV_{ORD} = $10,000,000, IY = 4.5%, CY = 2, PY = 2, Years = 30, PV = $0

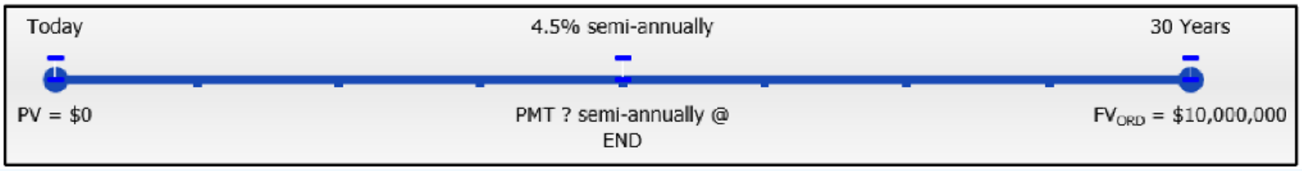
How You Will Get There

Step 3:

Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT).

Step 4:

Calculate the annual cost of the bond debt using Formula 14.8.



Perform

Step 3:

$i = 4.5\%/2 = 2.25\%; N = 2 \times 30 = 60$ payments

$$\$10,000,000 = PMT\left[\frac{\left[(1+0.0225)^{\frac{2}{2}}\right]^{60}-1}{(1+0.0225)^{\frac{2}{2}}-1}\right]$$

$$PMT = \frac{\$10,000,000}{\left[\frac{\left[(1+0.0225)^{\frac{2}{2}}\right]^{60}-1}{(1+0.0225)^{\frac{2}{2}}-1}\right]} = \$80,353.27$$

Step 4:

$$\begin{aligned} ACD &= (\$10,000,000 \times 0.051) + (\$80,353.27 \times 2) \\ &= \$510,000 + \$160,706.54 \\ &= \$670,706.54 \end{aligned}$$

Calculator Instructions

Mode	N	I/Y	PV
END	60	4.5	0

PMT	FV	P/Y	C/Y
Answer: -80,353.27482	10000000	2	2

Each year, the Bank of Montreal pays $510,000 in interest to its bondholders. Additionally, it deposits $160,706.54 to its sinking fund. Thus, the annual cost of the bond debt is $670,706.54 every year for the next 30 years.

Example 14.4.2: The Book Value of the Bond Debt

Using Example 14.4.1, calculate the book value of the bond debt after 10 years.

Solution

You must calculate the book value of the bond debt (BVD) after 10 years.

What You Already Know

Step 1:

Face Value = $10,000,000

Steps 2 and 3:

The sinking fund balance is unknown. From the previous example, the sinking fund payment is PMT = $80,353.27. The timeline for calculating the sinking fund after 10 years appears below.

FV_{ORD} = $10,000,000, IY = 4.5%, CY = 2, Years = 10, PMT = $80,353.27, PY = 2, PV = $0

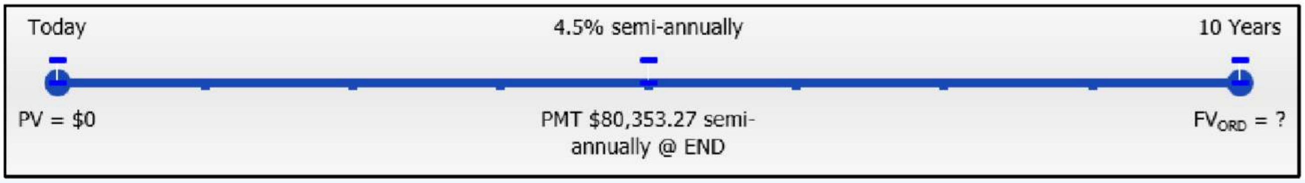
How You Will Get There

Step 4:

Solve for future value of the ordinary sinking fund, or FV_{ORD}, after 10 years using Formulas 9.1, 11.1, and 11.2.

Step 5:

Calculate the book value of the bond debt using Formula 14.9.



Perform

Step 4:

$i = 4.5\%/2 = 2.25\%; N = 2 \times 10 = 20$ payments made over 10 years

$$FV_{ORD} = \$80,353.27\left[\frac{\left[(1+0.0225)^{\frac{2}{2}}\right]^{20}-1}{(1+0.0225)^{\frac{2}{2}}-1}\right] = \$2,001,722.10$$

Step 5:

$$BVD = \$10,000,000 - \$2,001,722.10 = \$7,998,277.90$$

Calculator Instructions

Mode	N	I/Y	PV	PMT
END	20	4.5	0	-80353.27

FV	P/Y	C/Y
Answer: 2,001,722.095	2	2

After 10 years, the Bank of Montreal will accumulate $2,001,722.10 in its sinking fund. With a $10,000,000 debt, the book value of the bond debt remaining equals $7,998,277.90.

Amortization of Bond Premiums and Accrual of Bond Discounts

When a bond sells for a price different from its face value, the bond premium or bond discount has accounting implications. Take a look at each scenario.

Bond Premium

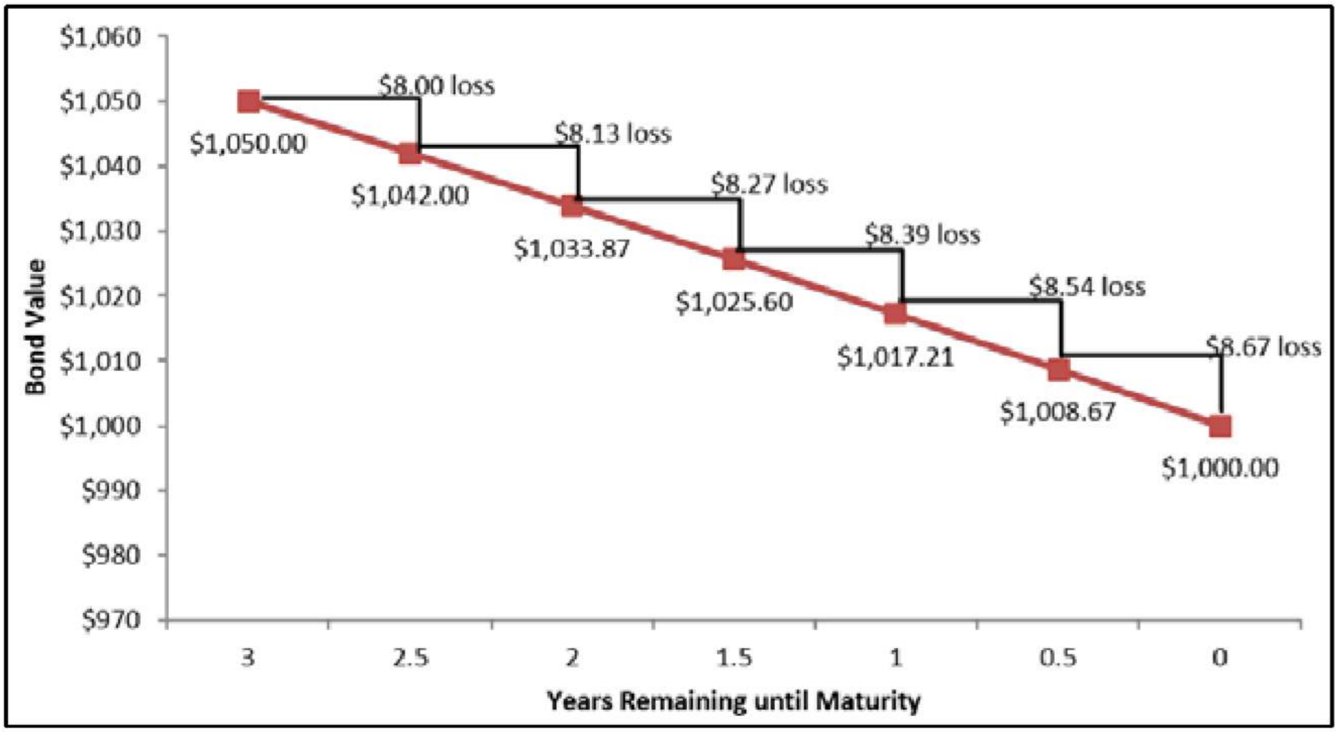


Figure 14.4.3

In these situations, the investor pays more for the bond, say $1,050 for a $1,000 bond. If the investor holds onto the bond until maturity, only the redemption price of $1,000 is returned. The bond premium of $50 represents a capital loss for the investor. A **capital loss** is the amount by which the current value of an asset falls short of the original purchase price. Commonly accepted practices allow the investor to amortize the $50 capital loss over the period of time that the bond is held and not just in the period during which the capital loss actually occurs (at maturity). The figure illustrates the amortization of a capital loss assuming three years remain until maturity on a $1,000 bond carrying a 5% coupon purchased when the market rate was 3.23775%. Note that the total capital loss of $50 is spread throughout the entire three-year time frame.

Bond Discount

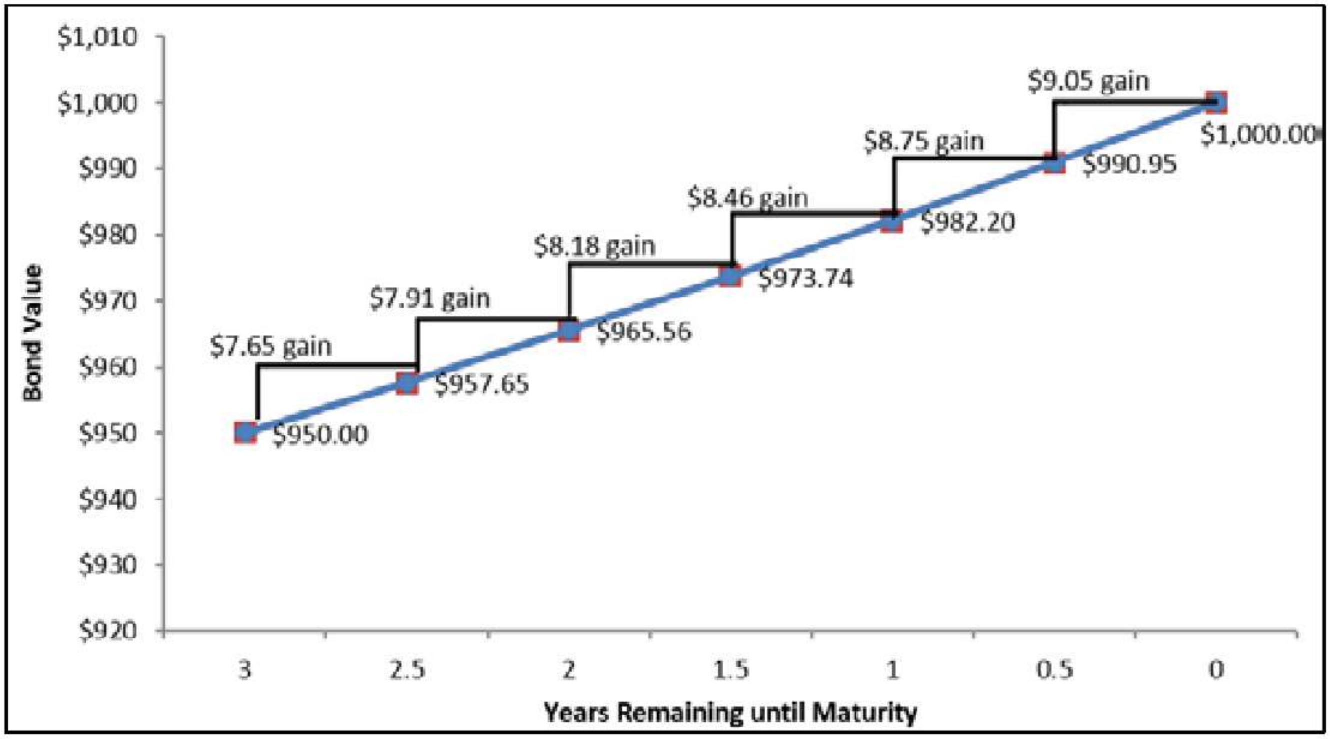


Figure 14.4.4

In these situations, the investor pays less for the bond, say $950 for a $1,000 bond. If the investor holds onto the bond until maturity, the investor receives the full redemption price of $1,000. The bond discount of $50 represents a capital gain for the investor. A **capital gain** is the amount by which the current value of an asset exceeds the original purchase price. Commonly accepted practice allows the investor to accrue the $50 capital gain over the period of time that the bond is held and not just in the period during which the capital gain actually occurs (at maturity). For example, assuming three years remain until maturity on a $1,000 bond carrying a 5% coupon purchased when the market rate was 6.8729%,the figure illustrates the accrual of a capital gain of $50. Note that the total gain is spread throughout the three-year time frame.

In either situation, the gain or loss has tax implications for the investor. These amounts appear on tax forms and either raise the amount of taxes paid by the investor (for gains) or lower the amount of taxes (for losses). As well, for companies these amounts appear on financial statements.

The Formula

To allow the investor to understand these capital gains or losses, you must develop either an amortization table for premiums or an accrual table for discounts. The table here illustrates the standard format of either table. You should note the following in the table:

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium or Discount Accrued	Bond Value (Cash Price)
0	N/A	N/A	N/A	Purchase Price
1	Formula 14.2	Formula 10.1	Premium: $PMT_{BOND} - PMT$ Discount: $PMT - PMT_{BOND}$	Premium: Previous Value − Premium Discount: Previous Value + Discount
...				

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium or Discount Accrued	Bond Value (Cash Price)
Last Payment				
Total	Total Interest Received	Net Income Realized	Premium: Capital Loss Discount: Capital Gain	N/A

- A suitable header of discount or premium is used for the fourth column
- In the last two columns, the method of calculation depends on whether the table is for a premium or discount.

How It Works

Follow these steps to develop a premium amortization table or discount accrual table:

Step 1: Draw a timeline for the bond. Identify all of the time value of money variables for the bond ($N, IY, FV, PMT_{BOND}, PY, CY$). Use Formulas 14.2 and 14.3 as needed to solve for any unknowns.

Step 2: Set up a table, using the appropriate headers depending on the bond discount or premium. The number of payment intervals is the N identified in step 1.

Step 3: Fill in the purchase price of the bond under the “Bond Value” column for payment 0.

Step 4: Fill in the bond annuity payment, or PMT_{BOND}, for every payment in the “Bond Interest Payment Amount” column.

Step 5: Based on the market yield at which the bond was purchased, calculate the interest portion of the current bond payment based on the prior “Bond Value” using Formula 10.1. Round the number to two decimals for the table but retain all decimals for future calculations.

Step 6: Calculate the amortized premium or discount accrued using one the two techniques shown below. Either way, round the result to two decimals for the table but retain all decimals for future calculations.

- For a bond premium, calculate the amortized premium of the current bond payment by taking the second column and subtracting the third column, or $PMT_{BOND} - PMT$.
- For a bond discount, calculate the discount accrued by taking the third column and subtracting the second column, or $PMT - PMT_{BOND}$.

Step 7: Calculate the new bond value for the current payment using one of the two techniques shown below. Either way, round the result to two decimals for the table but retain all decimals for future calculations.

- For a bond premium, calculate the new bond value by taking the previous unrounded “Bond Value” on the line above and subtracting the unrounded amortized amount from step 6.
- For a bond discount, add the two numbers to calculate the new bond value.

Step 8: Repeat steps 5 through 7 for each payment interval until you reach bond maturity.

Step 9: Since all numbers are rounded to two decimals throughout the table, check the table for the "missing penny” using the same method as for amortization tables.

- The previous bond values plus or minus the amortized or discount amount must equal the new bond value.
- The amortized or discount amount plus the interest portion must equal the bond payment amount. Note that on the final line of the table the balance should equal the face value. If necessary, adjust the penny to exactly match the face value.

Step 10: Sum the bond interest payments received. This is the total interest payments received by the investor.

Step 11: Sum the bond interest earned at the yield rate.

Step 12: Sum the amortized premium or discount accrued to arrive at the total capital loss or capital gain, respectively. You calculate a nominal net income by taking the total bond interest and either subtracting the total capital loss or adding the total capital gain.

Important Notes

This section introduces how to spread the capital gain or capital loss on a bond across different time periods. Therefore, it sticks to premium amortization tables and discount accrual tables where the bond is purchased on its interest payment date. If the bond is purchased on some other date, this adds complications that are better left for more in-depth texts.

✓ Example 14.4.3: Bond Premium Amortization

When posted market rates were 4%, Baseline Industries acquired a $10,000 bond carrying a 6% coupon rate with three years remaining until maturity. Develop a complete bond premium amortization table. Assume that all payments and interest are semi-annual.

Solution

You need to construct a complete bond premium amortization table.

What You Already Know

Step 1:

The timeline of the bond purchase appears below.

Coupon Interest Payment: CPN = 6%, CY = 2, Face Value = $10,000

Bond: FV = $10,000, IY = 4%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 3

How You Will Get There

Step 1 (continued):

Apply Formula 14.2 to determine the periodic bond interest payment. Then apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date.

Step 2:

Set up the bond premium amortization table.

Steps 3 and 4:

Fill in the purchase price and the bond payment amount column.

Steps 5 to 8:

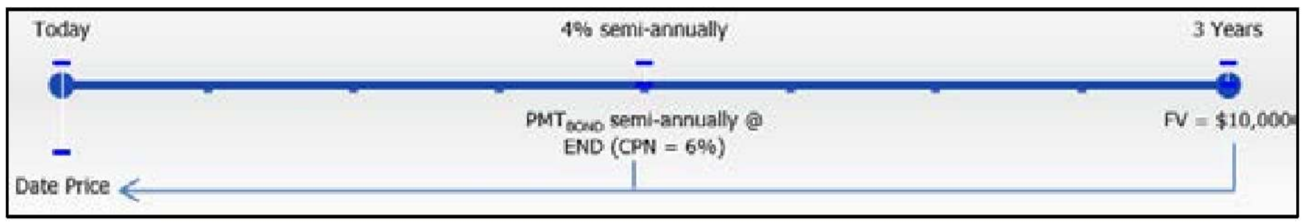
Use Formula 10.1 and for each line calculate the last two columns.

Step 9:

Check for the "missing penny."

Steps 10 to 12:

Total up the bond payments, interest payments, and amortized amounts.



Perform

Step 1:

$$PMT_{BOND} = \$10,000 \times \frac{0.06}{2} = \$300$$

$$i = 4\%/2 = 2\%; N = 2 \times 3 = 6 \text{ payments}$$

$$\text{Date Price } = \frac{\$10,000}{(1+0.02)^6} + \$300\left[\frac{1-\left[\frac{1}{1+0.02}\right]^6}{0.02}\right] = \$10,560.14$$

Step 2 to 8 (with some calculations) are detailed in the table below:

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium	Bond Value (Cash Price)
0				$10,560.14
1	$300	(1) $211.20	(2) $88.80	(3) $10,471.34
2	$300	$209.43	$90.57	$10,380.77
3	$300	$207.61	$92.38	$10,288.39
4	$300	$205.77	$94.23	$10,194.15
5	$300	$203.88	$96.12	$10,098.04
6	$300	$201.96	$98.04	$10,000.00
Total				

(1) $10,560.14 × 0.02 = $211.2028

(2) $300 − $211.2028 = $88.7972

(3) $10,560.14 − $88.7972 = $10,471.3428

Steps 9 to 12:

Adjust for the "missing pennies" (noted in red) and total the bond payment amount, interest at yield rate, and amortized premiums.

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium	Bond Value (Cash Price)
0				$10,560.14
1	$300	(1) $211.20	(2) $88.80	(3) $10,471.34
2	$300	$209.43	$90.57	$10,380.77
3	$300	$207.61	$92.38	$10,288.39
4	$300	$205.77	$94.23	$10,194.15
5	$300	$203.88	$96.12	$10,098.04
6	$300	$201.96	$98.04	$10,000.00
Total	$1,800	$1,239.86	$560.14	

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	6	4	Answer: -10,560.14309	300	10000	2	2

By purchasing the bond at a premium price of $10,560.14 and holding it until maturity, when it has a redemption price of $10,000, Baseline Industries takes a $560.14 capital loss. It receives $1,800 in bond payments, loses $560.14, and realizes nominal net income of $1,239.86.

✓ Example $14.4.4$: Bond Discount Accrual

Recalculate Example $14.4.3$ by changing the market rate to 8%. Construct a bond discount accrual table.

Solution

You need to construct a complete bond discount accrual table.

What You Already Know

Step 1:

The timeline for the bond purchase appears below.

Coupon Interest Payment: CPN = 6%, CY = 2, Face Value = $10,000

Bond: FV = $10,000, IY = 8%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 3

How You Will Get There

Step 1 (continued):

Apply Formula 14.2 to determine the periodic bond interest payment. Then apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date.

Step 2:

Set up the bond discount accrual table.

Steps 3 and 4:

Fill in the purchase price and the bond payment amount column.

Steps 5 to 8:

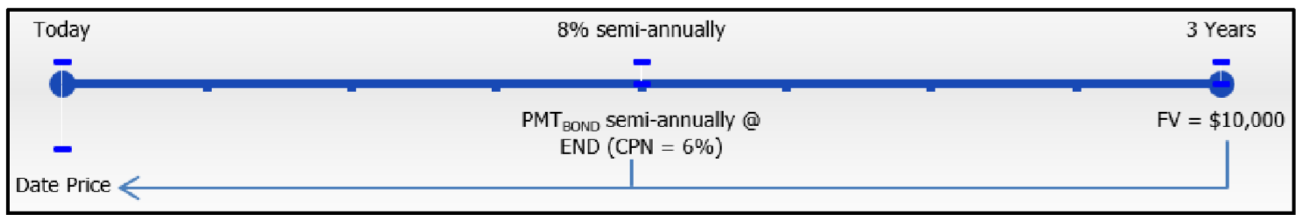
Use Formula 10.1 and for each line calculate the last two columns

Step 9:

Check for the "missing penny."

Steps 10 to 12:

Total up the bond payments, interest payments, and accrued amounts.



Perform

Step 1:

$$PMT_{BOND} = \$10,000 \times \frac{0.06}{2} = \$300$$

$$i = 8\%/2 = 4\%; N = 2 \times 3 = 6 \text{ payments}$$

$$\text{Date Price } = \frac{\$10,000}{(1+0.04)^6} + \$300\left[\frac{1-\left[\frac{1}{1+0.04}\right]^6}{0.04}\right] = \$9,475.79$$

Step 2 to 8 (with some calculations) are detailed in the table below:

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium	Bond Value (Cash Price)
0				$9,475.79
1	$300	(1) $379.03	(2) $79.03	(3) $9,554.82
2	$300	$382.19	$82.19	$9,637.01
3	$300	$385.48	$85.48	$9,722.50
4	$300	$388.90	$88.90	$9,811.39
5	$300	$392.46	$92.46	$9,903.85
6	$300	$396.15	$96.15	$10,000.00
Total				

(1) $9,475.79 × 0.04 = $379.0316

(2) $379.0316 − $300 = $79.0316

(3) $9,475.79 + $79.0316 = $9,554.8216

Steps 9 to 12:

Adjust for the "missing pennies" (noted in red) and total the bond payment amount, interest at yield rate, and discounts accrued.

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium	Bond Value (Cash Price)
0				$9,475.79
1	$300	(1) $379.03	(2) $79.03	(3) $9,554.82
2	$300	$382.19	$82.19	$9,637.01
3	$300	$385.48	$85.48	$9,722.50
4	$300	$388.90	$88.90	$9,811.39
5	$300	$392.46	$92.46	$9,903.85
6	$300	$396.15	$96.15	$10,000.00
Total	$1,800	$2,324.21	$524.21	

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	6	8	Answer: -9,475.786314	300	10000	2	2

By purchasing the bond at a discounted price of $9,475.79 and holding it until maturity, when it has a redemption price of $10,000, Baseline Industries earns a $524.21 capital gain. It receives $1,800 in bond payments, gains $524.21, and realizes nominal net income of $2,324.21.

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14.S: Summary

Key Concepts

14.1: Determining the Value of a Bond (What Does It Sell for?)

- Bond definition, characteristics, and key terminology
- Calculating a marketable bond price when the selling date is the interest payment date
- Calculating premiums and discounts
- Calculating a marketable bond price when the selling date is a noninterest payment date

14.2: Calculating a Bond’s Yield (Know When to Hold ’em, Know When to Fold ’em)

- Calculating the yield on the bond to maturity
- Calculating the yield realized by an investor if a bond is purchased and sold before maturity

14.3: Sinking Fund Schedules (You Need to Show Responsibility)

- Sinking funds and their purposes
- How to construct complete ordinary sinking fund schedules
- How to construct partial ordinary sinking fund schedules
- Adapting any ordinary sinking fund into an annuity fund due schedule

14.4: Debt Retirement and Amortization (Balancing the Books)

- The financial implications of retiring bond debt
- How to amortize a bond premium or accrue a bond discount

The Language of Business Mathematics

annual cost of the bond debt

The annual total of the bond interest payments and the bond sinking fund payments.

bond

A debt that is secured by a specific corporate asset and that establishes the issuer’s responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

bond accrued interest

The amount of interest that the bond has earned but has not yet paid out since the previous interest payment date.

bond cash price

Also known as the purchase price or flat price, this is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and the accrued interest.

bond coupon rate

Also known as the bond rate or nominal rate, this is the nominal interest rate paid on the face value of the bond.

bond discount

The amount by which a bond's selling price falls short of its face value.

bond face value

Also called the par value or denomination of the bond, this is the principal amount of the debt that the investor lent to the bond-issuing corporation.

bond issue date

The date that a bond is issued and available for purchase by creditors

bond market price

Also known as the quoted price, this is the actual value of the bond excluding any accrued interest.

bond market rate

This is the prevailing nominal rate of interest in the open bond market.

bond maturity date

Also known as the redemption date or due date, this is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.

bond premium

The amount by which a bond's selling price exceeds its face value.

bond redemption price

Also called the redemption value or maturity value, this is the amount the bond issuer will pay to the bondholder upon maturity of the bond.

bond selling date

This is the date that a bond is actively traded and sold to another investor through the bond markets.

book value of the bond debt

The difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time.

capital gain

The amount by which the current value of an asset exceeds the original purchase price.

capital loss

The amount by which the current value of an asset falls short of the original purchase price.

debenture

A debt that is not secured by a specific corporate asset and that establishes the issuer’s responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

sinking fund

A special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds will be on hand by a specified date to meet a future savings goal or debt obligation.

sinking fund schedule

A table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity.

yield to maturity

A bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price and the redemption price.

The Formulas You Need to Know

Symbols Used

ACD = Annual cost of the bond debt

AI = Accrued interest

BAL = Principal balance

BVD = Book value of the bond debt

Cash Price = Price paid for a bond including the market price and accrued interest

CPN = Nominal bond coupon rate of interest

CY = Compounding per year or compounding frequency

Date Price = Price of the bond on the interest payment date preceding the sale date

Discount = Bond discount amount

Face Value = Bond face value

i = Periodic interest rate

INT_{SFDUE} = Interest portion of any single annuity due sinking fund payment

N = Number of annuity payments

PMT = Annuity payment amount

PMT_{BOND} = The bond's annuity payment as determined by the coupon rate

Premium = Bond premium amount

PRI = Market price

t = Time ratio (using the actual number of days)

Formulas Introduced

Formula 14.1 The Cash Price for Any Bond: $\text{Cash Price } = PRI + AI$

Formula 14.2 Bond Coupon Annuity Payment Amount: $PMT_{BOND} = \text{ Face Value } \times \frac{CPN}{CY}$

Formula 14.3 Bond Price on an Interest Payment Date: $\text{Date Price } = \frac{FV}{(1+i)^N} + PMT_{BOND}\left[\frac{1-\left[\frac{1}{1+i}\right]^N}{i}\right]$

Formula 14.4 Bond Premium or Discount: $\text{Premium or Discount } = \text{ PRI - Face Value}$

Formula 14.5 Bond Cash Price on a Noninterest Payment Date: $\text{Cash Price } = (\text{ Date Price })(1+i)^t$

Formula 14.6 Accrued Interest on a Noninterest Payment Date: $Al = PMT_{BOND} \times t$

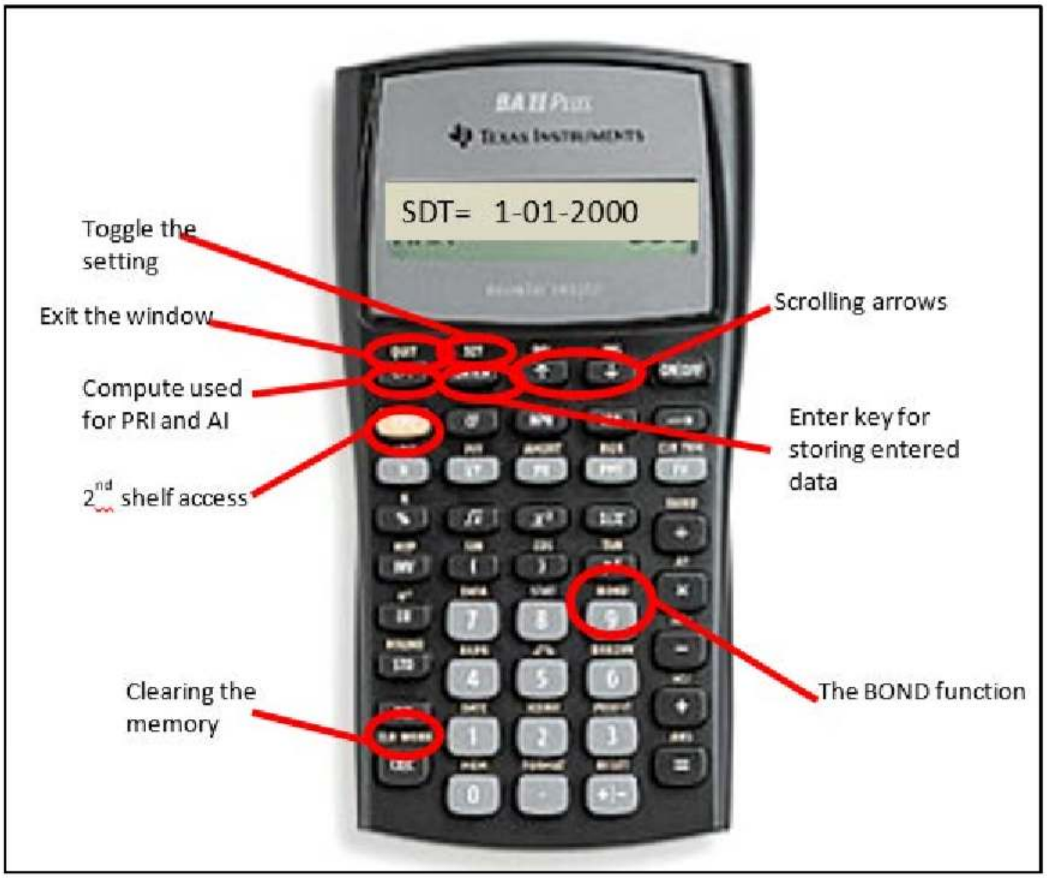
Formula 14.7 Interest Portion of a Sinking Fund Single Payment Due: '_' allowed only in math mode

Formula 14.8 The Annual Cost of the Bond Debt: $ACD = (\text{ Face Value } \times CPN) + (PMT \times PY)$

Formula 14.9 The Book Value of the Bond Debt: $BVD = \text{ Bonds Outstanding } - BAL$

Technology

Calculator



The **DATE function** was used in this chapter. See the end of Chapter 8 for a full discussion of this function.

The **Bond Function**

1. The BOND function is located on the second shelf above the number 9 key and accessed by pressing 2nd BOND.
2. Scroll through using the ↓ and ↑ arrows.
3. There are nine lines in the function. The first seven lines are considered the data entry lines, while the last two are the output lines.
4. Upon opening the window, you should use the 2nd CLR Work function to erase previously entered data. The input lines are as follows:
5. SDT is the selling date. Enter it in the standard date format of MM.DDYY and press the ENTER key to store the information.
6. CPN is the nominal coupon rate. It is formatted as a percentage but without the percent sign, so you key in 5.5% as 5.5. You must press the ENTER key to store the information.
7. RDT is the redemption date or maturity date. Enter it in the standard date format and press ENTER to store the information.
8. RV is the redemption value or redemption price expressed as a percentage of the face value. Since the redemption price equals the face value, use the default setting of 100.
9. ACT/360 is a toggle that you change by pressing 2nd SET. ACT counts the actual number of days in the transaction, while 360 treats every month as having 30 days. In Canada, ACT is the standard.
10. 2/Y or 1/Y is a toggle that you change by pressing 2nd SET. 2/Y indicates a semi-annual compound for both the market rate and coupon rate, while 1/Y indicates an annual compound.
11. YLD is the nominal market rate for bonds at the time of sale. It follows the same format as the CPN. Press the ENTER key to store the information. The output lines are as follows:
12. PRI is the market price of the bond. Press the CPT button to calculate for this solution. The output is a percentage of the redemption price (which is the same as the face value).
13. AI is the accrued interest of the bond. It is automatically calculated when you press the CPT button in the PRI line. The output is a percentage of the redemption price.

Sinking Funds Using AMORT

14. The AMORT function was used in this chapter. See the end of Chapter 13 for a full discussion of this function. Specific sinking fund comments and adaptations are listed below.
    - Ordinary sinking funds: The principal grows instead of declining.
        - With respect to the cash flow sign convention, the PV (if not zero) and PMT are negative, since money is being invested into the account. The FV is a positive number since it can be withdrawn in the future.
        - The BAL and INT outputs are accurate and true to definition. The PRN output is also accurate, but its definition is changed to represent the total of the annuity payments made and the interest earned.
    - Sinking funds due: You must adapt the payment number.
        - P1 and P2: Always add one to the payment number(s) desired.
        - BAL: It has one payment too many. To correct for this, decrease the balance by removing one payment (or with BAL on your display, press − RCL PMT =).
        - INT and PRN: Both of these numbers are correct.

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14.CS: Case Study - Financing An Acquisition Through Bonds

The Situation

Business is booming at Lightning Wholesale! The robust economy and favorable exchange rates have greatly increased the demand for its products. This means that Lightning Wholesale has to increase its inventories immediately to continue providing good service to its customers. Believing this to be a permanent increase in business, the distribution manager decides that the company's existing distribution warehouse simply has too little room to accommodate future needs.

Finding herself in a difficult but positive situation, she asks her commercial property representative to seek a vacant warehouse in the area for immediate occupation. Within a few days, the real estate broker calls back with details. The amount of money required to purchase the new facility along with all installations and equipment is more than the liquid assets of Lightning Wholesale. After examining all available options, the distribution manager suspects that the low rates in the bond market offer the cheapest source of financing.

The Data

- Lightning Wholesale needs \$9.84 million immediately to proceed with the distribution warehouse acquisition.
- The finance department believes it prudent to set a five-year maturity on the bonds.
- To ensure the availability of funds and allow for any unplanned cost overruns, Lightning Wholesale will issue bonds in an amount that exceeds the requirement by 2%. All bonds are issued with a \$1,000 face value.
- At the time of issue, the bond market is expected to have a market rate of 3.37% compounded semi-annually.
- The bond is issued with a sinking fund provision that requires semi-annual deposits such that the full amount issued will be available upon maturity to redeem the bonds. Lightning Wholesale is also permitted to use the funds to redeem the bonds at any point if the market provides a favourable opportunity. Estimates from the fund's third-party manager place the interest rate on the fund at 4.15% compounded semi-annually.

Important Information

Unbeknownst to Lightning Wholesale today:

1. The market rate will rise to 5.65% after three years.
2. The sinking fund's management will find a better fund earning 4.75% compounded semi-annually after two years.

Your Tasks

1. Examine the initial structure of the bond issuance and the financial commitments required.
    a. How many bonds will be issued?
    b. What is the amount of the sinking fund payment?
    c. Develop a complete sinking fund schedule and total the expected interest earned. For each payment, note the book value of the bond debt.
    d. Calculate the annual cost of the bond debt.
2. Advance two years into the future and determine the implications of the rate increase for the sinking fund.
    a. What is the balance in the sinking fund at this time?
    b. What is the new sinking fund payment?
    c. Including the past two years' interest, what total interest is now earned?
    d. By what amount does the annual cost of the bond debt decrease?
3. Advance another year into the future where the rate in the bond market has risen substantially to 5.65%.
    a. The sinking fund management company decides to use the savings in the sinking fund to redeem as many bonds as possible. Based on your calculations from the previous question, how many bonds will it be able to redeem?
    b. After the bonds are redeemed, what is the new sinking fund payment for the last two years?

   c. Construct a revised sinking fund schedule for the last two years based on your answers to the above questions.
   d. By what amount has the annual cost of the debt increased or decreased (compared to where it was the previous year)?
   e. If it is assumed the last two years are uneventful and the bonds are redeemed at maturity, how did the overall cost of the bond issuance change from its initial plan?
4. Examine this bond issuance from the investor's side.
   a. For an investor who acquired 23 of the bonds at the time of the bond issuance, what investor's yield did she realize when the company redeemed the bonds after the three years?
   b. An investor acquired 46 of the bonds when it had two years left until maturity. Construct a complete table detailing either the amortization or accrual of the premium or discount.

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14.E: Bonds and Sinking Funds (Exercises)

14.1: Determining the Value of a Bond

For all questions, assume all interest rates or yields and payment frequencies are compounded semi-annually. Also assume that the redemption price equals the face value.

Mechanics

For each of the following bonds and the indicated selling date, calculate the cash price, accrued interest, market price, and determine the amount of the bond premium or discount.

Face Value	Coupon Rate	Issue Date	Selling Date	Maturity Date	Market Rate on Selling Date
1. $1,000	7.16%	January 15, 1994	January 15, 1997	January 15, 2024	7.16%
2. $10,000	9.4%	April 28, 1987	October 28, 1990	April 28, 2022	10.71%
3. $5,000	7.64%	November 3, 1995	May 3, 2009	November 3, 2015	3.96%
4. $100,000	5.81%	February 20, 2000	February 20, 2010	February 20, 2015	4.07%
5. $250,000	6.27%	July 5, 1997	March 8, 2007	July 5, 2017	4.21%
6. $50,000	16.33%	January 12, 1982	May 2, 2009	January 12, 2032	4.19%
7. $2,500	10.41%	November 19, 1986	September 13, 1990	November 19, 2036	11.49%
8. $30,000	8.99%	January 30, 1992	February 12, 1995	January 30, 2042	8.99%

Applications

For each of questions 9–14, calculate the cash price, accrued interest, market price, and the amount of the premium or discount.

9. With 17 years until maturity, Julio purchased an $8,000 Government of Saskatchewan bond with a coupon rate of 5.55%. The market yield was 8.65%.
10. A $5,000 Vancouver Internal Airport bond issued on June 7, 2004, with 14 years to maturity has a coupon rate of 4.42%. It was sold on December 7, 2009, when market rates were 4.07%.
11. A $25,000 City of Kingston bond issued on July 17, 2007, with five years to maturity has a coupon rate of 5.12%. It was sold on January 10, 2010, at a market rate of 4.18%.
12. On January 2, 1990, when market rates were 10.1%, a $50,000 Province of Prince Edward Island bond maturing on December 17, 2012, with a coupon rate of 9.75% was sold.
13. A $100,000 Nova Scotia Power Corporation bond with a coupon rate of 11.25% matures on April 27, 2014. It was sold on February 27, 2006, when market rates were 4.09%.
14. A $75,000 Government of Canada bond due to mature on September 5, 2020, was sold on September 5, 2009, at a market rate of 3.85%. It carries a coupon rate of 3.84%.
15. The original holder of a $10,000 Province of Manitoba bond issued December 1, 2006, with a 2% coupon and 30 years to maturity sells her bond on June 1, 2010, when market rates were 5.25%. By what amount did the market price increase or decrease for this investor?

Challenge, Critical Thinking, & Other Applications

16. The Alberta Capital Finance Authority issued a 20-year $100,000 bond on December 15, 2005, with a coupon rate of 4.45%. If Mirabelle purchased the bond on June 15, 2007, at a market rate of 4.56% and subsequently sold the bond on March 31, 2009, at a market rate of 3.74%, determine the amount by which the market price increased or decreased for Mirabelle.
17. A $125,000 Province of Ontario 50-year bond issued on March 1, 1995, carries at 9.5% coupon. Amy purchased the bond on September 1, 1999, at a market rate of 5.91% and later sold the bond on September 30, 2008, when

posted rates were 6.13%. Based solely on the market price, determine the gain or loss that Amy experienced.

18. On August 16, 2012, a bond had a market price of $8,240.66 and accrued interest of $157.95 when the market rate was 8%. What is the bond's face value if it matures on May 15, 2033?
19. A 30-year maturity $10,000 face value Government of Canada bond with an 8% coupon was issued on June 1, 1997. Calculate the market price and cash price in the year 2006 on each of June 1, July 1, August 1, September 1, October 1, November 1, and December 1 if the market rate remains constant at 4.5%. Comment on the pattern evident in both the cash price and market price.
20. Assume a $10,000 bond has 20 years until maturity with a coupon rate of 5%.
    a. Determine its market price at rates of 3%, 4%, 5%, 6%, and 7%.
    b. Do the market prices of the bond equally change for each 1% change in the market rate? Why or why not?
    c. Does a market rate that is 1% higher than the coupon rate result in the same premium or discount as a rate that is 1% lower than the coupon rate? Why or why not?
    d. Assume the bond now only has 10 years until maturity and repeat part (a). e. Compare your answers to parts (a) and (d). What impact does the time until maturity have on the change in the bond price?

14.2: Calculating a Bond’s Yield

For all questions, assume that all interest rates or yields and payment frequencies are compounded semi-annually. Also assume that the redemption price equals the face value.

Mechanics

For each of the following bonds, calculate the yield to maturity.

Face Value	Purchase Price	Coupon Rate	Years Remaining until Maturity
1. $1,000	$1,546.16	6%	25
2. $5,000	$3,824.41	5,75%	18.5

Face Value	Purchase Date	Purchase Price	Maturity Date	Coupon Rate
3. $2,500	January 2, 2009	$1,671.47	January 2, 2025	6.18%
4. $10,000	April 13, 2003	$14,480.91	October 13, 2017	7.15%

For each of the following bonds, calculate the investor's yield.

Purchase Price	Selling Price	Years Held	Coupon Rate	Face Value
5. $23,123.77	$21,440.13	6	7.25%	$25,000
6. $106,590.00	$109,687.75	11 years, 6 months	8.65%	$100,000

Purchase Date	Market Rate at Time of Purchase	Selling Date	Market Rate at Time of Sale	Maturity Date	Coupon Rate	Face Value
7. September 29, 2005	9%	March 29, 2010	7%	March 29, 2022	8.65%	$500,000
8. May 21, 1998	6.355%	November 21, 2008	6.955%	May 21, 2033	7.165%	$50,000

Applications

9. A $495,000 face value Province of Ontario bond issued with a coupon rate of 7.5% is sold for $714,557.14. If 20 years remain until maturity, what was the yield to maturity?

10. A $100,000 face value bond is issued with a 5% coupon and 22 years until maturity. If it sells for $76,566.88 9½ years later, what was the yield to maturity?
11. A $15,000 face value bond matures on August 3, 2024, and carries a coupon of 6.85%. It is sold on February 3, 2009, for $20,557.32. Determine the yield to maturity.
12. A $50,000 face value Government of Canada bond is purchased for $48,336.48 and sold nine years later for $51,223.23. If the coupon rate is 4.985%, calculate the investor's yield.
13. Usama acquired a $30,000 face value City of Hamilton bond carrying a 4.95% coupon for $32,330.08 when market rates were 4%. Six years later, market rates are posted at 4.25% and he can sell his bond for $30,765.05. If he doesn't want his investor's yield to be less than his original yield to maturity, should he sell his bond? Provide calculations to support your answer.
14. Great-West Life (GWL) issued a $100,000 face value bond carrying a coupon rate of 6.14% and 20 years to maturity. Eight years later, GWL decides to buy back some of its outstanding bonds when current market rates are 7.29%. If the bond was held from the issue date until the selling date, what yield did the bondholder realize on her investment?
15. Island Telephone Corporation issued a $250,000 face value bond on March 1, 1988, carrying a 9.77% coupon and 30 years to maturity. Global Financial Services purchased the bond on March 1, 1995, at which point the market rate was 8.76%, and sold it on September 1, 2007, for $359,289.44 when market rates were 4.5%. What yield did Global Financial Services realize on its investment?
16. Briar Rose is considering selling her bond but will do so only if she can realize an investor's yield of no less than 3.75%. She purchased the $20,000 face value bond carrying a 6.35% coupon with 25 years to maturity when market rates were 4.85%. Today, 11 years and 6 months later, the market rate is 6.88%. Should Briar Rose sell her bond? Provide calculations to support your answer.

Challenge, Critical Thinking, & Other Applications

17. A $50,000 face value TransCanada Corporation bond carrying a 5.1% coupon matures on January 11, 2017. The bond is purchased on July 11, 2008, for $53,336.24 and later sold on January 11, 2010, for $53,109.69.
    a. Calculate the yield to maturity on the purchase date.
    b. Calculate the yield to maturity on the selling date.
    c. Calculate the investor's yield and express it as a percent change from yield to maturity on the purchase date.
18. A $75,000 face value Canada Post Corporation bond carrying a 4.08% coupon will mature on July 16, 2025. The bond was purchased on January 16, 2006, when market rates were 4.2% and later sold on July 16, 2009, when market rates were 4.05%.
    a. Calculate the purchase price of the bond along with the amount of the premium or discount.
    b. Calculate the selling price of the bond along with the amount of the premium or discount.
    c. Calculate the investor's yield on the bond and express it as a percent change of the market rate on the purchase date.
    d. Convert the purchase date market rate and the investor's yield to their effective rates. Express the investor's yield as a percent change of the original effective rate.
19. The finance manager for Global Holdings Corporation has some funds to invest in bonds. The following three $50,000 face value bond options are available:
    a. A bond carrying a coupon rate of 5.07% with 13 years until maturity and selling for $51,051.79.
    b. A bond carrying a coupon rate of 4.99% with 8.5 years until maturity and selling for $50,552.33.
    c. A bond carrying a coupon rate of 3.96% with 20 years and 6 months until maturity and selling for $44,022.81. Based solely on each bond's yield to maturity, rank the bonds and recommend the one in which the manager should invest the money.
20. Thirty years ago the Mario Brothers invested in a $100,000 face value bond carrying a 7.55% coupon. They have held onto the bond the whole time until today, when the bond matures. In looking at historical bond prices, they wonder what yield they might have realized by selling the bond somewhere along the way. At five-year intervals, the selling price of the bond would have been $105,230.26, $108,133.36, $120,755.53, $141,716.98, and $130,443.72. Calculate the investor's yield throughout the history of the bond and rank the five-year intervals from highest to lowest.

14.3: Sinking Fund Schedules

Mechanics

Create complete sinking fund schedules for each of the following sinking funds.

Future Value	Payment Timing	Payment Frequency	Nominal Interest Rate	Term (Years)

Future Value	Payment Timing	Payment Frequency	Nominal Interest Rate	Term (Years)
1. $50,000	End	Annually	8.2% annually	5
2. $75,000	Beginning	Semi-annually	4.25% semi-annually	4
3. $10,000	End	Quarterly	6.2% annually	2
4. $30,000	Beginning	Monthly	3.8% quarterly	7 months

Create partial sinking fund schedules for the indicated payments for each of the following sinking funds.

Future Value	Payment Timing	Payment Frequency	Nominal Interest Rate	Term (Years)	Payment #
5. $5,000,000	End	Semi-annually	7% semi-annually	6	7 to 10
6. $250,000	Beginning	Annually	10% annually	10	4 to 7
7. $800,000	End	Monthly	5% quarterly	8	Fifth year
8. $2,000,000	Beginning	Quarterly	6% monthly	5	Third year

Applications

For questions 9–12, create a complete sinking fund (due) schedule. Calculate the total payments and the total interest earned.

9. A $500,000 face value bond with semi-annual interest payments is issued with four years until maturity. The bond comes with a sinking fund provision that requires a sinking fund deposit timed to match the interest payments. The sinking fund will earn 4% compounded semi-annually.
10. A $125,000 face value bond is issued with two years until maturity. The ordinary sinking fund provision requires quarterly deposits and can earn an interest rate of 7.4% compounded semi-annually.
11. Jamie and Athena want to take their children to Disney World one year from today. The estimated cost of their stay is $8,600, for which they will save up the funds in advance. Their savings plan can earn 5% compounded quarterly and they will make their first quarterly deposit today.
12. Because of rising demand, a manufacturer predicts that in three years' time it will need to acquire land on which to build a new production plant. It estimates the value of the land at $5 million. The company makes its first semi-annual deposit today into a sinking fund earning 7.35% compounded monthly. For questions 13–15, create a partial sinking fund schedule.

For the specified time frame, calculate the total payments and the total interest earned.

13. A $10 million face value bond is issued with 20 years until maturity. The sinking fund provision requires semi-annual deposits, and the fund can earn an interest rate of 8.5% compounded semi-annually. The company would like detailed information on the sixth through eighth years.
14. A sinking fund is set up for a newly issued $750,000 face value bond with 25 years until maturity. Quarterly deposits will be made into a fund earning 10% compounded annually. Interest and principal information is needed for the third year.
15. Revisit the opening situation discussed in this section. Recall that you had your eye on a home with a list price of $325,000 that requires a 5% down payment. At the end of every month for the past four years, you have been contributing an amount rounded up to the nearest dollar into a five-year savings plan earning 5% compounded monthly. You have found a new home for $250,000 and wonder if you have enough to meet the 5% down payment requirement.
    a. Create a schedule for the 45th through 51st payments into your savings plan.
    b. Can you buy the $250,000 home today? Justify your answer.

Challenge, Critical Thinking, & Other Applications

16. Gillette (a division of Procter and Gamble) projects it will need $500 million today to fund the research and development for a new product line. This money will be charged interest at 6.5% compounded annually. Financing is arranged such that the first end-of-month payment to the sinking fund will start three years and one month from today with the aim of repaying the debt in

total after seven years from today. The fund can earn 7.8% compounded monthly. For the fund itself, create a sinking fund schedule for the first six months of its third year.

17. Yogi is saving up for a $10,000 down payment on his new car, which he plans to purchase one year from now. He will make deposits at the end of every three months into a fund earning 4% compounded quarterly. He already has saved $3,000 today. Create a complete sinking fund schedule and calculate the total payments and interest earned.
18. In total, $10 million in bonds carrying a coupon rate of 5% semi-annually with 10 years until maturity was issued seven years ago. The denomination of each bond is $10,000. A sinking fund provision has required semi-annual deposits into a fund earning 6.6% compounded semi-annually. Today, the market rate on bonds has risen to 7% compounded semiannually. The company decides to use the money in its sinking fund to purchase its bonds on the open market.
    a. If it uses the maximum funds available, how many bonds can it purchase today?
    b. Assuming it purchased the bonds, revise and create a complete sinking fund schedule for the last three years.
19. Every year, a National Car Rental retail outlet needs to purchase its rental fleet of vehicles to reflect current year models. The estimated purchase price of its fleet is $500,000. However, it does not need to save up this entire amount since it sells the previous fleet to automotive dealerships for 45% of the purchase price. Create a complete annual sinking fund schedule for the retail outlet, assuming its sinking fund earns 3.6% compounded monthly and it makes end-of-month deposits. Determine the total payments made along with the total interest earned.
20. 20. A $100 million face value bond is issued with 30 years until maturity. The sinking fund provision requires semi-annual payments into a fund earning 8% compounded semi-annually.
    a. Calculate the total interest earned and total payments.
    b. For the 10th through 15th year inclusive, calculate the total interest earned.

14.4: Debt Retirement and Amortization

For all questions, assume that all interest rates or yields and payment frequencies are compounded semi-annually and that the redemption price equals the face value.

Mechanics

For questions 1–4, assume the bond has a sinking fund requirement. Calculate the following for each question:

a. The annual cost of the bond debt
b. The book value of the bond debt after the payment number specified in the last column

Bond Face Value	Coupon Rate	Years to Maturity	Sinking Fund Rate	Find Book Value after Payment #
1. $500,000	8%	5	5.5%	6
2. $2,500,000	4.4%	25	5%	20
3. $1,000,000	6.5%	15	7.25%	25
4. $750,000	5.25%	10	4%	9

For questions 5–6, create a complete bond premium amortization table.

Bond Face Value	Coupon Rate	Years to Maturity	Market Interest Rate at Time of Purchase
5. $50,000	8%	2	5%
6. $100,000	7%	4	4.5%

For questions 7–8, create a complete bond discount accrual table.

Bond Face Value	Coupon Rate	Years to Maturity	Market Interest Rate at Time of Purchase

Bond Face Value	Coupon Rate	Years to Maturity	Market Interest Rate at Time of Purchase
7. $20,000	5%	3	6.75%
8. $250,000	6.6%	3.5	7.8%

Applications

9. The City of Toronto has an outstanding $5 million face value bond carrying a 7% coupon. It makes payments of $436,150 to its sinking fund every six months. Calculate the annual cost of the bond debt.
10. The Province of Nova Scotia issued a $50 million face value bond on June 1, 2007, carrying a coupon rate of 6.6% with 20 years until maturity. A sinking fund earning 4.3% compounded semi-annually with payments at the end of every six months was established. Calculate the book value of the bond debt on December 1, 2012.
11. Newfoundland and Labrador Hydro issued a $100 million face value bond on February 27, 1996, carrying an 8.4% coupon with 30 years until maturity. A matching sinking fund earning 5.1% compounded semi-annually was set up to retire the debt in full upon maturity.
    a. Calculate the annual cost of the bond debt.
    b. Calculate the book value of the bond debt on August 27, 2011.
12. The Province of Ontario issued a $175 million face value bond on March 1, 1995, carrying a 9.5% coupon with 50 years until maturity. The matching sinking fund, with the payment rounded up to the next $100, is expected to earn 6.3% compounded semi-annually and will retire the debt in full upon maturity.
    a. Calculate the annual cost of the bond debt.
    b. Calculate the book value of the debt on September 1, 2015.
13. Three years before maturity, a $55,000 face value bond carrying a 5.5% coupon is acquired when posted market rates are 4.77% compounded semi-annually. Create a complete bond premium amortization table. How much total interest is received? What is the amortized bond premium total, and what nominal net income was realized?
14. Jaycee works in the finance department. His company just acquired 12 $5,000 face value bonds carrying a 6.1% coupon with four years remaining until maturity. Current market yields are 7.65% compounded semi-annually. Construct a complete bond discount accrual table. How much interest does the company receive on its investment? What is the accrued bond discount total, and what nominal net income was realized?
15. When the current bond market is yielding 5.89% compounded semi-annually, Jennifer purchases six $10,000 face value bonds carrying a 4.2% coupon with three years until maturity for her RRSP. Construct a complete table for the premium or discount. What total capital gain or capital loss will she record for the second year?
16. On March 20, 2011, Hussein decides to invest in a $15,000 face value Ville de Gatineau bond carrying a 4.1% coupon when current market yields are 3.76%. The bond matures on March 20, 2014. Construct a complete table for the premium or discount. What total capital gain or capital loss will be recorded in 2013?

Challenge, Critical Thinking, & Other Applications

17. For a large project, New Flyer Industries considers issuing a $15 million face value bond with a 10-year maturity date. When issued, the finance manager estimates it will have a coupon rate of 6.2%. The matching sinking fund provision will require the full repayment of the amount upon maturity and is estimated to earn 5.45% compounded semi-annually. In examining future cash flows, the manager estimates that the company will be able to afford no more than $2 million per year toward the financing of this project. Based on financial calculations, should the company proceed with the project as planned? Provide appropriate calculations to support your answer.
18. In Canada, 50% of an individual's capital gains are taxed at the person's current tax rate. Sylvester has a tax rate of 24%. He considers purchasing a $100,000 face value bond carrying a 7.15% coupon with five years remaining until maturity. Current market yields are posted at 8.55% compounded semi-annually. He will purchase the bond only if, in any given year, the capital gains tax amount remains under $150. Should he purchase the bond? Provide the necessary calculations to support your answer.
19. The annual cost of bond debt depends on many factors, including the time to maturity and the interest rate achieved by the sinking fund. Assume a $10 million face value bond carrying a coupon rate of 7% with 10 years until maturity.
    a. Calculate the annual cost of the bond debt if the matching sinking fund can earn semi-annually compounded rates of 4% and 7%. What percentage more is the annual cost of the debt at 4% than 7%?

   b. Leaving the sinking fund rate at 5%, calculate the annual cost of the bond debt if the time to maturity is either 8 years or 12 years. What percentage more is the annual cost of the debt at 8 years than 12 years?

20. The current market rate is 5% compounded semi-annually. Two bonds with four years until maturity are acquired. The first bond has a $50,000 face value and carries a coupon rate of 6.4%. The second bond has a $30,000 face value and carries a coupon rate of 4.6%.
    a. Construct the appropriate premium or discount table for each bond.
    b. For each year, determine the net capital gain or net capital loss recorded for both bonds combined.

Review Exercises

For all questions, assume that all interest rates or yields and payment frequencies are compounded semi-annually and that the redemption price equals the face value.

Mechanics

1. A Province of Alberta $100,000 face value bond carrying a 5.03% coupon was issued on December 17, 1998, with 20 years until maturity. What was its purchase price on June 17, 2005, when market yields were 4.29%?
2. A Government of Canada $50,000 face value bond carrying a 5.75% coupon was issued on June 1, 2008, with 25 years until maturity. What was its market price on November 21, 2009, when market rates were 3.85%?
3. A $35,000 face value bond carrying a 7% coupon will mature on October 3, 2019. If it is purchased on April 3, 2006, for $46,522.28, what is its yield to maturity?
4. A $55,000 face value bond carrying a 4% coupon is purchased for $33,227.95 on March 29, 1997. The bond is later sold on September 29, 2007, for $60,231.63. Calculate the semi-annual investor's yield.
5. A Province of Manitoba $250,000 face value bond carrying a 2% coupon was issued on December 1, 2006, with 30 years until maturity. On January 10, 2010, when market yields were 4.19%, what were its market price, accrued interest, cash price, and premium or discount?
6. Carlyle needs to save up $20,000 to meet the down payment requirement on his new home. He wants to make semiannual deposits at the beginning of each six months for the next four years, putting them into a fund earning 6.12% compounded semi-annually. Construct a complete sinking fund due schedule.
7. A $15 million face bond carrying an 8.5% coupon has nine years until maturity. The bond issue has a sinking fund provision requiring semi-annual payments, and the full bond value must be saved by its maturity date. If the fund can earn 7.35% compounded semi-annually, calculate the annual cost of the bond debt.
8. When market yields are 16%, a $65,000 face value bond carrying a 6.75% coupon is purchased three years before maturity. Construct a complete bond discount accrual table for the bondholder.

Applications

9. When posted market rates are 9.65%, a $75,000 face value bond carrying a 7% coupon is purchased with 23½ years to maturity. With eight years remaining until maturity the bond is then sold, when posted market rates are 3.5%. Calculate the investor's yield.
10. A $275,000 face value Province of British Columbia bond carrying a 10.6% coupon is issued on September 5, 1990, with 30 years until maturity. The bond is purchased on March 5, 2002, when posted rates are 5.98%. Calculate the purchase price of the bond. What is the amount of its premium or discount?
11. 11. A $40,000 face value bond carrying a 7.6% coupon is purchased with four years until maturity. Posted rates are then 4.9%. Construct an appropriate complete table for the capital gain or loss. What is the total gain accrued or loss amortized in the third year?
12. A $50 million face value bond carrying a 4.83% coupon with 25 years until maturity is issued. The bond has a sinking fund requirement with semi-annual payments designed to retire the full face value upon maturity. If the sinking fund is expected to earn 3.89% compounded semi-annually, calculate the annual cost of the bond debt. What is the book value of the debt after 10 years?
13. A $62,000 face value bond carrying an 8.88% coupon is purchased on July 15, 2011. The bond matures on November 13, 2027. At the time of purchase, the market rate on the bond was 4.44%. Calculate the market price, accrued interest, and cash price of the bond. Determine the amount of the bond premium or discount.
14. A bond is purchased on September 17, 1997, for $28,557.25 with 14 years until maturity. The 6% coupon pays $967.50 every six months. Calculate the yield to maturity.

15. A $300,000 face value bond carrying a 4% coupon is issued with four years until maturity. A sinking fund with semiannual payments is set up and is expected to earn 6.35% compounded semi-annually. Construct a complete sinking fund schedule. Calculate the annual cost of the debt. What is the book value of the debt after the fifth payment?
16. A $500,000 face value bond makes semi-annual payments of $15,900 and will mature on January 2, 2014. The bond is purchased on July 14, 1997, when posted market rates are 7.77%. Calculate the market price, accrued interest, cash price, and the amount of the bond premium or discount.

Challenge, Critical Thinking, & Other Applications

17. In Canada, 50% of an individual's capital gains or losses are taxed or deducted respectively at the individual’s current tax rate. Tammy has a tax rate of 12.76%. She just purchased a $120,000 face value bond carrying a 9.89% coupon with six years remaining until maturity. Current market yields are posted at 6.14% compounded semi-annually. Calculate her taxes owing or deducted on the capital gain or loss in the fourth year.
18. 18. A $75 million face value bond carrying a 4.45% coupon is issued with 35 years until maturity. The sinking fund provision requires 80% of the face value to be saved up by the maturity date. The sinking fund is projected to earn 4.95% compounded semi-annually.
    a. Create a partial sinking fund schedule detailing the first two years, last two years, and the 10th and 11th years.
    b. Calculate the total interest earned by the sinking fund.
    c. Calculate the annual cost of the bond debt.
    d. Determine the book value of the bond debt after the 29th payment.
19. A $400,000 face value bond carrying a 5.6% coupon is issued on March 23, 2007, and expected to mature on March 23, 2011. The sinking fund provision requires semi-annual payments such that the full amount is saved upon maturity. The fund is expected to earn 3.7% compounded semi-annually.
    a. Create a complete sinking fund schedule for the issuing company. Calculate the total interest earned.
    b. Suppose an investor purchased $50,000 of the bond on September 23, 2007, at a market rate of 5.45% and later sells the bond on March 23, 2009, at a market rate of 3.74%. Calculate the investor's yield.
    c. If an investor purchased $25,000 of the bond on March 23, 2008, for $25,991.24, what would be the yield to maturity?
    d. For each investor in parts (b) and (c), construct a complete table detailing the amortized gain or discount accrued. Assume the investor in part (b) held onto the bond until maturity instead of selling it.
20. 20. A $650,000 face value bond carrying a 4.59% coupon is issued on March 30, 2005, and expected to mature on March 30, 2010. The sinking fund provision requires semi-annual payments such that the full amount is saved upon maturity. The fund is expected to earn 4.25% compounded semi-annually.
    a. Create a complete sinking fund for the issuing company. Calculate the total interest earned.
    b. Suppose an investor purchased $100,000 of the bond on September 30, 2005, at a market rate of 4.75% and later sells the bond on September 30, 2008, at a market rate of 4.27%. Calculate the investor's yield.
    c. If an investor purchased $34,000 of the bond on September 30, 2007, for $34,167.47, what would be the yield to maturity?
    d. For each investor in parts (b) and (c), construct a complete table detailing the amortized gain or discount accrued. Assume the investor in part (b) held onto the bond until maturity instead of selling it.

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CHAPTER OVERVIEW

15: Making Good Decisions

15.0: Introduction
15.1: Net Present Value
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15.E: Making Good Decisions (Exercises)

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15.0: Introduction

You must make financial decisions throughout your professional career and everyday personal life. Some of these decisions are easy. To make the best choice you need a little intuition and simple calculation. Other decisions are very challenging, confronting you with a great array of competing options, each of which is bundled with numbers projected from now until ... whenever. If you tried just to wing it in these latter scenarios, you could make a catastrophic mistake.

For example, let’s say you are the production manager for a company that soon needs to replace a critical machine that is nearing the end of its useful life. At your invitation, salespeople from three competing companies have paid you a visit this week. Each of them showed you an impressive replacement machine. The three machines appear to be equal in design and performance, but each carries a different price tag. Each also differs widely in operating costs such as power consumption, consumables, and labour costs. Maintenance costs follow different timetables. In all cases, you can either purchase the machines or lease them through the supplier’s leasing plan.

These machines are not cheap. Because the one you select represents such a significant investment on your company’s part, good financing for it is an integral part of the decision. Will the money to pay for it come from a bank loan, a bond issuance, or by issuing some common shares? Perhaps the finance department can withdraw some money from your organization’s current investments. Each of these funding sources is associated with a different interest rate.

Your company relies on you to choose the best machine at the lowest possible cost. From a strictly financial perspective, and assuming that all machines are equally productive, which of the three machines is your best choice?

This investment decision should not be guesswork. In previous chapters you have already learned many of the fundamental financial skills required to make effective monetary decisions. In this chapter you will amalgamate and apply your existing skills in dealing with compound interest, ordinary annuities, annuities due, leases, loans, and much more. Section 15.1 introduces the concept of net present value and its application to decision making. You also see a new concept known as cash flows. Using these techniques you will work through the basic kinds of decision-making scenarios. Section 15.2 then examines other decision characteristics along with an important decision-making measure called the internal rate of return.

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15.1: Net Present Value

If you decide to pursue a course of action, what is the net benefit or cost of making that decision? How can you tell systematically which choice is financially superior?

Let’s say that your marketing department thinks it has stumbled onto the next hot product to hit the market. The estimated research and development costs are $10 million today with an expected $5 million revitalization cost in five years. The product hits shelves in two years. Estimated year-end profits are $4 million in the first three years and $3 million for the subsequent two years. At that time, the rapid pace of technology will end the life cycle of the new product. This last year will earn an estimated $1 million in profits. The funds needed for this project will be obtained at 12% compounded annually. Does this project make financial sense? Your co-worker argues that the decision is clear: You would spend $15 million to earn $19 million, so the project should go forward. What do you think?

Your co-worker’s analysis summed the nominal numbers. This clearly violates the fundamental concept of time value of money. In this section, we will explore different types of investment decisions and study techniques for making optimal financial decisions.

What Is Involved in Making Monetary Business Decisions?

Investment decisions must consider both the type of decision being made along with the source of funding, which influences the interest rate used in financial calculations.

What Type of Decision Is Being Made?

While there is no such thing as a “standard” or “normal” decision, many business decisions seem to fall into the same three categories:

1. Decision #1: Making One Choice from Multiple Options. When you face a variety of options that will solve your problem, which option is financially the best?
2. Decision #2: Deciding Whether to Pursue One Course of Action. When you face only one particular course of action, do you do it or not?
3. Decision #3: Making Multiple Choices under Constraints. If you have limited resources and many different opportunities from which you can choose more than one, how do you select which opportunities to pursue and which to let pass by?

What Monetary Source Is Being Used and What Does It Cost?

The structure of many decisions follows the old adage that you have to spend money to make money. Many courses of action require an investment up front (costs) to receive the reward (profits) in the future. This means you need to tap a source of money before you can go forward with any course of action. Individuals and corporations can access many of the same monetary sources, but businesses have additional options that are not available to individuals.

- **Personal Funding Sources**. You have limited options for obtaining money. Typically, you can either use debt financing such as a loan from a bank or a line of credit, or you can withdraw money from an investment fund or savings account.
    1. If you borrow, the financial cost of the project is the interest rate that is being charged on the borrowed funds. Therefore, if you fund the project using a loan at 8.5% compounded annually, the loan's interest rate becomes the nominal rate you should use in the required time value of money calculations.
    2. If you withdraw money from an investment, the cost of this project is the interest rate you forego on those funds. For example, if you take the money out of a GIC earning 4% compounded annually, you should use the GIC's interest rate in the decision calculations.
- **Business Funding Sources**. Businesses have substantially more choices than individuals when obtaining financing. In addition to debt financing and current investments, businesses also can issue bonds or use equity financing, such as offering company shares. Bartering with business partners is also possible. The same principle applies, though: Whatever interest rate the funding source involves carries forward into any decision-making calculations.

The fundamental concept of time value of money states that all money must be moved to the same date before you can compare alternatives fairly. As mentioned above, the source of funding you use determines the interest rate in these calculations. If you use multiple sources to gather the needed funds, your interest rate is a weighted average of all the debt and equity financing rates, and of course if you use a single source then the weighted average is simply the one source’s rate. The weighted interest rate is known as the cost of capital. This textbook assumes the **cost of capital** in calculations is compounded annually unless otherwise specified.

How to Make the Decision

Businesses most commonly make decisions by performing one of two methods: net present value or internal rates of return. Net present value methods are based on all of the cash flows of a decision expressed in today's dollars. This is the focus of this section. Internal rates of return methods are based on the percentage gains of an investment or the yield of the project, which is discussed in the next section.

The **net present value** (NPV) method calculates the difference between all benefits and costs for any given project in today's dollars. This decision-making technique calculates the value of all future income and expense streams brought back to today (which serves as the focal date). The word net means that the technique considers the invested money, all costs or expenses, along with all rewards such as profits in its calculations. Remember these criteria for making smart NPV financial decisions:

1. If the net present value is greater than or equal to $0, the project makes financial sense since it will add value to the company or at minimum break even. It is financially logical to pursue these types of projects.
2. If the net present value is less than $0, the project does not make financial sense since it will lose money for the company. A company has no financial reason to pursue these types of projects.

Of course, businesses often have many non-financial reasons for pursuing or not pursuing certain courses of action. These reasons are beyond the scope of this textbook.

Observe how NPV relates to each of the three common types of business decision categories:

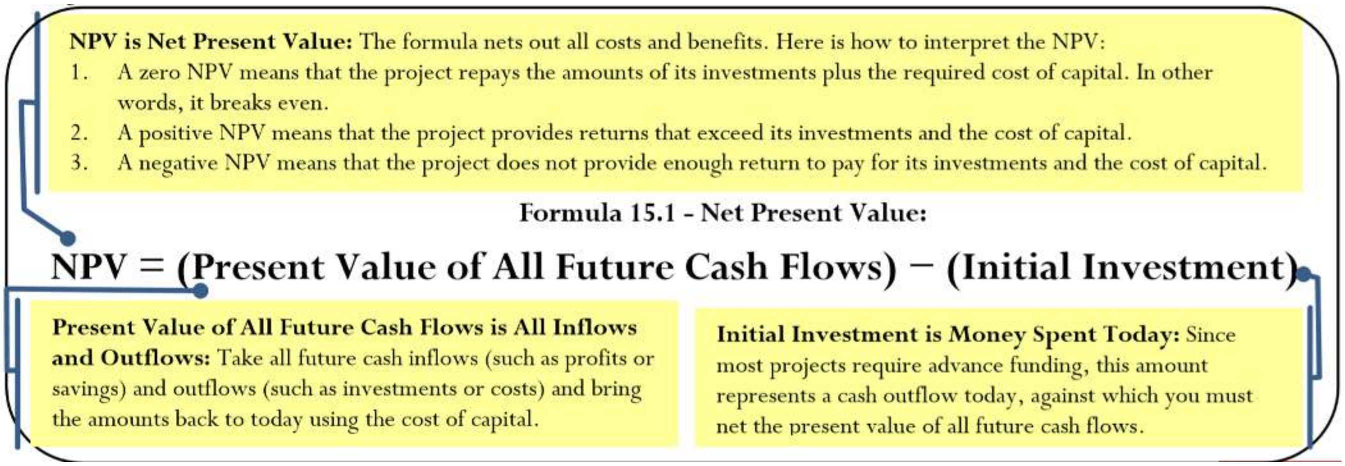
1. **Decision #1: Making One Choice from Multiple Options**. You should choose the project with the best NPV. In this section you will learn a technique that helps you choose between projects of identical length. Section 15.2 introduces another technique for choosing between projects of differing length. For example, you cannot simply use NPV to compare project #1, which has a four-year life, against project #2, which has a six-year life.
2. **Decision #2: Choosing Whether to Pursue One Course of Action**. You should pursue the project if the NPV is positive or at a minimum zero.
3. **Decision #3: Making Multiple Choices under Constraints**. You should choose the projects that maximize the total NPV.

This table summarizes the relationship between the three decision types and the financial criteria.

Decision Type	Financial Criteria
Making One Choice from Multiple Options	Best NPV
Pursuing One Course of Action	NPV must be $0 or higher
Making Multiple Choices under Constraints	Combination of choices that efficiently maximizes total NPV

The Formula

There is no one formula for calculating the net present value. However, you can represent the NPV conceptually through Formula 15.1.



NPV is Net Present Value: The formula nets out all costs and benefits. Here is how to interpret the NPV:

1. A zero NPV means that the project repays the amounts of its investments plus the required cost of capital. In other words, it breaks even.
2. A positive NPV means that the project provides returns that exceed its investments and the cost of capital.
3. A negative NPV means that the project does not provide enough return to pay for its investments and the cost of capital.

Formula 15.1 - Net Present Value:

$$\text{NPV} = (\text{Present Value of All Future Cash Flows}) - (\text{Initial Investment})$$

Present Value of All Future Cash Flows is All Inflows and Outflows: Take all future cash inflows (such as profits or savings) and outflows (such as investments or costs) and bring the amounts back to today using the cost of capital.

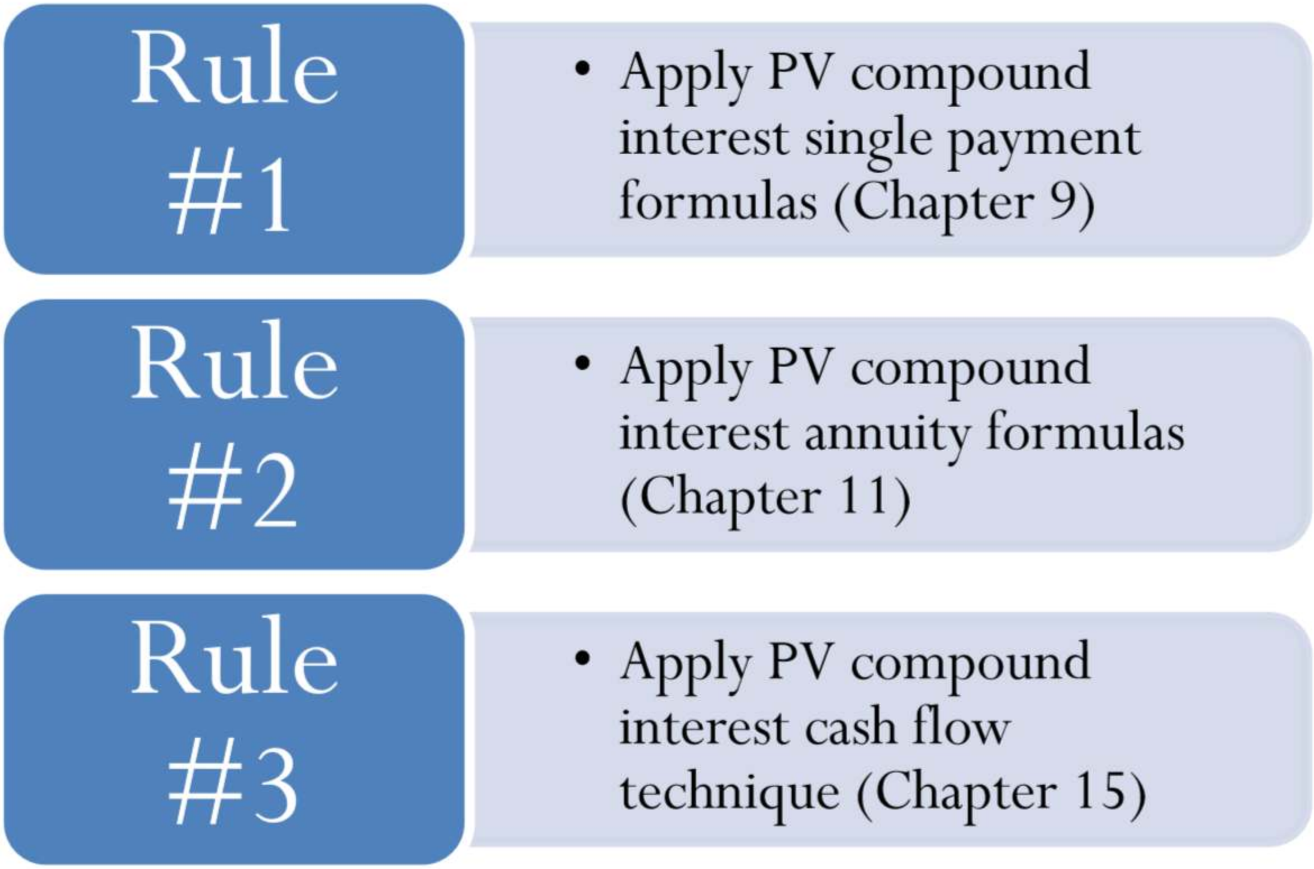
Initial Investment is Money Spent Today: Since most projects require advance funding, this amount represents a cash outflow today, against which you must net the present value of all future cash flows.

Formula 15.1

How It Works

Every decision is different—there is no one definable method to solve a problem. If the solution was obvious, everyone would always make smart decisions! Instead, three rules for moving money will help you choose well:

1. **Rule #1**: If there is no regularity to the payments, then each amount represents a single payment. You can calculate the present value of all future cash flows only through the techniques you learned in Chapter 9 for compound interest present value for single payments. Example 15.1.1 demonstrates this rule.
2. **Rule #2**: If there are regular payments continuously in the same amount, you have an annuity situation. You can calculate the present value of all future cash flows using the techniques from Chapter 11 for compound interest present value for either ordinary annuities or annuities due. Example 15.1.1 also demonstrates this rule.
3. **Rule #3**: If there are regular payments continuously but in differing amounts, then to use a formula you would have to compute the present value of compound interest single payments as per Rule #1. What distinguishes Rule #3 from Rule #1, though, is the "regularity" characteristic. Under this condition, and assuming that you have access to a financial calculator or spreadsheet, this particular situation presents a shortcut technique called cash flows. Example 15.1.2 and Example 15.1.3 illustrate these cash flows, which are explained later in this section.


Rule #1
• Apply PV compound interest single payment formulas (Chapter 9)
Rule #2
• Apply PV compound interest annuity formulas (Chapter 11)
Rule #3
• Apply PV compound interest cash flow technique (Chapter 15)

Important Notes

When you calculate net present value, most financial forecasts of costs and benefits are best estimates. Therefore, you do not need to be precise to the penny in the final solution. Depending on the dollar amounts involved, companies may commonly round these calculations to the nearest hundred or thousand dollars. While there is no standard practice, this textbook rounds final solutions to the nearest dollar.

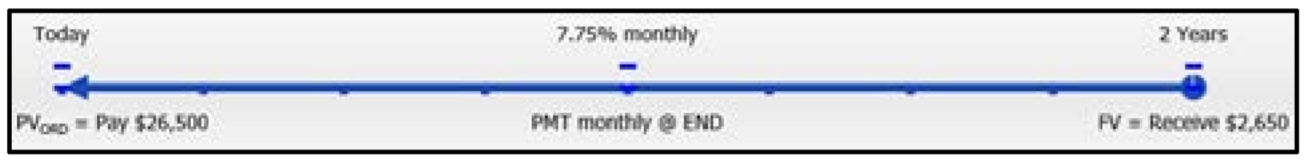
Saggitt Accounting Services needs to regularly update its existing computer workstations. If it leases the workstations for two years, it will make beginning-of-month payments of $1,125 with no residual value. If it purchases the workstations, it will need to pay $26,500 today and it can sell the equipment after two years for 10% of the purchase price. It will need to borrow these funds for two years at 7.75% compounded monthly. Strictly on a financial basis, which option should Saggitt Accounting pursue? How much better is that option in today's dollars?

Solution

Since you need to decide today, you should calculate the net present value (NPV) for each alternative. Each of these choices represents a cost, so you choose the option with the lowest NPV.

What You Already Know

The timeline for the purchase appears in the first timeline below. With respect to the bank loan itself, present value calculations are not necessary since the present value of all loan payments equals the original loan principal. PV_{ORD} = $26,500, IY = 7.75%, CY = 12, FV = 10% × $26,500 = $2,650, Years = 2 The timeline for the lease appears in the second timeline below. Use the interest rate from the loan in this calculation since Saggitt Accounting would otherwise have to get a loan if it chose not to lease. Therefore, the loan rate represents an appropriate cost of capital. PMT = $1,125, PY = 12, IY = 7.75%, CY = 12, FV = $0, Years = 2



How You Will Get

There Purchase Timeline:

Step 1:

You only need to move the revenue from the end of the second year to today. This is decision Rule #1. Therefore, apply Formulas 9.1, 9.2, and 9.3 rearranging for PV.

Step 2:

Calculate NPV using Formula 15.1.

Lease Timeline:

Step 3:

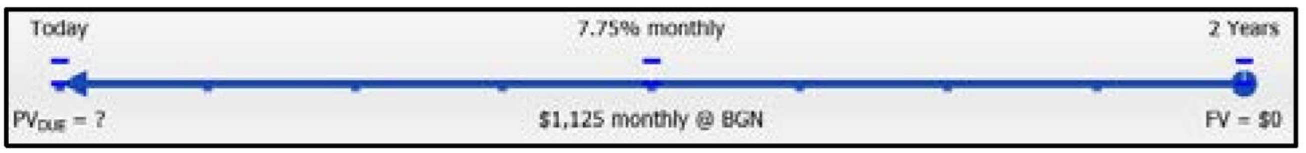
Take the annuity due and move it to today. This is decision Rule #2. Apply Formulas 11.1 and 11.5.

Step 4:

There is no initial investment. Formula 15.1 is then $NPV = PV_{DUE}$.

Step 5:

Calculate the savings by choosing the better option.



Step 1:

$$i = 7.75\%/12 = 0.6458\overline{3}\%; N = 12 \times 2 = 24 \text{ compounds}$$

$$\$2,650 = PV(1 + 0.06458\overline{3})^{24}$$

$$PV = \$2,650 \div (1 + 0.06 + 58\overline{3})^{24} = \$2,270.63$$

Step 2:

$$NPV = \$2,270.63 - \$26,500.00 = -\$24,229.37 \text{ or a cost of } \$24,229.37$$

Step 3:

$$N = 12 \times 2 = 24 \text{ payments}$$

$$PV_{DUE} = \$1,125 \left[\frac{1 - \left[\frac{1}{\left(1 + 0.006458\overline{3}\right)^{\frac{12}{12}}} \right]^{24}}{\left(1 + 0.006458\overline{3}\right)^{\frac{12}{12}}} \right] \times \left(1 + 0.006458\overline{3}\right)^{\frac{12}{12}} = \$25,098.24$$

Step 4:

$$NPV = \text{ cost of } \$25,098.24$$

Step 5:

$$\$25,098.24 - \$24,229.37 = \$868.87 \text{ savings}$$

Calculator instructions:

Transaction	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Purchase	END	24	7.75	Answer:-2, 270.631567	0	2650	12	12
Lease	BGN	√	√	Answer: -25,098.235 08	-1125	0	√	√

If Saggitt Accounting Services purchases the machine, it will cost $24,299.37 in today's dollars. If it leases the machine, it will cost $25,098.24 in today's dollars. The smart choice is to purchase the machine, realizing a savings of $868.87 in today's dollars.

A Regular Stream of Cash Flows

A **cash flow** is a movement of money into or out of a particular project. Therefore, any time you incur a cost or receive a benefit on a particular project, you have a cash flow. Under the three rules for moving money, both single payments and annuities are cash flows. You can move money from a future date to a present date using the existing formulas or calculator techniques.

The third decision rule is for the situation of continuous regular payments that are variable in amount. As indicated in the rule, to solve the problem by formula you would need to calculate compound interest present value (Rule #1) repeatedly. This procedure requires many repetitive calculations in which you calculate N and PV for each cash flow. This is quite cumbersome, since every cash flow introduces two more calculations and another present value to net out. Both the calculator and Excel template simplify this process, reducing both the effort and the chance of error.

As you work with these continuous cash flows, where to put an amount on the timeline depends on whether it is a cost or a benefit.

- **Costs:** Unless otherwise noted, capital costs are spent at the beginning of any time segment. Thus, if there are capital costs in year two, you can assume you incur them at the beginning of year two, which is the same as the end of year one.
- **Benefits:** Revenues, net profits (product sales minus product costs), or other benefits are realized at the end of any time segment. Thus, if there are net profits in year two, you can assume you realize them at the end of year two.

How It Works

Follow these steps when working with regular cash flows of varying amounts:

Step 1: Draw a timeline illustrating all of the cash flows, cost of capital, and any initial investment. Always draw your timeline in an ordinary annuity format with all transactions occurring at the end of each time interval. Thus, any costs you incur at the beginning of a time segment are placed at the end of the prior time segment (unless noted otherwise).

Step 2: For any particular point on the timeline at which more than one cash flow exists, net all cash flows into a single number.

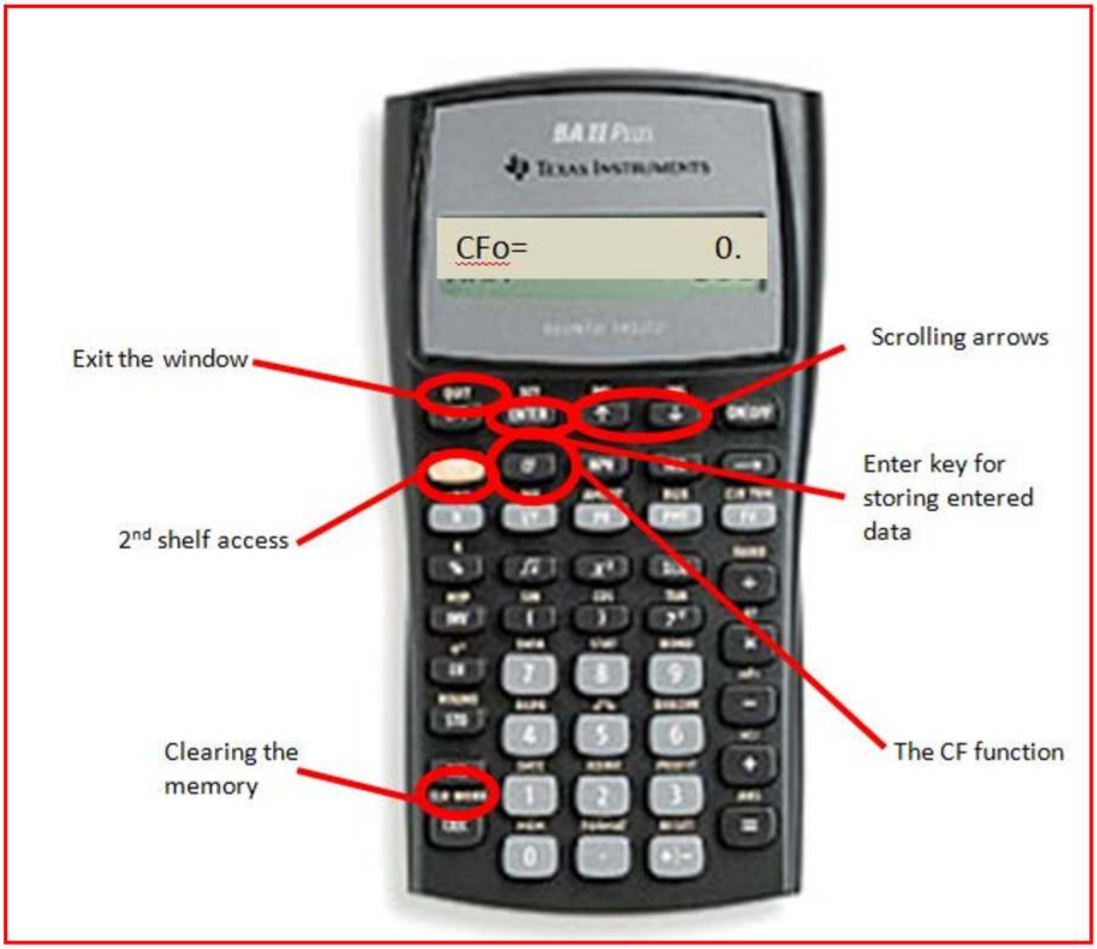
Step 3: Move each cash flow to its present value. If using formulas, calculate the periodic interest rate from Formula 9.1. Follow this calculation by repeatedly applying Formulas 9.2 (Number of Compound Periods for Single Payments) and 9.3 (Compound Interest for Single Payments). Alternatively, if using technology then apply the technique discussed in the appropriate section below.

Step 4: Net the results using Formula 15.1 to arrive at the NPV.

Important Notes

Using the BAII Plus Cash Flow Worksheet.

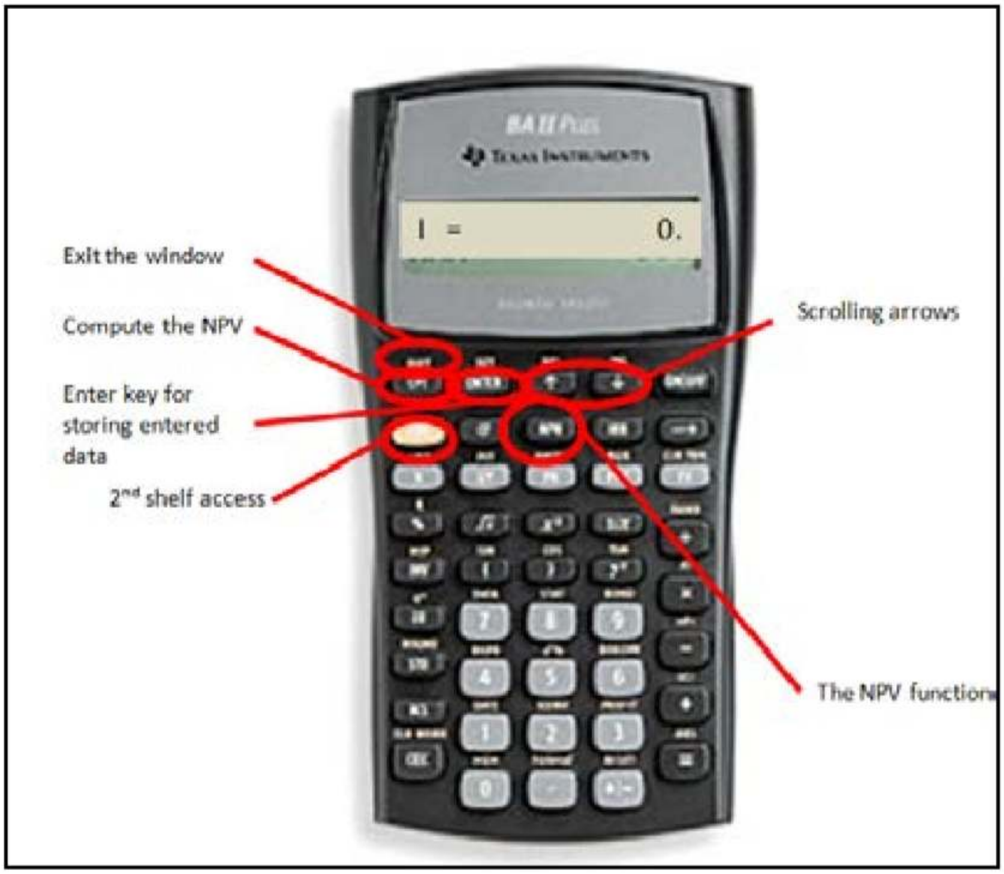
The cash flow function of the calculator requires the cash flows to be regular (such as every month or every quarter or every year). The payment amounts can be the same or variable. If the regularity assumption is not met, do not use the cash flow function; instead solve the problem using Rule #1 for compound interest single payments.



The photo illustrates the buttons used in the cash flow function of your BAII Plus calculator. Access the cash flow function by pressing CF on the keypad. Immediately clear the memory using 2nd CLR Work to delete any previously entered data. Use the ↑ and ↓ arrows to scroll through the various lines. You must strictly obey the cash flow sign convention and press Enter after keying in any data.

Here are the various lines appearing in the cash flow function:

- CF_o = any cash flow today
- CXX = a particular cash flow, where XX is the cash flow number starting with 01. You must key in the cash flows in order from the first time segment to the last. You cannot skip a time segment even if it has a value of zero. Each time segment is a placeholder on the timeline.
- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times in a row the corresponding cash flow amount occurs. This allows you to enter recurring amounts instead of keying them in separately. By default, the calculator sets this variable to 1.



Once you have entered all cash flows, you must access other functions to generate the output. For net present value, press NPV on the keypad as illustrated in the photo. This window has two lines:

- I = the matching periodic interest rate for the interval of each time segment. If the timeline is drawn with yearly intervals, then this must be the annual cost of capital. If the timeline is drawn with semi-annual intervals, this needs to be the matching semi-annual periodic cost of capital.
- NPV = the net present value. Press CPT to calculate this amount. The output nets all future cash flows against any initial investment. This key executes Formula 15.1.

Things To Watch Out For

The most common mistake in using the BAII Plus calculator is failing to enter the matching periodic interest rate in the NPV window. Always check the following:

1. The compounding on the cost of capital must match the interval between cash flows. If not, convert the rate to match. For example, if your intervals are semi-annual but your cost of capital is annual, you must convert the cost of capital to its equivalent semi-annually compounded rate.
2. Apply Formula 9.1, being sure to enter the periodic rate and not the nominal rate of interest.

Paths To Success

Since cash flows can be inflows or outflows, it avoids confusion to write all numbers with their cash flow sign conventions onto the timeline. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.

? Give It Some Thought

If a negative net present value is calculated, does that mean the project is unprofitable?

Answer

> No, it means the project cannot recover its investments and the cost of capital. If the company can sufficiently lower its cost of capital, then the net present value may in fact become positive

✓ Example 15.1.2: Decision #2: Pursuing a Particular Option

Revisit the new product launch discussed in the opening scenario in this section. Recall that research and development costs are $10 million today with an expected revitalization cost of $5 million in the fifth year. The product is expected to go on sale starting in two years and is forecast to produce year-end profits of $4 million for the first three years and another $3 million per year for the subsequent two years. Its last year will earn a projected $1 million in profit. The cost of capital for this project is 12% compounded annually. From a financial perspective only, should this new product be launched?

Solution

You have cash flows on many different dates. In order to decide today, you should move all monies to today and calculate the net present value (NPV).

What You Already Know

Steps 1 and 2:

The timeline for the proposed product launch appears below.

$$IY = 12\%, CY = 1, PV = -\$10M$$

Future cash flows are shown in the timeline. Note that the cost in the fifth year occurs at the beginning of the year, which is the same point in time as the end of the fourth year.

How You Will Get There

Step 3:

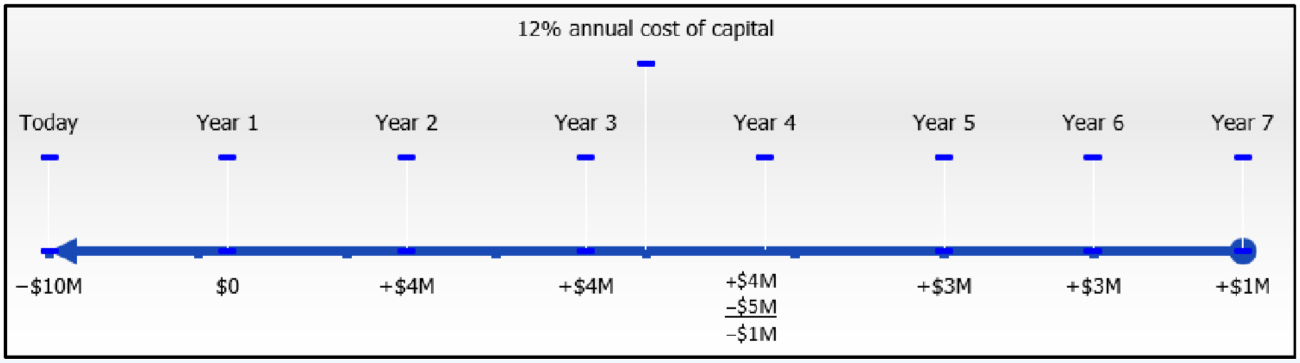
Calculate the periodic interest rate by using Formula 9.1. For each cash flow, calculate the present value using Formulas 9.2 and 9.3, rearranging for PV.

$$\begin{aligned}
&i = 12\%/1 = 12\% \\
&\text{ Cash Flow } 1 : N = 1 \times 2 = 2 \text{ compounds; } PV = \$4,000,000 \div (1+0.12)^2 = \$3,188,775.51 \\
&\text{ Cash Flow } 2 : N = 1 \times 3 = 3 \text{ compounds; } PV = \$4,000,000 \div (1+0.12)^3 = \$2,847,120.991 \\
&\text{ Cash Flow } 3 : N = 1 \times 4 = 4 \text{ compounds; } PV = -\$1,000, \\
&000 \div (1+0.12)^5 = \$1,702,280.567 \\
&\text{ Cash Flow } 4 : N = 1 \times 5 = 5 \text{ compounds; } PV = \$3,000,000 \div (1+0.12)^6 = \$1,519,893.367 \\
&\text{ Cash Flow } 6 : N = 1 \times 7 = 7 \text{ compounds; } PV = \$1,000,000 \div (1+0.12)^7 = \$452,349.2153
\end{aligned}$$

Step 4:

Calculate NPV by using Formula 15.1.

$$\begin{aligned}
NPV &= (\$3,188,775.51 + \$2,847,120.991 - \$635,518.0784 + \$1,702,280.567 + \$1,519, \\
&\quad 893.364 + \$452,349.2153) - \$10,000,000 \\
&= \$9,074,901.569 - \$10,000,000 \\
&= -\$925,098.43 \\
&==> -\$925,098
\end{aligned}$$



Perform

Step 3:

i = 12%/1 = 12%

$$\begin{array}{l}
\text{Cash Flow 1}: N = 1 \times 2 = 2 \text{ compounds; } PV = \$4,000,000 \div (1+0.12)^2 = \$3,188,775.51 \\
\text{Cash Flow 2}: N = 1 \times 3 = 3 \text{ compounds; } PV = \$4,000,000 \div (1+0.12)^3 = \$2,847,120.991 \\
\text{Cash Flow 3}: N = 1 \times 4 = 4 \text{ compounds; } PV = -\$1,000,000 \div (1+0.12)^4 = -\$635,518.0784 \\
\text{Cash Flow 4}: N = 1 \times 5 = 5 \text{ compounds; } PV = \$3,000,000 \div (1+0.12)^5 = \$1,702,280.567 \\
\text{Cash Flow 5}: N = 1 \times 6 = 6 \text{ compounds; } PV = \$3,000,000 \div (1+0.12)^6 = \$1,519,893.364 \\
\text{Cash Flow 6}: N = 1 \times 7 = 7 \text{ compounds; } PV = \$1,000,000 \div (1+0.12)^7 = \$452,349.2153
\end{array}$$

Step 4:

$$\begin{aligned}
NPV &= (\$3,188,775.51 + \$2,847,120.991 - \$635,518.0784 + \$1,702,280.567 + \$1,519, \\
&\qquad 893.364 + \$452,349.2153) - \$10,000,000 \\
&= \$9,074,901.569 - \$10,000,000 \\
&= -\$925,098.43 \\
&==> -\$925,098
\end{aligned}$$

Calculator Instructions

Cash Flows

Cash Flow	**Amount (CXX**	**Frequency (FXX)**
CFO	-10000000	N/A
C01 & F01	0	1
C02 & F02	4000000	2
C03 & F03	-1000000	1
C04 & F04	3000000	2
C05 & F05	1000000	1

NPV

I	NPV
12	Answer: -925,098.4309

If the forecasted costs and profits are reasonably accurate, then the new product should not be launched under the current financial structure since its net present value is −$925,098. This means that the profits are unable to recover the investments required along with the cost of capital.

Example 15.1.3: Decision #1: Choosing between Options with Cash Flows

Lethbridge Community College is considering purchasing new industrial photocopier equipment and has narrowed the choices down to two comparable Xerox and Canon machines. The Xerox machine can be acquired for $81,900 today and is expected to have a life of four years with savings of $31,000 in labour and materials each year. A $10,000 maintenance package is expected to be purchased at the beginning of year three, and the machine can be sold for $16,000 after the four years. The Canon machine retails for $74,800 and is forecasted to save $26,000 in labour and materials each year. A $4,000 maintenance procedure is needed at the beginning of both years three and four. The salvage value of the machine is estimated at $11,750. The cost of capital is 11%. Which machine should be purchased, and how much money will it save over the alternative?

Solution

For each photocopier, you have cash flows on many different dates. In order to decide today, you should move all monies to today and calculate the net present value (NPV) for each photocopier.

What You Already Know

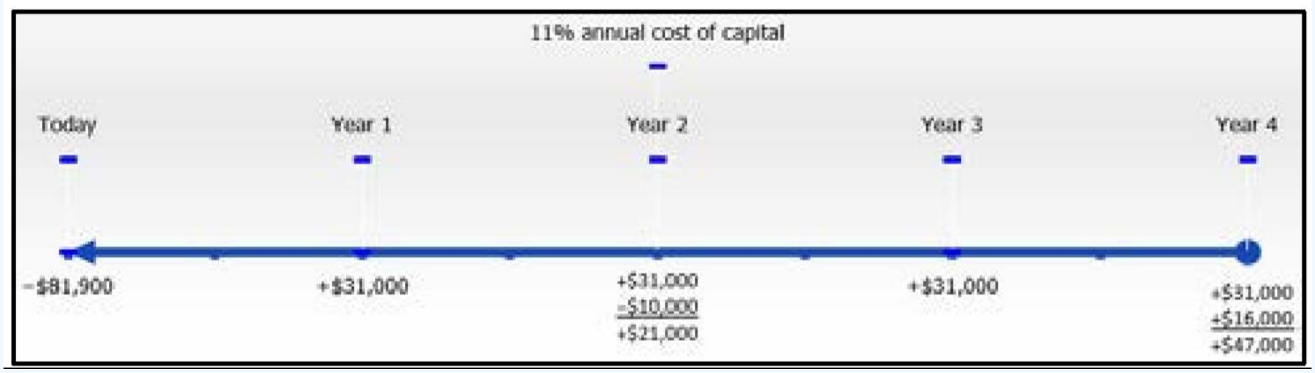
Steps 1 and 2:

The timelines for each photocopier appear below (Xerox on left, Canon on right).

Xerox: $IY = 11\%, CY = 1, PV = -\$81,900$, future cash flows are shown in the timeline.

Canon: $IY = 11\%, CY = 1, PV = -\$74,800$, future cash flows are shown in the timeline.

Note that costs occurring at the beginning of the year are recorded at the end of the previous year.



How You Will Get There

Step 3:

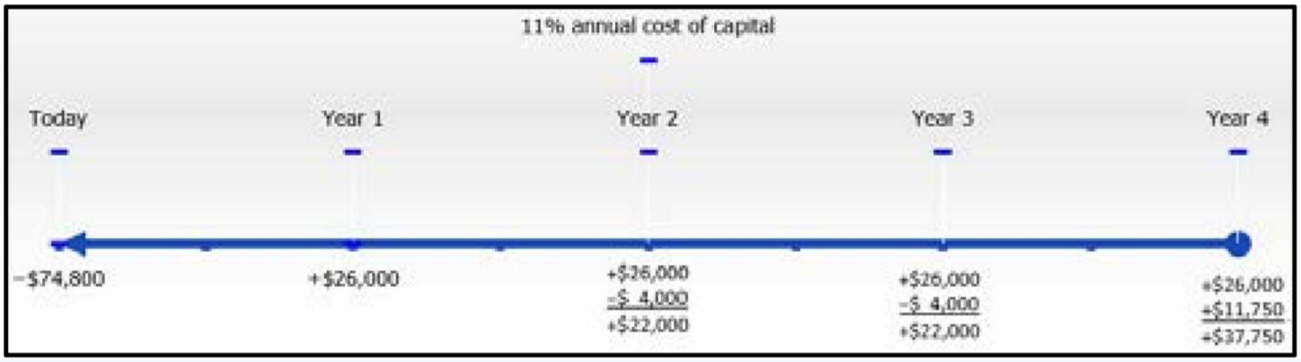
Calculate the periodic interest rate by using Formula 9.1.For each photocopier, calculate the present value of each cash flow using Formulas 9.2 and 9.3, rearranging for PV.

Step 4:

For each photocopier, calculate NPV by using Formula 15.1.

Step 5:

Calculate the savings by choosing the better option.



Perform

Step 3:

$$i = 11\%/1 = 11\%$$

Xerox:

$$\begin{aligned}
&\text{Cash Flow 1}: N = 1 \times 1 = 1 \text{ compounds; } PV = \$31,000 \div (1+0.11)^1 = \$27,927.\overline{927}\\
&\text{Cash Flow 2}: N = 1 \times 2 = 2 \text{ compounds; } PV = \$21,000 \div (1+0.11)^2 = \$17,044.0711\\
&\text{Cash Flow 3}: N = 1 \times 3 = 3 \text{ compounds; } PV = \$31,000 \div (1+0.11)^3 = \$22,666.93282\\
&\text{Cash Flow 4}: N = 1 \times 4 = 4 \text{ compounds; } PV = \$47,000 \div (1+0.11)^4 = \$30,960.35578
\end{aligned}$$

Step 4:

$$\begin{aligned}
NPV &= (\$27,927.\overline{927} + \$17,044.0711 + \$22,666.93282 + \$30,960.35578) - \$81,900\\
&= \$98,599.28763 - \$81,900\\
&= \$16,699.29\\
&==> \$16,699
\end{aligned}$$

Canon:

$$\begin{aligned}
&\text{Cash Flow 1}: N = 1 \times 1 = 1 \text{ compounds; } PV = \$26,000 \div (1+0.11)^1 = \$23,423.\overline{423}\\
&\text{Cash Flow 2}: N = 1 \times 2 = 2 \text{ compounds; } PV = \$22,000 \div (1+0.11)^2 = \$17,855.69353\\
&\text{Cash Flow 3}: N = 1 \times 3 = 3 \text{ compounds; } PV = \$22,000 \div (1+0.11)^3 = \$16,086.21039\\
&\text{Cash Flow 4}: N = 1 \times 4 = 4 \text{ compounds; } PV = \$37,750 \div (1+0.11)^4 = \$24,867.09427
\end{aligned}$$

Step 4:

Canon:

$$\begin{aligned}
NPV &= (\$23,423.\overline{423} + \$17,855.69353 + \$16,086.21039 + \$24,867.09427) - \$74,800\\
&= \$82,232.42162 - \$74,800\\
&= \$7,432.42\\
&==> \$7,432
\end{aligned}$$

Xerox has higher savings by $\$16,699 - \$7,432 = \$9,267$

Step 5:

Xerox has higher savings by \$16,699 − \$7,432 = \$9,267

Calculator Instructions

Cash Flows

	Xerox		Canon	
Cash Flow	**Amount (CXX**	**Frequency (FXX)**	**Amount (CXX**	**Frequency (\ (FXX\))**

Cash Flows

CFO	-81900	N/A	-74800	N/A
C01 & F01	31000	1	26000	1
C02 & F02	21000	1	22000	2
C03 & F03	31000	1	37750	1
C04 & F04	47000	1		

NPV

	Xerox	Canon
I	11	11
NPV	Answer: 16,699.28763	Answer: 7,432.421617

Based on the estimated costs, maintenance, and savings, the Xerox photocopier should be purchased because it results in an expected savings of $8,867 in today's dollars. The Xerox machine represents a current cash flow of $16,299, while the Canon is $7,432.

Working with Limited Resources

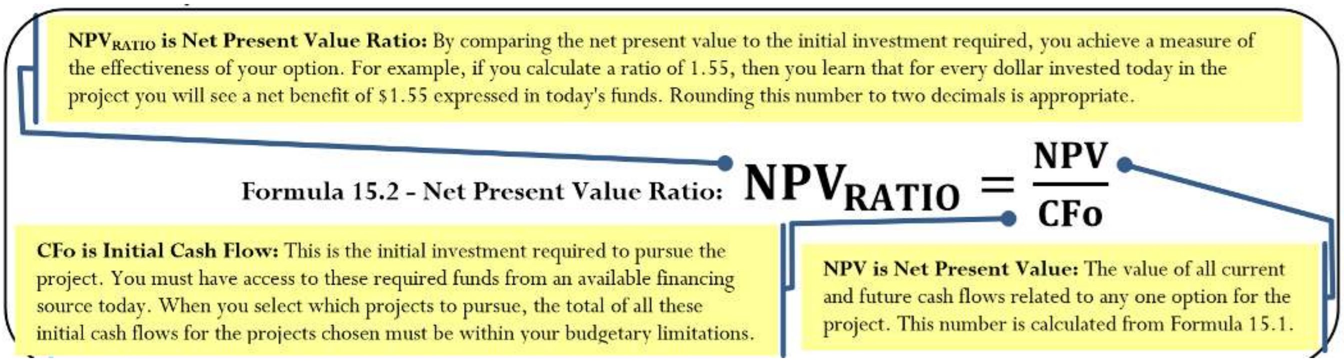
In an ideal financial world, you should always pursue any project with a positive net present value since that decision results in a financial gain. However, no one, whether an individual person or a large corporation or a government, actually has access to unlimited resources to fund all its potential projects. If such unlimited resources existed, the world's problems could be solved in a day!

There are usually many paths in life that you can choose, each with its own merits. These paths are not necessarily mutually exclusive, so you could choose more than one. However, having limited resources means that you cannot pursue all of these paths. For example, perhaps PepsiCo has learned through market research that it could launch 15 new flavours and types of soft drinks. Each of these flavours and types requires vast production resources, labour demands, and marketing support to bring it to market. But even a company as huge as PepsiCo does not have the resources to pursue all of these opportunities. Which one or ones does it select?

A process called hard capital rationing is used to decide. This process requires you to calculate the net present value of each possible project and then allocate your limited capital resources to the projects efficiently to maximize the combined net present value. From a financial perspective, you do not consider any projects with a negative net present value.

The Formula

The goal in choosing multiple options is to maximize the net present value. In selecting these various options, it is unfair to compare projects of different magnitudes simply by looking at their NPV. One would expect that a project requiring a $10 million investment should produce a much larger NPV than a project requiring a $10,000 investment. Instead, it is proper to examine the efficient usage of these limited resources, or to look at the "bang for your buck." This requires you to calculate the net present value ratio shown in Formula 15.2.



NPV_{RATIO} is Net Present Value Ratio: By comparing the net present value to the initial investment required, you achieve a measure of the effectiveness of your option. For example, if you calculate a ratio of 1.55, then you learn that for every dollar invested today in the project you will see a net benefit of $1.55 expressed in today's funds. Rounding this number to two decimals is appropriate.

Formula 15.2 - Net Present Value Ratio: $NPV_{RATIO} = \frac{NPV}{CFo}$

CFo is Initial Cash Flow: This is the initial investment required to pursue the project. You must have access to these required funds from an available financing source today. When you select which projects to pursue, the total of all these initial cash flows for the projects chosen must be within your budgetary limitations.

NPV is Net Present Value: The value of all current and future cash flows related to any one option for the project. This number is calculated from Formula 15.1.

Formula 15.2

How It Works

Follow these steps to perform hard capital rationing:

Step 1: Identify the budget constraints.

Step 2: If not already known, calculate the net present value (NPV) for each of the available projects using Formula 15.1.

Step 3: Calculate the net present value ratio for each project using Formula 15.2.

Step 4: Rank the projects from the highest to lowest ratios, noting the initial cash flow required under each project.

Step 5: Starting with the project with the highest net present value ratio and progressing through the ranked list of projects, assess different combinations of projects and their total NPV such that the total initial cash flows remain within the budget constraint.

Step 6: Choose the combination that maximizes the NPV.

Important Notes

This technique makes the assumption that each of the projects being considered is independent of the others. In other words, choosing one particular project or path does not prevent another project or path from being chosen. There are much more advanced techniques for optimizing a list of projects. These techniques include management sciences or linear programming methods. However, these techniques are beyond the scope of this textbook, which focuses simply on the basic principles behind this decision.

Paths To Success

When completing step 5 of the process, always select the highest NPV_{RATIO} projects in order until you reach a point where the remaining budget becomes an issue. In other words, only at the point where the selection of the next ranked project eliminates a lower-ranked project from consideration do you need to start examining the different possible combinations. In most typical situations, the optimal combination includes all projects selected before this critical point.

✓ Example $15.1.4$: Decision #3: Working with Limited Resources

Nygard International has identified several projects that it can pursue, as listed in the table below. The current finances of the company permit a capital budget of $1.25 million. Which projects should be undertaken?

Project	Initial Cash Flow	Net Present Value
Automate a production process	$428,000	$214,000
Launch new fashion A	$780,000	$550,000
Replace machinery in distribution centre	$150,000	$175,000
Launch new fashion B	$395,000	$610,000
Expand a retail outlet	$145,000	$245,000

Project	Initial Cash Flow	Net Present Value
Revise delivery processes	$282,000	$420,000

Solution

You need to perform hard capital rationing to maximize your NPV within your limited budget. This requires you to calculate the net present value ratio (NPV_{RATIO}) for each project.

What You Already Know

Step 1:

The total budget available is $1.25 million.

Step 2:

The net present values are provided in the table.

How You Will Get There

Step 3:

Calculate the net present value ratio for each project using Formula 15.2.

Steps 4 to 6:

Rank the NPV_{RATIO} from highest to lowest, assess the budget and combinations available, and then choose the possibility that maximizes NPV.

Perform

Steps 3 to 5:

The NPV_{RATIO} is shown in the table below. Projects are sorted into descending order. A few calculations listed below the table show how the NPV_{RATIO} is calculated. Note that after you select the first project, if you were to select the second project then this would eliminate the fifth project since its initial cash flow exceeds the remaining budget ($1,250,000 − $145,000 − $395,000 = $710,000 remaining). Therefore, you always include the first project in your combinations and must explore various arrangements of the next five projects.

Project	Initial Cash Flow	Net Present Value	Net Present Value Ratio	Some Possible Combinations				
				1	2	3	4	5
Expand a retail outlet	$145,000	$245,000	(1)1.69	.	.	.	.	.
Launch new fashion B	$395,000	$610,000	(2)1.54	.	.	.		
Revise delivery processes	$282,000	$420,000	1.49	.	.		.	.
Replace machinery in distribution centre	$150,000	$175,000	1.17	.		.	.	
Launch new fashion A	$780,000	$550,000	0.71					.

Project	Initial Cash Flow	Net Present Value	Net Present Value Ratio	Some Possible Combinations				
Automate a production process	$428,000	$214,000	0.50		.	.	.	

(1) $NPV_{RATIO} = \frac{\$245,000}{\$145,000} = 1.69$

(2) $NPV_{RATIO} = \frac{\$610,000}{\$395,000} = 1.54$

	Combinations				
	1	2	3	4	5
Total Budget	(1) $972,000	$1,250,000	$1,118,000	$1,005,000	$1,207,000
Total NPV	(2) $1,450,000	$1,489,000	$1,244,000	$1,054,000	$1,215,000

(1) $145,000 + $395,000 + $282,000 + $150,000 = $972,000

(2) $245,000 + $610,000 + $420,000 + $175,000 = $1,450,000

Step 6:

Combination #2 offers the highest NPV of $1,489,000.

After the projects are ranked from highest to lowest NPV ratio, the "Expand a retail outlet" offered the highest ratio and is selected. Combination #2, consisting of "Expand a retail outlet," "Launch new fashion B," "Revise delivery processes," and "Automate a production process" offers the highest NPV of $1,489,000 and uses the full $1,250,000 capital budget.

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15.2: Other Measures For Making Decisions

The previous section explored the three main types of decisions and introduced various techniques for making smart investment decisions. This section revisits Decision #1 (making one choice from multiple options) and Decision #2 (pursuing one course of action) and introduces new characteristics:

- Decision #1: Examine situations where the timelines are not equal in length.
- Decision #2: Determine another profit-focused method of reaching the same decision.

Making Choices for Unequal Timelines

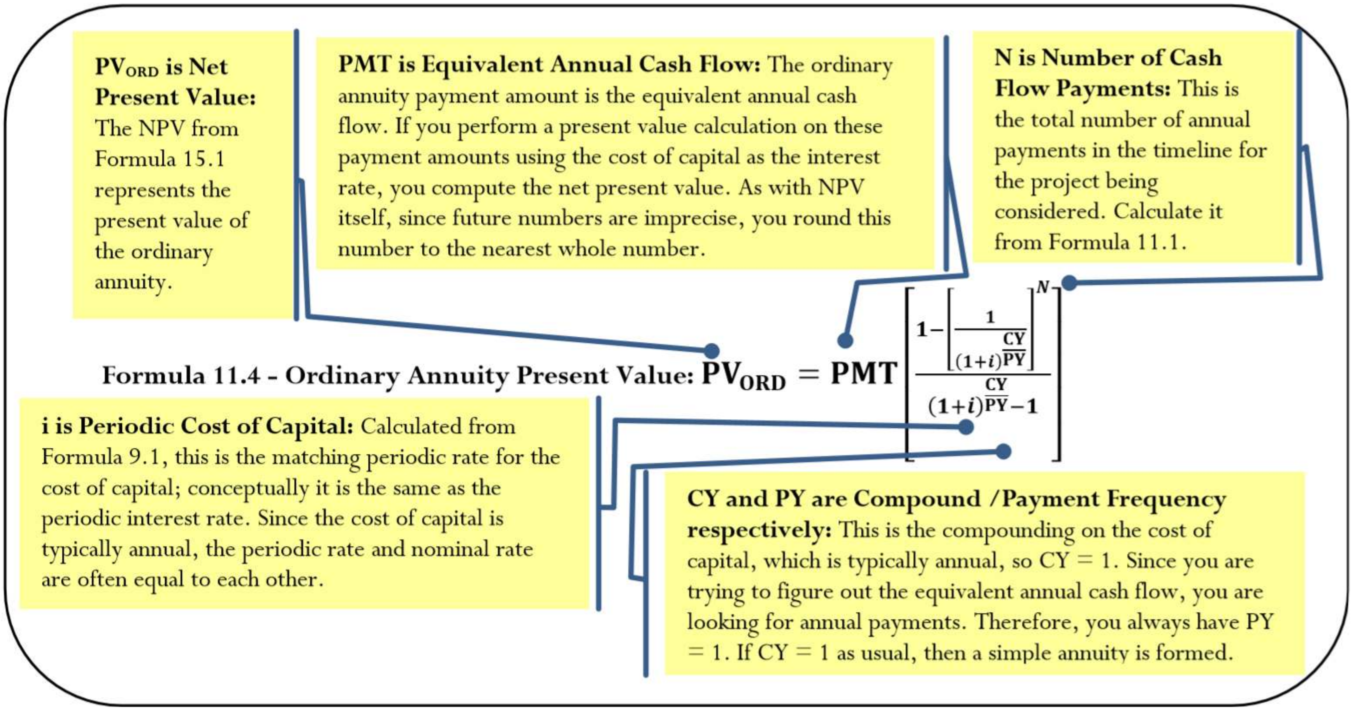
Consider the situation in which you need to buy only one of two $50,000 machines that solve the same problem. Machine #1 has savings of $20,000 per year, while machine #2 has savings of $14,000 per year. However, machine #1 has a life expectancy of four years, while machine #2 has a life expectancy of eight years. Calculating the net present value on these two machines at a 15% cost of capital reveals the NPV of machine #1 equals $7,100 while the NPV of machine #2 equals $12,823. Is this a fair comparison? Should machine #2 be selected based on its higher NPV? The answer is no. The NPV analysis does not factor in that if you chose machine #1 it must be replaced after four years and would once again have the opportunity to produce more savings over the subsequent four years, thereby offsetting the original difference in NPV.

To fairly choose between the alternatives, you need a calculation that can equate two (or more) timelines of different lengths. This requires you to convert the net present value for each alternative into its equivalent annual cash flow. This is an annual annuity payment amount that, when present valued using the cost of capital, arrives at the same NPV as all of the original cash flows.

The Formula

To arrive at the equivalent annual cash flow, you need to apply two formulas:

1. Formula 15.1 calculates the net present value for each alternative you are considering.
2. To convert each calculated NPV into an annual cash flow payment, convert the net present value into an annuity possessing an annual cost of capital and annual end-of-interval payments—or an ordinary simple annuity! Formula 11.4 is reprinted below to illustrate how you can adapt this formula to suit the purposes of the equivalent annual cash flow.



PV$_{\text{ORD}}$ is Net Present Value: The NPV from Formula 15.1 represents the present value of the ordinary annuity.

PMT is Equivalent Annual Cash Flow: The ordinary annuity payment amount is the equivalent annual cash flow. If you perform a present value calculation on these payment amounts using the cost of capital as the interest rate, you compute the net present value. As with NPV itself, since future numbers are imprecise, you round this number to the nearest whole number.

N is Number of Cash Flow Payments: This is the total number of annual payments in the timeline for the project being considered. Calculate it from Formula 11.1.

Formula 11.4 - Ordinary Annuity Present Value:

$$\mathbf{PV_{ORD}} = \mathbf{PMT}\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right]$$

i is Periodic Cost of Capital: Calculated from Formula 9.1, this is the matching periodic rate for the cost of capital; conceptually it is the same as the periodic interest rate. Since the cost of capital is typically annual, the periodic rate and nominal rate are often equal to each other.

CY and PY are Compound / Payment Frequency respectively: This is the compounding on the cost of capital, which is typically annual, so CY = 1. Since you are trying to figure out the equivalent annual cash flow, you are looking for annual payments. Therefore, you always have PY = 1. If CY = 1 as usual, then a simple annuity is formed.

How It Works

Follow these steps to calculate the equivalent annual cash flow:

Step 1: For each alternative project, draw a timeline and calculate the net present value using Formula 15.1 and the techniques discussed in Section 15.1.

Step 2: For each alternative project, calculate the periodic interest rate using Formula 9.1 and the number of annuity payments using Formula 11.1. Then solve Formula 11.4 for PMT.

Step 3: Compare the equivalent annual cash flow of each alternative and make the best decision.

Important Notes

Factoring in the unequal life expectancies of projects is important for situations in which you can choose only one out of many mutually exclusive projects. This is the basis for Decision #1. With respect to the other decisions:

- There is only one timeline to consider in Decision #2 (pursuing one course of action), so the issue of unequal life expectancies does not apply.
- In Decision #3 (making multiple choices under constraints), the net present value ratio provides an adequate means of equating different timelines. You do not need the equivalent annual cash flow.

Exercise $15.2.1$: Give It Some Thought

For each of the following decisions where alternative projects solve the same problem and only one can be chosen (a Decision #1 situation), indicate whether the decision should be made through the comparison of NPV or equivalent annual cash flows.

1. Project #1 with a seven-year life; Project #2 with a seven-year life
2. Project #1 with a five-year life; Project #2 with a seven-year life

Answer

1. NPV; each alternative has the same time frame.
2. Equivalent annual cash flow; each alternative has a different time frame

Example $15.2.1$: Which Machine to Purchase when Timelines Are Unequal

Recall the earlier situation in which you can buy only one of two equal $50,000 machines. Machine #1 has savings of $20,000 per year, while machine #2 has savings of $14,000 per year. However, machine #1 has a life expectancy of four years, while machine #2 has a life expectancy of eight years. The cost of capital is 15%. Determine which machine should be purchased and the annual benefit of your choice.

Solution

Notice that you are in a Decision #1 situation and need to choose from these two alternatives. Also notice that the cost of capital is known but the timelines are different. You need to use the equivalent annual cash flow to make the decision.

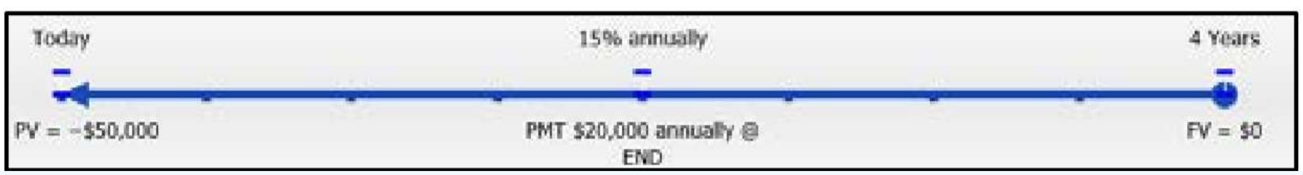
What You Already Know

Step 1:

The timelines for machine #1 and machine #2 appear below, respectively.

Machine #1: PV = −$50,000, IY = 15%, CY = 1, PMT = $20,000, PY = 1, FV = $0, Years = 4

Machine #2: PV = −$50,000, IY = 15%, CY = 1, PMT = $14,000, PY = 1, FV = $0, Years = 8



How You Will Get There

Step 1 (continued):

Each timeline represents an ordinary simple annuity. Calculate the net present value by applying Formulas 9.1, 11.1, 11.4, and Formula 15.1 to each alternative.

Step 2:

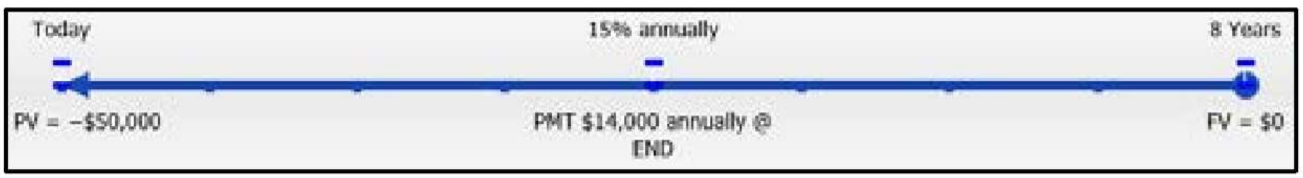
For each alternative, calculate the equivalent annual cash flow using Formula 11.4 (rearranging for PMT).

Step 3:

Make the decision.

Step 4:

Determine on an annual basis how much better your decision is by taking the equivalent annual cash flow of your chosen alternative and subtracting the equivalent annual cash flow of the other alternative.



Perform

Machine #1:

Step 1:

$$i = 15\%/1 = 15\%; N = 1 \times 4 = 4 \text{ payments}$$

$$PV_{ORD} = \$20,000\left[\frac{1-\left[\frac{1}{(1+0.15)^{\frac{1}{1}}}\right]^{4}}{(1+0.15)^{\frac{1}{1}-1}}\right] = \$57,099.56725$$

$$\begin{aligned} NPV &= \$57,099.56725 - \$50,000 \\ &= \$7,099.56725 \\ &==> \$7,100 \end{aligned}$$

Step 2:

$$\begin{aligned} PMT = &\frac{\$7,100}{\left[\frac{1-\left[\frac{1}{(1+0.15)^{\frac{1}{1}}}\right]^{4}}{(1+0.15)^{\frac{1}{1}}-1}\right]} \\ &= \$2,486.883996 \\ &==> \$2,487 \end{aligned}$$

Machine #2:

Step 1:

$$i = 15\%/1 = 15\%; N = 1 \times 8 = 8 \text{ payments}$$

$$PV_{ORD} = \$14,000\left[\frac{1-\left[\frac{1}{(1+0.15)^{\frac{1}{1}}}\right]^{8}}{(1+0.15)^{\frac{1}{1}-1}}\right] = \$62,822.50111$$

$$
\begin{aligned}
NPV &= \$62,822.50111 - \$50,000 \\
&= \$12,822.50111 \\
&==> \$12,823
\end{aligned}
$$

Step 2:

$$
\begin{aligned}
PMT &= \frac{\$12,823}{\left[\frac{1-\left[\frac{1}{(1+0.15)^{\frac{1}{1}}}\right]^{8}}{(1+0.15)^{\frac{1}{1}}-1}\right]} \\
&= \$2,857.606699 \\
&==> \$2,858
\end{aligned}
$$

Step 3:

The best choice is machine #2 because it has a higher equivalent annual cash flow of $2,858.

Step 4:

Annual benefit = Machine #2 − Machine #1 = $2,858 − $2,487 = $371

Calculator Instructions

Cash Flows				
	Machine #1		Machine #2	
Cash Flow	Amount (CXX)	Frequency (FXX)	Amount (CXX)	Frequency (FXX)
$CF0$	-50000	N/A	-50000	N/A
C01 & F01	20000	4	14000	8

NPV		
	Machine #1	Machine #2
I	15	15
NPV	Answer: 7,099.567254	Answer: 12,822.50111

Machine	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	END	4	15	-7100	Answer: 2,486.883996	0	1	1
2	√	8	√	-12823	Answer: 2,857.606699	0	√	√

The smart decision is to purchase machine #2 because it produces the highest equivalent annual cash flow of $2,858, which represents savings of $371 more per year than machine #1.

Internal Rate of Return

Another method of reaching a decision when choosing whether to pursue a single course of action (Decision #2) involves percentages. While the NPV calculations in Section 15.1 provide an exact monetary magnitude of the project, the common mindset in business focuses on profitability as a percentage and not a dollar amount. Thus, decisions are based on the **internal rate of**

return for a project, or IRR for short. The IRR is the annual percentage rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.

To interpret the IRR, examine the NPV decision criteria and the relationship to the IRR:

1. If the net present value is greater than or equal to \$0, pursue the project.
   1. If the NPV is more than zero, the definition of IRR requires you to find a rate of return such that your present value becomes zero. Mathematically this means that a higher discount rate must be used to calculate your present value. In other words, the IRR is greater than the cost of capital.
   2. If the NPV equals zero, by definition the cost of capital and the IRR are the same value.
2. If the net present value is less than \$0, do not pursue the project. The IRR requires you to find a rate of return where the present value becomes zero. Mathematically this means that a lower discount rate must be used to calculate your present value. In other words, the IRR is less than the cost of capital.

This table summarizes how to decide whether to pursue a single course of action using the IRR method instead of the NPV method.

If...	So...	Then...	Decision
$NPV > 0$	IRR > Cost of Capital	It is profitable since it makes enough money to cover the costs	Pursue the project
$NPV = 0$	IRR > Cost of Capital	It breaks even and just pays the bills	This is the minimum financial condition to pursue the project
$NPV < 0$	IRR > Cost of Capital	It is unprofitable and does not make enough money to cover the costs	Do not pursue the project

The Formula

Solving for the internal rate of return requires you to calculate the annually compounded interest rate for the project. For annuities, substituting and rearranging Formula 15.1 produces:

$$NPV = \text{ (Present Value of All Future Cash Flows) - (Initial Investment)}$$

$$\$0 = PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right] - (\text{ Initial Investment })$$

$$\text{Initial Investment } = PMT\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{CY}{PY}}}\right]^{N}}{(1+i)^{\frac{CY}{PY}}-1}\right]$$

The only algebraic method to solve this general formula for the periodic interest rate is through trial and error, which is time consuming and inefficient. The same algebraic problem exists if your cash flows consist of multiple lump-sum amounts at different points in time. Assume you have inflows of \$15,000 and \$10,000 at the end of years one and two, respectively. Taking Formula 15.1 you have:

$$NPV = \text{ (Present Value of All Future Cash Flows) - (Initial Investment)}$$

$$\$0 = \left(\$15,000/(1+i)^{1} + \$10,000/(1+i)^{2}\right) - (\text{ Initial Investment })$$

$$\text{Initial Investment } = \$15,000/(1+i)^{1} + \$10,000/(1+i)^{2}$$

It is algebraically difficult to solve this formula for the periodic interest rate.

Therefore, using the same process as in Section 11.6, you should let the BAII Plus calculator perform the trial and error and arrive at the solution. Note that Excel can also perform this calculation, and the formula is pre-programmed into the chapter template.

How It Works

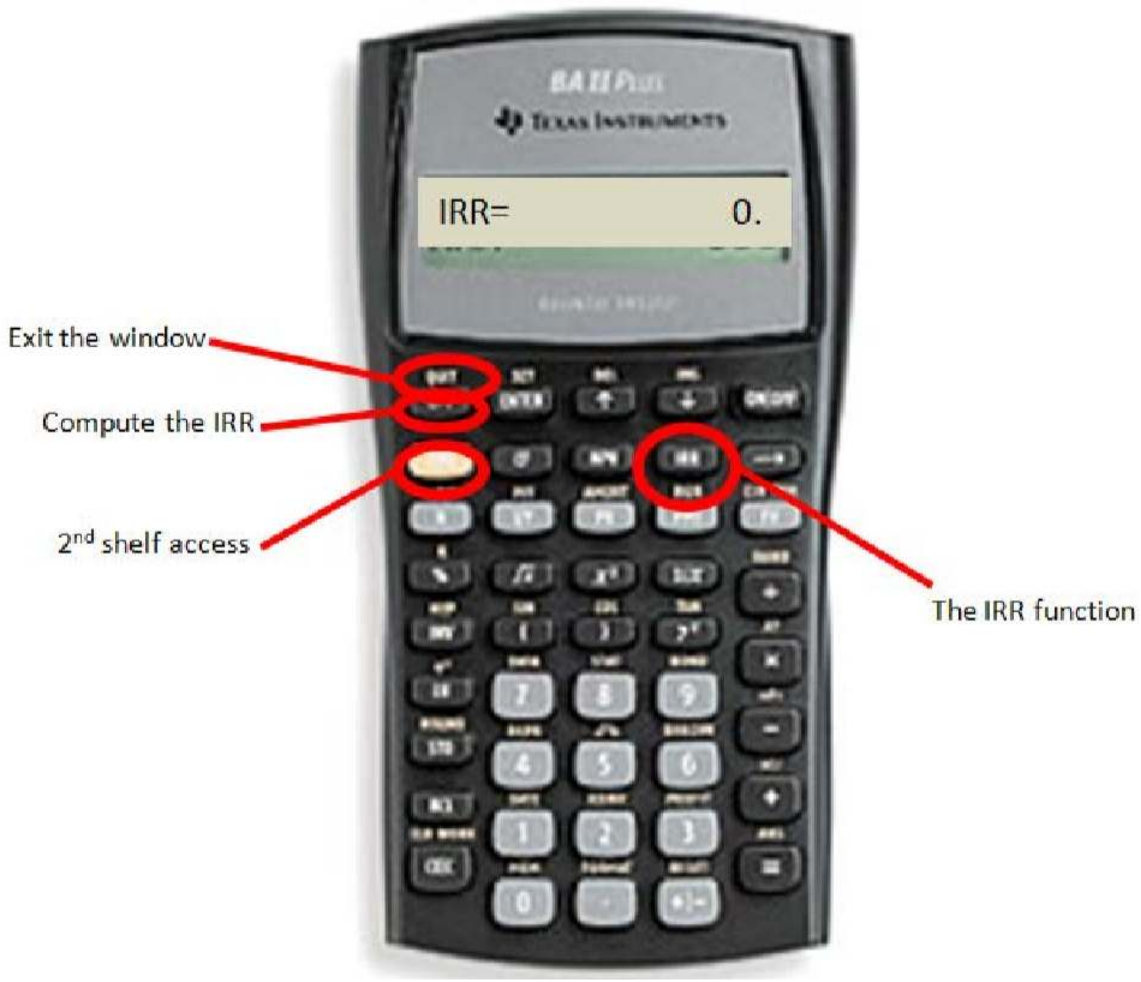
Follow these steps to solve for the internal rate of return:

Step 1: Draw a timeline to illustrate the cash flows involved in the project.

Step 2: If using manual trial and error, set up the appropriate algebraic formula to arrive at an NPV of $0 and start the sequence of iterations to generate the solution. Alternatively, use technology such as the BAII Plus calculator by entering the cash flows and solving for the IRR.

Step 3: Compare the IRR to the cost of capital and make a decision.

Important Notes



Using the IRR Function on the BAII Plus Calculator. Use the IRR function in conjunction with the CF (cash flow) function. Once you have entered all cash flows, activate the IRR function by pressing the IRR key followed by the CPT button to perform the calculation. The output is a percentage in percent format. To exit the window, press 2nd Quit. Recall that because of the trial-and-error method required, the calculator may briefly hesitate before displaying the solution.

Things To Watch Out For

When making decisions, you use the internal rate of return only to figure out if one particular path should be followed or not (which is Decision #2). The internal rate of return should not be used when making one choice from multiple options (Decision #1) or when making multiple choices under constraints (Decision #3). This rule is in place for two reasons:

1. **The Cost of Capital Is Ignored**. The IRR does not factor in the cost of capital in its computations. Recall that the fundamental concept of time value of money requires all money to be on the same date using an appropriate rate of interest—the cost of capital—before any decision can be made. Therefore, if you have not factored in the cost of capital, then your analysis is incomplete and choosing between different alternatives based solely on the IRR is flawed.
2. **The Magnitude of the Decision Is Ignored**. It can easily happen that an alternative has a high IRR but a low NPV. For example, using a cost of capital of 10% consider two alternatives. Alternative A invests $1 and one year later returns $1.50. The IRR is 50%, while the NPV is $0.36. Alternative B invests $1,000 and one year later returns $1,250. The IRR is 25%, while the NPV is $136.36. If choosing between these two options based on the IRR, you select Alternative A, resulting in a net present value that is $136 lower than for Alternative B.

? Give It Some Thought

In each of the following situations, determine whether the project should be pursued or not.

a. Cost of capital = 15%; IRR = 17%
b. Cost of capital = 12%; IRR = 9%
c. Cost of capital = 14%; IRR = 14%

Answer

a. Pursue it; IRR > cost of capital
b. Do not pursue it; IRR < cost of capital
c. Minimum condition to pursue it; IRR = cost of capital (breaks even

✓ Pursuing a Project Using the IRR Criterion

Tim Hortons has purchased the lease on a three-year onsite concession space in the cafeteria at a local college for $750,000. The franchise is expected to earn $400,000, $500,000, and $600,000 in profits per year for the first three years, respectively.

a. What is the investment's internal rate of return?
b. If the cost of capital is 20%, did Tim Hortons make a smart financial decision?

Solution

You need to calculate the internal rate of return (IRR) for this project. Once calculated, you can compare it to the provided cost of capital to evaluate the decision.

What You Already Know

Step 1:

The timeline for this project appears below.

PV = $750,000, CY = 1

$C01$ = $400,000, Years = 1

$C02$ = $500,000, Years = 2

$C03$ = $600,000, Years = 3

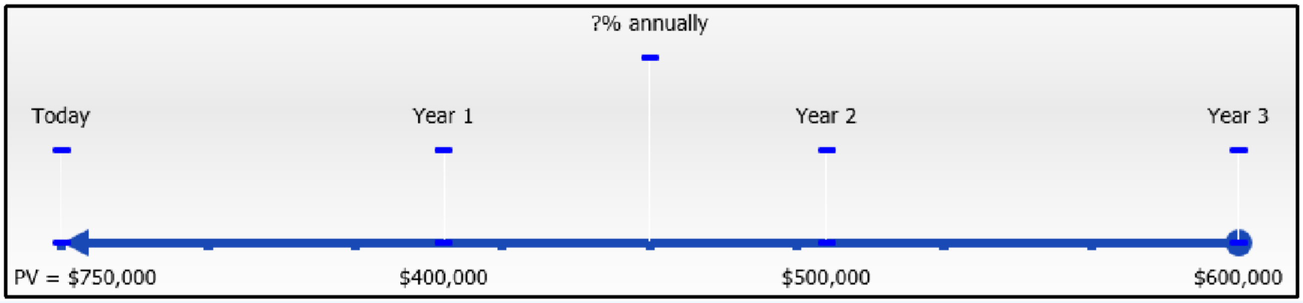
How You Will Get There

Step 2:

This project involves multiple lump-sum cash flows, so apply Formulas 9.2 and 9.3 (rearranged for PV). Substitute into the rearranged Formula 15.1. Algebraically, this must be solved through trial and error for i (note that $i = IY$ since the compounding frequency is 1). Alternatively, use the cash flow and internal rate of return function on your calculator.

Step 3:

Compare the IRR to the cost of capital to make the decision.



Perform

Step 2:

Cash Flow 1: $N = 1 \times 1 = 1 \text{ compound};\ PV = \$400,000 \div (1+i)^1$

Cash Flow 2: $N = 1 \times 2 = 2 \text{ compounds};\ PV = \$500,000 \div (1+i)^2$

Cash Flow 3: $N = 1 \times 3 = 3 \text{ compounds};\ PV = \$600,000 \div (1+i)^3$

$\$750,000 = \$400,000 \div (1+i)^1 + \$500,000 \div (1+i)^2 + \$600,000 \div (1+i)^3$

Through trial and error or by using the calculator (see instructions below), the calculated solution is: IY = 40.9235%

Step 3:

40.9235% > 20% ==> smart decision

Calculator Instructions

Cash Flow	Amount (CXX)	Frequency (FXX)
CF0	-750000	N/A
C01 & F01	400000	1
C02 & F02	500000	1
C03 & F03	600000	1

IRR

Answer: 40.923472

Since the internal rate of return on this project is 40.9235%, which far exceeds the cost of capital of 20%, Tim Hortons made a very smart financial decision in pursuing this project.

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15.S: Summary

Key Concepts

Section 15.1: Net Present Value (What Is the Value of My Decision?)

- The characteristics of making decisions including decision types, monetary sources, and interest rates
- Making decisions through net present value
- Handling calculations involving regular cash flows
- Making multiple choices when resources are limited

Section 15.2: Other Measures for Making Decisions (How Else Can I Do It?)

- Choosing between projects of different lengths
- Using internal rate of return to choose whether to pursue one course of action

The Language of Business Mathematics

cash flow

A movement of money into or out of a particular project

cost of capital

A weighted average of all of the debt and equity financing rates used to provide needed funds for a project.

equivalent annual cash flow

An annual annuity payment amount that, when present valued using the cost of capital, arrives at the same NPV as all of the original cash flows.

hard capital rationing

A process that requires the calculation of the net present value for each possible project and then the allocation of limited capital resources to these projects efficiently to maximize the combined net present value.

internal rate of return (IRR)

The annual rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.

net present value (NPV)

The difference in today's dollars between all benefits and costs for any given project.

The Formulas You Need to Know

Symbols Used

NPV = net present value

NPV_{RATIO} = net present value ratio

CF_o = initial cash flow

Formulas Introduced

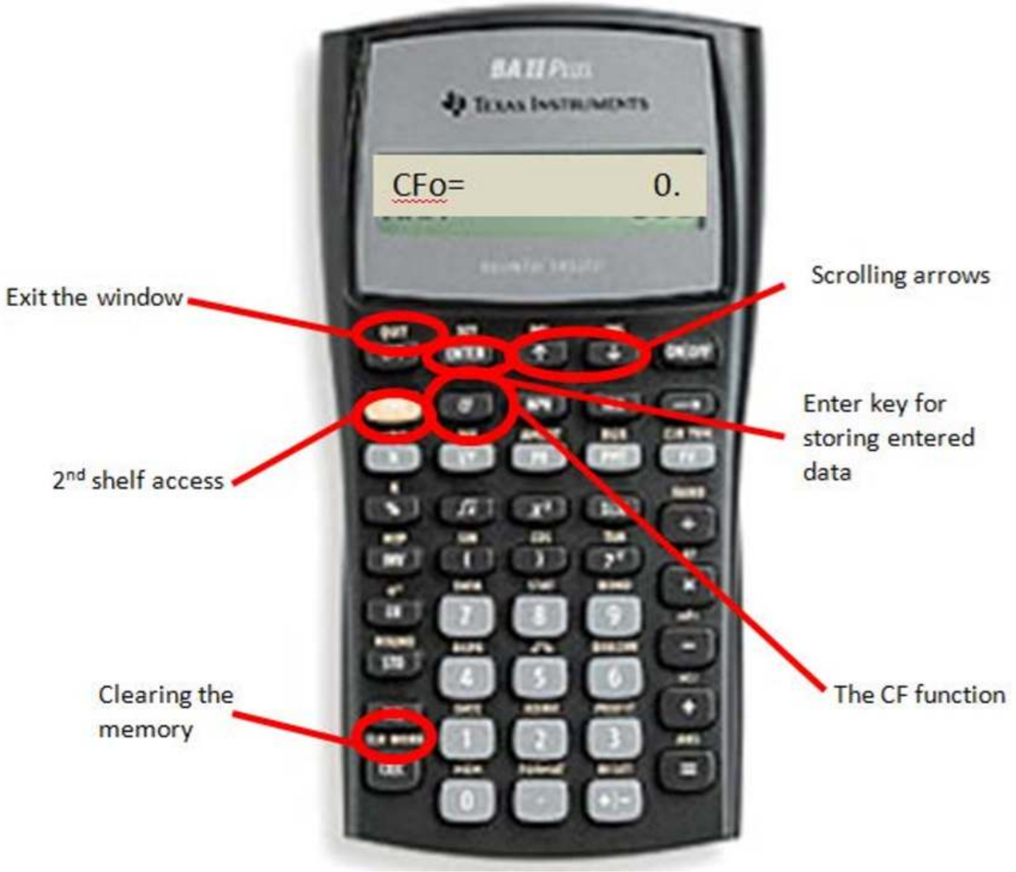
Formula 15.1 Net Present Value: $\text{NPV} = (\text{Present Value of All Future Cash Flows}) \text{ - } (\text{Initial Investment})$

Formula 15.2 Net Present Value Ratio: $\text{NPV}_{\text{RATIO}} = \dfrac{\text{NPV}}{\text{CFo}}$

Technology

Calculator

Cash Flow Worksheet (CF)

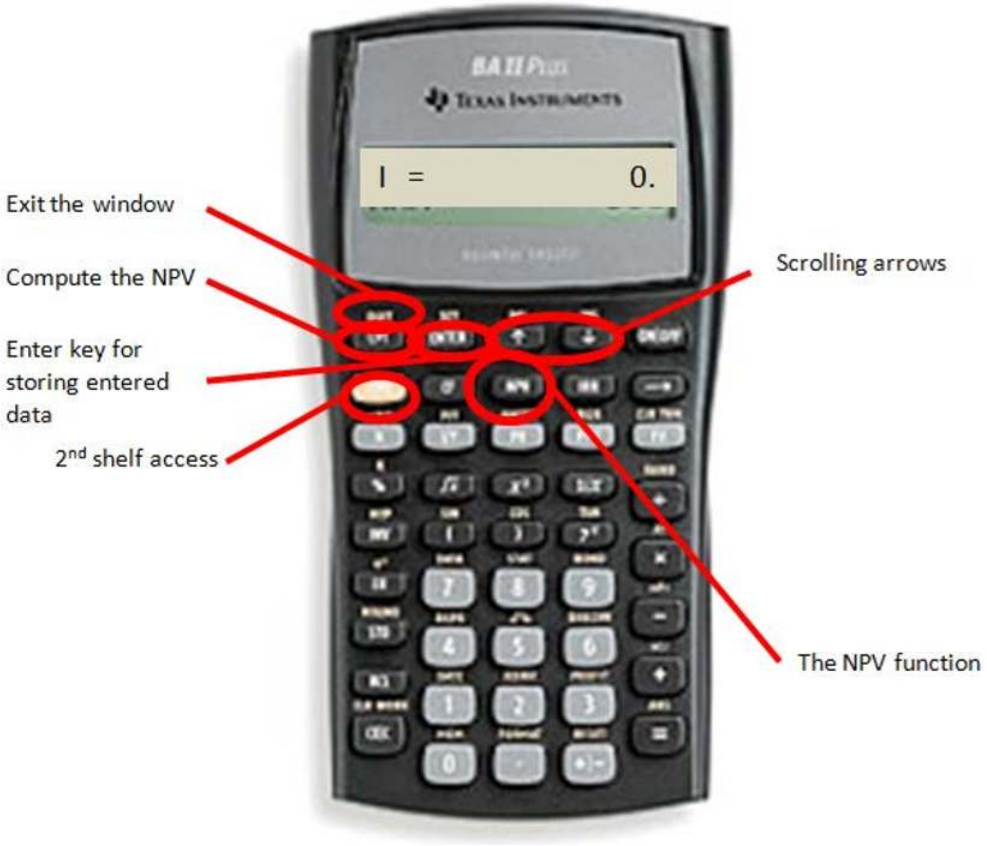

CFo=
0.
Exit the window
Scrolling arrows
Enter key for storing entered data
2nd shelf access
Clearing the memory
The CF function

- Access the cash flow function by pressing CF on the keypad.
- Always clear the memory using 2nd CLR Work so that any previously entered data is deleted.
- Use the ↑ and ↓ arrows to scroll through the various lines.
- Strictly adhere to the cash flow sign convention when using this function and press Enter after keying in the data.
- To exit the window, press 2nd Quit.

The various lines are summarized below:

- CF_o = any cash flow today.
- CXX = a particular cash flow, where XX is one of a series of cash flow numbers starting with 01. You must key in cash flows in order from the first time segment to the last. You cannot skip a time segment, even if it has a value of zero, because each time segment is a placeholder on the timeline.
- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times in a row the corresponding cash flow amount occurs. This allows you to enter recurring amounts together instead of keying them in separately. By default, the calculator sets this variable to 1.

Net Present Value (NPV)

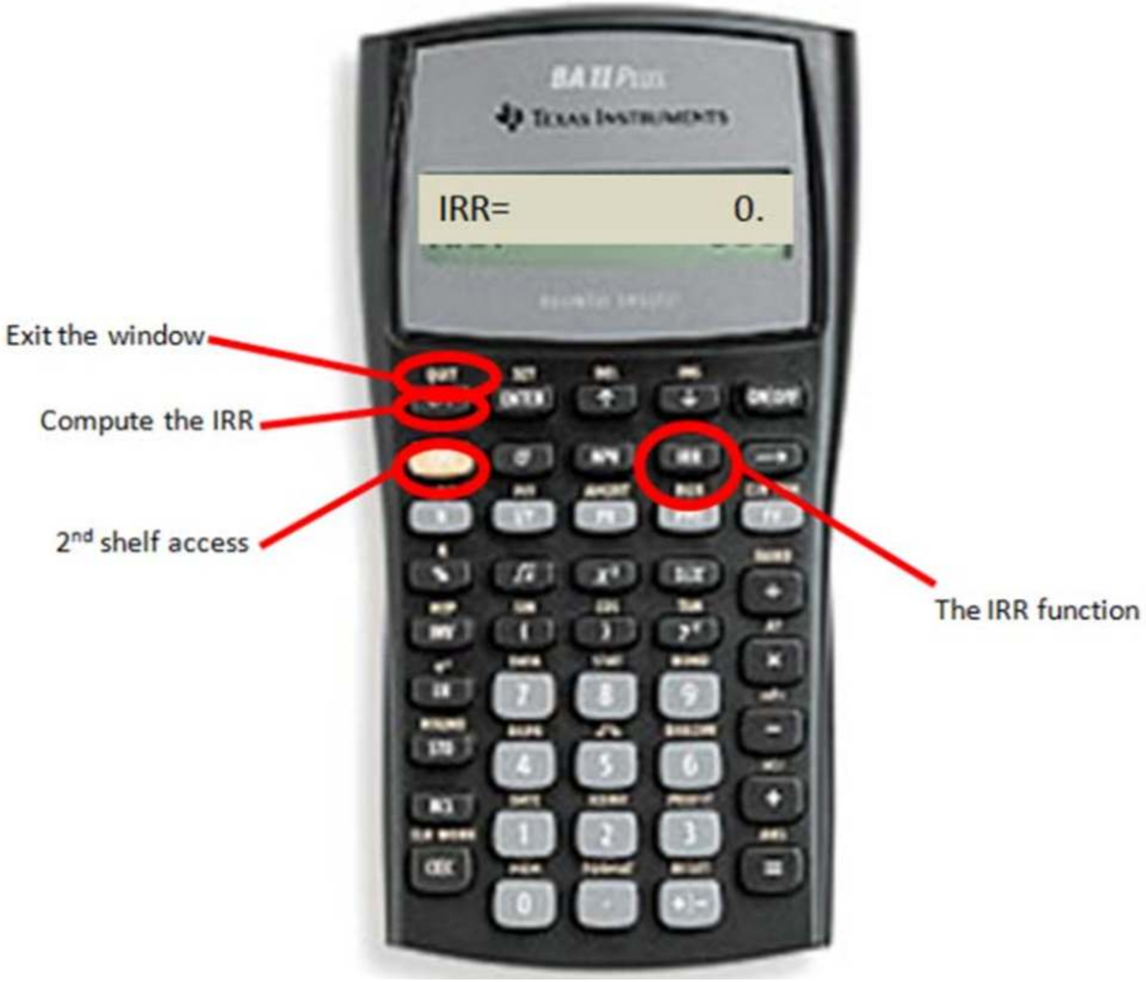

I = 0.
Exit the window
Compute the NPV
Enter key for storing entered data
2^{nd} shelf access
Scrolling arrows
The NPV function

- Use this function after you have entered all cash flows.
- Press NPV on the keypad to access the function.
- Use the ↑ and ↓ arrows to scroll through the window.
- To exit the window, press 2nd Quit.

This window has two lines:

- I = the matching periodic interest rate for the interval of each time segment.
- NPV = the net present value. Press CPT to calculate this amount.

Internal Rate of Return (IRR)


IRR=
0.
Exit the window
Compute the IRR
2nd shelf access
The IRR function

- Use this function after you have entered all cash flows.
- Press IRR on the keypad to access the function.
- Press CPT button to perform the calculation. The output is in percent format.
- To exit the window, press 2nd Quit.

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15.CS: Case Study - Choosing a New Supplier

The Situation

Lightning Wholesale is always looking for profitable opportunities to expand its business. This usually means adding more product lines that will increase its product mix offering to its retail clients. The retail clients appreciate the offerings, since the diversity results in increased customer satisfaction along with higher profits. Lightning Wholesale finds itself in a position to add another product line to its mix. Four different suppliers have provided information to management. To accommodate any of these manufacturers, management knows that modifications must be made to their distribution centre in terms of the needed equipment, storage space, sorting machines, and other additional resources.

The Data

Supplier	Initial investment required	Storage space required in distribution centre	Additional labour hours required annually by the product	Cost of the product today	Forecasted unit sales in the first year and expected annual increase in sales	Lightning Wholesale's price for the product today	Years until the product line is discontinued
#1	$220,000	450	520	$54.00	2,000; 10%	$92.00	4
#2	$253,000	685	390	$76.00	3,000; 8%	$102.00	5
#3	$190,000	265	295	$33.00	4,200; 5%	$54	3
#4	$280,000	540	440	$48.00	1,500; 13%	$97	5

Important Information

- Utility costs are estimated at $5.40 per square foot of storage space and forecasted to rise 5% each year.
- Labour costs are estimated at $14.25 per hour and forecasted to rise 3% each year.
- All suppliers will increase the cost of their product in line with inflation each year at 3% annually.
- Lightning Wholesale will increase the selling price of any product $2 annually.
- Lightning Wholesale's cost of capital is 11%.
- The storage, labour, and product costs are incurred throughout the year.
- The profits are earned throughout the year.

Your Tasks

Based on the information provided, recommend only one supplier that Lightning Wholesale should select to add to its offerings. Perform the necessary calculations and write a brief report to the marketing manager about your decision. For simplicity in any given year, since the costs and profits are incurred throughout the year, you can net the storage, labour, and product costs against sales of the product to determine the net profit at the end of each year. When making any annual changes, be sure to round all costs and prices to two decimals as well as any units to integers.

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15.E: Making Good Decisions (Exercises)

15.1: Net Present Value

Mechanics

For questions 1–4, calculate the net present value and the net present value ratio.

Problem Number	Investment (today)	Cost of Capital (annually)	Cash Outflows (at beginning)	Cash Inflows (at end)
1.	\$500,000	13%	Year 2: \$100,000	Years 2-3: \$500,000
2.	\$1,000,000	16%	Year 3: \$250,000	Years 1-4: \$400,000
3.	\$750,000	15%	Years 3 & 5: \$100,000	Years 1-6: \$200,000
4.	\$300,000	11%	Year 2: \$25,000 Year 6: \$40,000	Years 1-3: \$150,000 Year 4: \$125,000 Years 5-6: \$75,000

For questions 5 and 6, given the capital budget indicated determine the combination of projects that should be chosen, the total net present value, and total budget spent.

		5.	6.
Project #1	Initial Cash Flow	\$300,000	\$325,000
	NPV	\$360,000	\$357,500
	NPV_{RATIO}	1.20	1.10
Project #2	Initial Cash Flow	\$150,000	\$400,000
	NPV	\$165,000	\$500,000
	NPV_{RATIO}	1.10	1.25
Project #3	Initial Cash Flow	\$185,000	\$315,000
	NPV	\$240,500	\$378,000
	NPV_{RATIO}	1.30	1.20
Capital Budget Available		\$500,000	\$650,000

Applications

7. Citywide Courier Services projects that demand for its services will rise for a period of three years before subsiding. It can lease additional communication equipment for \$1,000 at the beginning of every quarter for three years. Alternatively, it can purchase the equipment for \$13,995 at 7.5% compounded quarterly. The salvage value of the equipment after three years is expected to be \$4,000. Which option would you recommend? How much better is that option in today's dollars?
8. A new diamond deposit has been found in northern Alberta. Your researchers have determined that it will cost \$2.5 million to purchase the land and prepare it for mining. At the beginning of both the second and third years, another \$1 million investment will be required to establish the mining operations. Starting at the end of the second year, the deposit is expected to earn net profits of \$3 million, which will be sustained for three years before the deposit is depleted. If the cost of capital is 16%, should your company pursue this venture? Provide calculations to support your decision.
9. Toys “R” Us has identified several projects that it can pursue, as listed in the table below. A capital budget of \$500,000 has been set. Which projects should be undertaken? What total budget will be spent? What total net

present value will be realized?

Project	Initial Cash Flow	Net Present Value
#1: Renovate Manitoba outlets	$180,000	$125,000
#2: Renovate Alberta outlets	$230,000	$210,000
#3: Renovate Maritime outlets	$75,000	$60,000
#4: Renovate BC outlets	$245,000	$250,000

10. A company can pursue only one of two available projects, both of which require a $3 million investment today. In project A, another investment of $2 million is required at the beginning of the second year, and three years of $2.5 million in year-end profits will start in the second year. In project B, another investment of $2 million is required at the beginning of the fourth year, and four years of $1.7 million in year-end profits will start in the first year. If the cost of capital is 12%, which project should be chosen? How much better is your chosen alternative?
11. A new marketing project requires initial research and development costs of $550,000 today with further investments of $100,000 and $75,000 in the fourth and sixth year, respectively. Initially the project will lose $50,000 in its first year and then earn profits of $250,000 for the next four years followed by profits of $140,000 in the last two years. At the end of the project, its capital goods can be sold for an estimated $50,000. If the cost of capital is 9%, should the marketing project be pursued? Show calculations to support your recommendation.
12. Olfert has received three offers to purchase his used combine. Farmer A has offered him $95,000 today and $105,000 two years from now. Farmer B has offered him $50,000 today and $37,500 every six months for two years. Farmer C has offered him three annual payments of $67,500 starting today. If prevailing interest rates are 7.5% compounded annually, which offer should Olfert accept? Provide calculations to support your decision.
13. One of your company's clients has proposed a contract offering an estimated $150,000 in net profits for the next six years; however, your company would be required to invest $585,000 today to acquire the needed resources for the project. Determine whether the project should be accepted if the cost of capital is a. 10% b. 13% c. 16% Provide calculations to support your decisions.
14. A rural municipality has received various proposals from local business developers. The capital budget for the year is $4.5 million. In all projects, the city will initially develop lands that will then result in future returns. Which projects should be undertaken? What total budget will be spent? What total net present value will be realized?

Project	Initial Cash Flow	Net Present Value
Project A	$1,250,000	$1,100,000
Project B	$775,000	$900,000
Project C	$2,335,000	$1,750,000
Project D	$1,850,000	$2,010,000
Project E	$495,000	$335,000
Project F	$940,000	$1,000,000

15. Recalculate question 11 using a cost of capital of 20%. Why does the result differ from the answer to question 11?

Challenge, Critical Thinking, & Other Applications

16. A company has a $100,000 capital budget. After analyzing opportunities, management has narrowed down the choices to four projects. If the cost of capital is 14%, make your recommendation about which project(s) to pursue and provide calculations to support your choice(s).

Project	Investment (today)	Cash Outflows (at beginning)	Cash Inflows (at end)

Project	Investment (today)	Cash Outflows (at beginning)	Cash Inflows (at end)
A	$40,000	Year 2: $20,000	Years 1–4: $25,000
B	$26,000	Year 3: $35,000	Years 1–4: $25,000
C	$40,000	Year 4: $15,000	Years 1–2: $30,000; Years 3–4: $20,000
D	$34,000	Years 2 & 4: $25,000	Years 1–3: $35,000; Year 4: $15,000

17. To replace an aging production machine, Johnston Distributors is considering the purchase of a new robotic machine for $950,000. The machine is expected to have a service life of eight years and an estimated residual value of $75,000. In each of the first four years, the machine is expected to produce production efficiencies and labour savings of $250,000. In the last four years, the annual savings are expected to rise to $300,000. Hydro costs for the machine are estimated at $25,000 starting at the beginning of the second year and increasing by 5% each year thereafter. Maintenance costs are estimated at the beginning of the third, fifth, and seventh years in the amounts of $40,000, $60,000, and $80,000, respectively. If the cost of capital is 16%, determine if the new robotic machine should be purchased. Provide calculations to support your answer.
18. If the cost of capital changes, then the net present value changes as well. For illustration purposes, recalculate question 11 using costs of capital of 7.5%, 13.5%, and 17.5%.
    a. Rank the three offers.
    b. Determine how much better the chosen alternative is than the worst alternative.
    c. Summarize your findings about the cost of capital.
19. A company can choose only one of two comparably performing machines to purchase. Expected costs and savings are identified in the table below. If the cost of capital is 14% compounded quarterly, determine which machine should be purchased and show calculations to support your decision.

Machine	Investment (today)	Cash Outflows (at beginning)	Cash Inflows (at end)
A	$104,000	12 months: $10,000; 24 months: $15,000; 36 months: $20,000	Every 6 months for 4 years: $30,000
B	$92,000	18 months: $17,500; 42 months: $22,500	Every 6 months for 4 years: $28,000

20. Through market research, a company has identified eight different market opportunities. The capital available to pursue these opportunities is $10 million. Which projects should be undertaken? What total budget will be spent? What total net present value will be realized?

Project	Initial Cash Flow	Net Present Value
Project #1	$3,000,000	$3,500,000
Project #2	$2,500,000	$2,850,000
Project #3	$4,000,000	$3,750,000
Project #4	$1,500,000	$1,437,500
Project #5	$1,900,000	$2,050,000
Project #6	$2,700,000	$2,450,000
Project #7	$3,500,000	$3,550,000

Project #8	$1,200,000	$1,325,000

15.2: Other Measures For Making Decisions

Mechanics

For questions 1–4, calculate the equivalent annual cash flow for each project.

Problem Number	NPV	Annual Cost of Capital	Years
1.	$8,439	15%	4
2.	$15,783	12%	6
3.	$43,985	14%	8
4.	−$31,255	16%	5

For questions 5–8, calculate the internal rate of return.

Problem Number	Cash Outflow Today	Future Cash Outflows (at beginning)	Future Cash Inflows (at end)
5.	$50,000	Year 2: $25,000	Years 2–4: $40,000
6.	$350,000	Year 4: $100,000	Years 1–6: $150,000
7.	$100,000	Year 3: $40,000 Year 6: $25,000	Years 1–3: $45,000 Years 4–6: $35,000
8.	$800,000	Year 4: $150,000 Year 8: $125,000	Years 1–4: $200,000 Years 5–8: $175,000

Applications

9. Sanchez is looking at purchasing a new snow blower and has found three models he thinks are equivalent. A Brute model at Walmart retails for $458 with an expected life of five years. A Sno-Tek model at Home Depot retails for $749 with an expected life of 10 years. A YardWorks model at Canadian Tire retails for $649 with an expected life of eight years. If the money to acquire the snow blower is obtained at a rate of 8% compounded annually, which snow blower should be selected? (Hint: Since these are costs, look for the lowest value.)
10. A municipality is considering purchasing a police helicopter to assist its ground patrols. A Eurocopter EC120 model is being considered for a purchase price of $1.7 million with annual operating costs expected to be $150,000 at every yearend and a salvage value of $500,000 after six years, which is its predicted life expectancy. A Eurocopter AS350 model could also be purchased for $2 million with annual operating costs expected to be $120,000 at every year-end and with a salvage value of $550,000 after seven years, which is its predicted life expectancy. If both helicopters are deemed suitable, which model should be selected at a cost of capital of 10%? Provide calculations to support your answer.
11. A project requires an investment of $225,000 today and is projected to have annual profits of $34,000 for nine years. The capital assets can be sold at the end of the ninth year for $23,500. Calculate the IRR for this project. If capital could be acquired at a cost of 12%, should the project be pursued?
12. A new diamond deposit has been found in northern Alberta. Your researchers have determined that it will cost $4.5 million to purchase the land and prepare it for mining. Starting at the end of the second year, the lode is expected to earn net profits of $3 million, which will be sustained for three years before the deposit is depleted. Calculate the IRR on the diamond mine. If the cost of capital is 21%, should the deposit be acquired?
13. Schick is considering launching only one of two new products to compete with the Gillette Fusion ProGlide Power razor. For Product A, research and development costs are estimated at $10 million. Forecasted profits are expected to be $3 million in years two to four, $1.5 million in years five to seven, and $0.5 million in years eight to ten.

Alternatively, Schick can launch Product B, requiring only a $6.5 million investment and producing forecasted profits of $2.75 million in years one to two, $1 million in years three to four, and $0.35 million in years five to six. The cost of capital is estimated at 8.5% compounded semi-annually. Determine which product should be selected based on the equivalent annual cash flow.

14. The Toronto Transit Commission (TTC) is looking to purchase 75 new buses to replace some of its aging fleet. The selection has been narrowed down to two equivalent products from two different suppliers. New Flyer Industries offers a bus for $275,000 with an expected life of 15 years and an estimated residual value of $10,000. The bus requires $5,000 in maintenance at the end of every year, plus a $20,000 overhaul at the beginning of the fourth year and every three years thereafter except for the final year. Motorcoach Industries offers a bus for $250,000 with an expected life of 12 years and an estimated residual value of $13,500. The bus requires $6,000 in maintenance at the end of every year and requires a $30,000 overhaul at the beginning of the fifth, eighth, and eleventh years. The cost of capital is estimated at 11%. Which supplier should the TTC select? What total annual savings will it realize by choosing the supplier?
15. The Dragons (from CBC's Dragon's Den) are considering a venture from an Albertan entrepreneur. His proposal requires the Dragons to invest $500,000 immediately. Based on market conditions, the product is estimated to have year-end profits of $100,000, $200,000, $300,000, $200,000, and finally $100,000. The Dragons will invest only if the internal rate of return exceeds their cost of capital of 20%. Should they invest?
16. A Western Canadian producer is considering purchasing a five-year product licence from Sunkist Growers Inc. The product licence will cost $1,500,000, and the required equipment will cost $1 million. Equipment upgrades will be $100,000 at the beginning of the third and fifth years. Expected profits are $1 million in each of the first three years and $750,000 in the last two years. The producer will purchase product licences only if the IRR is greater than 25%. Should the product licence be purchased?

Challenge, Critical Thinking, & Other Applications

17. Revisit the robotic machine in question 17 of Section 15.1. What is the highest cost of capital that would still result in a decision to purchase the machine?
18. You use the equivalent annual cash flow technique to select between two options satisfying the same need but having different timelines. This technique can also be used even if the timelines are the same. Consider the following two projects, of which only one can be selected: Project A: Immediate investment of $225,000, profits in years four to six of $160,000. Project B: Immediate investment of $112,500, profits in years one to three of $55,000 followed by profits of $20,000 in years four to six.
    a. At a cost of capital of 12%, calculate the NPV for both projects and make your recommendation.
    b. Calculate the equivalent annual cash flow for both projects and make your recommendation.
    c. Comment on the findings.
19. Consider the following two projects with a cost of capital of 15%. Only one can be chosen. Project A: Immediate investment of $550,000; profits starting at $75,000 in the first year and rising by $25,000 for the next six years followed by decreasing profits of $25,000 per year in the subsequent seven years; costs of $50,000 at the beginning of years 4, 10, and 14. Project B: Immediate investment of $800,000; profits starting at $200,000 and rising by $50,000 for the next three years followed by decreasing profits of $75,000 per year in the subsequent four years; costs of $110,000 at the beginning of the fourth and seventh years.
    a. Calculate the equivalent annual cash flow for each project and recommend which project should be chosen.
    b. What annual benefit will be realized over the alternative project?
20. In this section, it was stressed that the IRR should be not be used when ranking projects or choosing between projects since the analysis does not factor in the cost of capital. Consider the same two projects from question 18.
    a. Calculate the IRR for both projects and make your recommendation.
    b. At a cost of capital of 15%, calculate the NPV for both projects and make your recommendation.
    c. At a cost of capital of 10%, calculate the NPV for both projects and make your recommendation.
    d. Comment on the findings. Which method makes more sense in making the decision—NPV or IRR?

Chapter Review Exercises

Applications

1. A database marketing firm can lease its computer equipment with beginning-of-quarter payments of $10,000 for three years. Alternatively, it can purchase the equipment for $131,200 on a loan at a rate of 9% compounded quarterly with an expected resale value after three years of $33,000. Which option would you recommend, and how much better in today's dollars is your chosen alternative?
2. Jeremy is thinking of starting up a candle manufacturing business. The initial outlay for equipment, moulds, and other required production equipment is $15,000. Working part time on this hobby business, Jeremy estimates that he will lose $2,000 in the first year, break even in the second year, and earn annual profits of $5,000, $10,000, and $15,000 in subsequent years. After the five years, he hopes to sell the business to an investor for $17,500. If his cost of capital is 8.25% compounded annually, should he pursue this venture? Provide net present value calculations to support your answer.
3. A company has a limited capital budget of $300,000. The following four projects are available. Which projects should be undertaken? What total budget will be spent? What total net present value will be realized?

Project	Initial Cash Flow	Net Present Value
Project A	$120,000	$130,000
Project B	$90,000	$105,000
Project C	$85,000	$94,000
Project D	$145,000	$163,000

4. You are looking to buy a car and own it for five years. A particular make and model has caught your eye. You could purchase the vehicle brand new for $30,000 and sell the vehicle for $11,000. Regular annual maintenance on the vehicle not covered by the three-year manufacturer warranty is expected to be $200, $800, $1,500, $2,250, and $3,000. Alternatively, you could buy a two-year-old used vehicle for $23,000 and sell it after five years for $6,300. Maintenance costs are expected to be $1,500, $2,250, $3,000, $3,500, and $4,500. The cost of financing is 6.75% compounded monthly. Should you purchase the new or used vehicle? How much will you save in current dollars by picking your chosen alternative?
5. A company pursues all projects that exceed its 15% cost of capital. Project "Affinity" is available to the company with an initial investment of $655,000. Forecasted profits are expected to be $200,000 for the first three years and $160,000 for the subsequent three years. Based on the IRR, should the company pursue Project "Affinity"?
6. Coca-Cola is considering two new flavours—cherry and vanilla—for expansion of one of its product lines. Based on historical and competitive information, the cherry flavour isn't quite as popular as the vanilla flavour, but it will endure in the market longer. Both options require an investment of $2.25 million in equipment and modifications. The cherry flavour is expected to have annual profits of $750,000 for the first two years and $500,000 for five more years. The vanilla flavour is expected to initially have annual profits of $950,000 for two years, $650,000 for two years, and $250,000 in its last year. The cost of capital is 16%. How much higher is the equivalent annual cash flow from choosing the financially preferable flavour?
7. A saleswoman informs the head librarian that investing in her new automated library product is a steal at $83,000. Before purchasing, the head librarian investigates the benefits of the automated product, which is expected to have a useful life of seven years and 15% cost of capital. Each year, $24,000 in labour savings and reduced book shrinkage is projected. However, the automated products require regular maintenance of $1,000 in the first four years and $2,000 in the last three years. Electricity bills will also rise by $1,500 in the first three years and $2,100 in the last four years. Is the saleswoman correct? Should the automated system be purchased? If so, what savings in today's dollars will be realized?
8. Hershiser has been researching different methods of management leadership that are designed to increase employee motivation and performance. If he pursues a new leadership style, it will require an investment of

$165,000 to take the training. A new executive assistant will be hired at a cost of $55,000 annually with expected raises of 5% each year. The techniques and tactics from the style are expected to have an impact on employees for five years before needing to be refreshed and updated with more modern techniques. Increased motivation and performance will produce net profits of $100,000 in the first year, rising by 10% in each subsequent year.

   a. Calculate the net present value using a cost of capital of 10%. Should Hershiser take the management leadership course?
   b. What is the maximum cost of capital that will still produce a decision to proceed with the course?

9. Mariners Inc. in Halifax has the option to purchase only one of two available offshore fishing rights. Both rights are available for purchase on a four-year contract before they have to be renewed. Fishing right #1 costs $600,000 to acquire, and $64,000 is needed in new equipment to harvest the fish in the area. For each of the first two years, $21,000 will be spent on equipment maintenance and replacement followed by $33,000 in both of the last two years. Annual profits are expected at $300,000. Fishing right #2 costs $400,000 to acquire, and $82,000 is needed in new equipment, which requires annual costs of $30,000 for each of the first two years and $41,000 for both of the last two years. Annual profits are expected to be $250,000.
   a. Calculate the net present value for both projects if the cost of capital is 14%. What decision do you recommend?
   b. Calculate the equivalent annual cash flow for both projects. What decision do you recommend?
10. A capital budget of $700,000 has been set. Five different capital projects are available for selection. Which projects should be undertaken? What total budget will be spent? What total net present value will be realized?

Project	Initial Cash Flow	Net Present Value
Project #1	$285,000	$305,000
Project #2	$240,000	$120,000
Project #3	$146,000	$135,000
Project #4	$211,000	$215,000
Project #5	$373,000	$375,000

Challenge, Critical Thinking, & Other Applications

11. The following projects are available, but only one can be chosen. The cost of capital in all cases is 16%. Project A: An initial investment of $180,000 followed by profits of $30,000 in years one to four, $40,000 in years five to seven, and $50,000 in years eight to ten. Project B: An initial investment of $335,000 followed by profits of $65,000 in years one to three, $85,000 in years four to six, and $110,000 in years seven to eight. Project C: An initial investment of $372,000 followed by profits of $150,000 in years four to nine and a residual value of $70,000 in the ninth year.
    a. Using an appropriate decision-making technique, recommend which project should be selected. How much better is the chosen alternative over the worst alternative?
    b. If a new lower cost of capital equal to 11% is used, what decision do you reach?
12. Recalculate question 11 with the following modifications: Project A: Add a residual value of $100,000 in year 10. Project B: Add a residual value of $60,000 in year 10. Project C: Move the residual value from the ninth year to the tenth year.
    a. At a 16% cost of capital and using an appropriate decision-making technique, recommend which project should be selected. How much better is the chosen alternative over the worst alternative?
    b. If a new lower cost of capital equal to 11% is used, what decision do you reach?
13. The Banff Gondola is looking to increase its operational capacity through modifications to its gondola equipment. If it invests $2.3 million it can upgrade the motors, strengthen the necessary supports, and add an additional cable car to its line. The expected life of the project is eight years before the equipment will need to be replaced. Annual increased maintenance costs are $50,000 at the end of every year for four years and then $100,000 at the end of every year during the last four years. Because of construction and down times, Banff Gondola is forecasting a loss

of $415,000 in the first year. Afterwards, it projects that the equipment will result in increased profits of $850,000 per year. What is the maximum cost of capital that will result in a break-even scenario?

14. For question 13, calculate the net present value (use a cost of capital of 15%), internal rate of return, and the equivalent annual cash flow.
15. A capital budget of $1 million has been set. Eight different capital projects are available for selection. Which projects should be undertaken? What total budget will be spent? What total net present value will be realized?

Project	Initial Cash Flow	Net Present Value
Project #1	$295,000	$203,000
Project #2	$286,000	$301,000
Project #3	$391,000	$550,000
Project #4	$172,000	$226,000
Project #5	$403,000	$453,000
Project #6	$107,000	$135,000
Project #7	$212,000	$101,000
Project #8	$168,500	$168,500

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CHAPTER OVERVIEW

16: Appendices

16.1: Exercise Solutions (Appendix A)
16.2: Bibliography
16.3: Rounding Rules
16.4: Formula Sheet
16.NS: Notation Summary

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16.1: Exercise Solutions (Appendix A)

Note: Some solutions are too big or space consuming to be placed into this appendix such as schedules and tables. Please see the instructor manual for these complete solutions.

Chapter 1

1. End of chapter or glossary
2. End of chapter
3. Chapter sections
4. Guided examples
5. End of Chapter or How It Works
6. Review or Lyryx
7. End of Chapter
8. Beginning of chapter OR End of chapter OR table of contents
9. Plan, Understand, Perform, Present; it is a systematic problem-solving process
10. Insufficient; proceed onto Applications to develop problem solving skills.
11. (i) unknown section from which question derived; (ii) integration of various chapter concepts.
12. to 18. Answers are student dependent.

Chapter 2

Section 2.1

1. 15
2. 10
3. 40
4. 100
5. 55
6. 10
7. $21.70
8. $7,645.52
9. $19,215.37
10. 400
11. $25,967.38
12. $1,345.87
13. $5,978.08
14. $1,056,767.41
15. $2,156,764.06
16. $41,271.46
17. $83,228.60
18. $2,339.07
19. $26,450.25
20. $14,561.43

Section 2.2

1.
    a. proper
    b. compound
    c. improper
    d. proper
    e. complex
    f. improper
    g. complex
    h. proper
2.
    a. $\frac{3}{4}=\frac{27}{36}$

   b. $\frac{2}{8} = \frac{16}{64}$
   c. $\frac{2}{5} = \frac{18}{45}$
   d. $\frac{5}{6} = \frac{75}{90}$
3. a. $\frac{5 \times 2}{10 \times 2} = \frac{10}{20}\ \ \frac{5 \div 5}{10 \div 5} = \frac{1}{2}$
   b. $\frac{6 \times 5}{8 \times 5} = \frac{30}{40}\ \ \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$
4. a. 0.875
   b. 16.25
   c. 2.6
   d. 135.5
5. a. 0.875
   b. 16.250
   c. 1.111
   d. 0.469
6. a. $0.08\overline{3}$
   b. $5.\overline{24}$
   c. $1.\overline{3}$
   d. $-0.\overline{309}$
7. a. 5.95
   b. 15.3125
8. a. 1.04
   b. 0.96
   c. 9.64
9. a. $3.\overline{20}$
   b. 0.5
10. $1,182.14
11. $14,005.26
12. $525,745.68
13. $2,693.13
14. $149,513.74
15. $4,450.29
16. $9,579.23
17. $138,934.38
18. $12,252.25
19. $267,952.30
20. $1,041.58

Section 2.3

1. a. 46.38%
   b. 315.79%
   c. 0.0138%
   d. 1.5%
2. a. 37.5%
   b. 53.125%
   c. 350%
   d. 280%
3. a. $383.\overline{3}\%$
   b. $22.\overline{2}\%$
   c. $27.\overline{27}\%$
   d. 51.6129%
4. a. 0.153
   b. 0.0003

   c. 1.53987
   d. 0.140005
5. 5. 67.5%
6. \$2,500
7. \$0.003061
8. 3,862,105
9. \$608,600
10. 8.2358%
11. 7,325
12. \$147.8 billion
13. \$8.50
14. 1.48%
15. 13.0235%
16. \$124,722.22
17. \$20,000,000
18. \$128,000.03
19. Share price = \$12.36; Money made = \$2,360
20. $116.\overline{6}\%$

Section 2.4

1. $5a + 1$
2. $20b^2 + 10b$
3. $2x^2 + 4x + 4.5$
4. $(1 + i)^{17}$
5. 4
6. I = \$55.48
7. $15r^2 + 3r + 5$
8. $1{,}024x^{40}$
9. $0.75t + 3.5t^3$
10. $2 + 3(1 + i)^3 - 5(1 + i)^6$
11. 3.995628R
12. 0.0256
13. PV=\$2,550
14. PMT=\$246.64
15. $a^4 - 9a^2 + 18$
16. $-2x^2 - 0.6xy + 4.8y^2 - 7.2x + 4.2y$
17. $-3{,}888z^{19}$
18. \$15,223.10
19. \$39,963.05
20. \$12,466.44

Section 2.5

1. x = 10
2. b = 1
3. m = 2
4. x = 4, y = 2
5. h=5, q=1
6. a=8, b=0.6
7. y = \$620.14
8. $t = 6.151\overline{3}$
9. Delay = 1.5 hours
10. Users in 2000 = 107,998,376
11. Direct cost = \$3,000

12. Gross income = \$140,000
13. Single tickets = 10,363, 3-pack Tickets = 1,879
14. A = 700 units, B = 550 units
15. Company shares = \$975,000
16. 240 minutes
17. Marianne = \$50,000; William = \$40,000; Hendrick = \$65,000; Charlotte = \$20,000
18. Regular stalls = 10,860; Handicap stalls = 240; Small car stalls = 720; RV stalls = 180
19. Z = \$25,000
20. q = 2.78; r = 9.31

Section 2.6

1. $e^{1.367188} = 3.9243$
2. $e^{-0.295042} = 0.7445$
3. $e^{0.609336} = 1.83921$
4. $e^{2.585852} = 12.2746$
5. $e^{-2.051398} = 0.128555$
6. 0.392393
7. 2.302585
8. 1.811554
9. 0.455460
10. 0.519446
11. 1.529094
12. 50.517191
13. 22.48254
14. 36.904108
15. N = 87.808096
16. N = 160.423164
17. 0.583198
18. 1.451535
19. 403.428793
20. 33.115452

Chapter 2 Review

1. 0.003016
2. $4.\overline{324}$
3. $71.\overline{3}\%$
4. x
5. r=1
6. 3 × ln(2) = ln(23) = 2.079442
7. \$632.86
8. 1.277142
9. 204ml
10. 43.1548%
11. \$3,417.44
12. $5.196152a^3b^3$
13. 28.186510PMT
14. 799 starts
15. Behind Plate tickets = 3,489; Base line & outfield tickets = 1,843
16.
    a. \$55,680
    b. \$51,782.40
    c. 112.01%
17. \$2,034.86
18. 6.1543%

19. Regular Tires = 205; Promotional Tires (sold in sets of 4) = 300
20. 102.9%

Chapter 3

Section 3.1

1. 5.0023%
2. $549.25
3. 5.346%
4. 5.0005%
5. $17.99
6. $2,500
7. 7.1773%
8. $218.34
9. 16.2624%
10. Δ% = −26.2937%; RoC = −3.0048%
11. a. −2.7248%
    b. $246,308.72
    c. 10.4833%
12. −28.5714%
13. a. 10%
    b. −9.5801%
    c. −0.5381%
14. 5.1452%
15. 15 years
16. 2.5148%
17. Amount saved = $193.18; Δ%=−70.0004%
18. −0.6155%
19. 61.9048% less time.
20. a. $8.\overline{3}\%$
    b. $4.08\overline{3}\%$ per period.

Section 3.2

1. 15
2. $1,779
3. 10
4. $3,795
5. 6.7910%
6. 2.6888%
7. $0.00802/ml
8. $6.46
9. 5.0821%
10. $15.88
11. $314.83
12. 2.74
13. $125,000
14. 1.8666%
15. $0.8082
16. 99.0805%
17. a. Pert better; Pert=$ $0.00591\overline{6}$/ml; H&S=$0.006680/ml
    b. Pert saves $1.74
18. Grocery 11.2853%; Cosmetics 9.4493%; Pharmaceuticals 8.9208%

19. $213.64
20. 10.2809%

Section 3.3

1. a. 11:2
   b. 6:13:10
   c. 2:3:7:9
2. a. 6:5
   b. 6:7:4
3. a. 35:4
   b. 5:6:7
   c. 17:9:27:8
4. a. 24:5
   b. 5:2:12
   c. 12:28:21:15
5. a. 1:1.10
   b. 2.14:1:1.12
   c. 1:16.64:8.33:3.12
6. a. y=1.87
   b. x=44.55
   c. q=0.92
7. a. r=$17.47; x=$18,775.25
   b. g=212,142.40; h = $66,294.50; i=$119,330.10
8. $6,820.95 : $4,547.30
9. $21,897.82 : $15,641.30 : $9,384.78
10. 5:7:4
11. 6:3:4:11
12. m=$778.88
13. $383.20
14. $44,916.67
15. Gingerale = 4.08L, Grenadine = 2.176L, Vodka = 2.244L
16. $46,965,000
17. 7-11: transactions, 4% savings; Tim Horton's: square metres, 12.963% savings; Quizno's: transactions, 26.3636% savings
18. a. Texts = 4,052; Video = 54, Cost = $3,381.18
    b. $887.73.
19. 1.64 : 1.29 : 3.21 : 1
20. Manager = $11,015.12; Supervisor = $5,507.56; Worker = $2,753.78

Chapter 3 Review

1. $2,135.32
2. 5.25
3. a. 16%
   b. $1,680,000
   c. 5.346%
4. a. 3.0776%
   b. −27.5684%
5. a. 7:2:9
   b. 12:7
   c. 10:16:25
   d. 68:81

6. a. 1:1.86
   b. 1.20:1.38:1
   c. 1.41:3.52:1:1.74
7. a. x=4
   b. q=25, r=31.25
8. $11,250:$10,000:$3,750
9. 10.39%
10. $160,000
11. Refrigerated = $105,322.68; Nonrefrigerated = $135,129.69; Non-food = $95,547.63
12. total sales = $997.50; average price = $1.27
13. 16:3:1
14. Three years ago = $1,500; Two years from now = $1,886.73
15. $153.45
16. Euros = $677.84; Pounds = 590.06; Yen = 77,104.02
17. Rachard = 10 slices for $6.25, Sergey = 2 slices for $1.25, Basilio = 4 slices for $2.50, Alphonso = 8 slices for $5.00
18. a. $1.27:$1.00
    b. $1.29:$1.00 then $1.25:$1.00
19. 2008-2009 Δ% = −1.6470%; 2009- 2010 Δ% = 1.4306%; 2008-2010 RoC = −0.1201%
20. New Zealand = 287.35; Indian Rupees = 15,932.06; South African Rand = 1,357.52

Chapter 4

Section 4.1

1. a. $5,000
   b. $2,500
   c. $2,307.69
   d. $1,153.85
2. $1,277.21
3. $57,600
4. $6,110
5. $468.24
6. $188
7. $3,767.58
8. $1,443
9. $19,162.50
10. $16,860
11. $850
12. Offer #2 Best = $3,412.50; Exceeds Offer #1 = $312.50; Exceed Offer #3 = $67.50
13. $5,902.89
14. $1,454.38
15. $43,550.45
16. $168,000
17. 2%
18. Option 1 = 1.47%; Option 2 = 1.05%
19. $3,342.35
20. $1,755.25

Section 4.2

1. $1,777.20
2. $34,410.41
3. $18,735.35
4. $5,478.46

5. $8,417.62
6. $4,731.86
7. $8.701.20
8. $16,981.20
9. $12,121.55
10. $4,604.80
11. −$817.22
12. British Columbia is $4,025.28 better
13. Saskatchewan - $78,120.88 (LOW); Ontario = $81,278.61 (HIGH); NWT = $81,081.29; Difference = $3,157.73
14. New Brunswick = $62,322.77 (MORE); Nunavut = $62,240.61
15. Federal = $5,808.87; Provincial = $2,692.38; Total = $8,501.25
16. Federal = $1,930.95; Provincial = $1,537.82; Total = $3,468.77
17. Federal = $380.56; Provincial = $262.50; Total = $643.06
18. 5.5097%
19. $46,846.57
20. Nunavut = $46,434.59 (HIGH); Quebec = $40,659.97 (LOW) which is −12.436%

Section 4.3

1. 103.7
2. $25,370.69
3. 100
4. $385.29
5. 92.6784%
6. 125.0
7. $37,500
8. $31,000
9. 107.1
10. Toronto=122.6; Montreal=145.6
11. $53,850.80
12. $105.22
13. $3,934.61
14. 3.6256%
15. $325,167.63
16. 8.7853% then 8.8743%
17. $72,968.21
18. $193,934.64
19. 100.0 (base year), 140.1, 198.4, 132.4, 206.3
20.
    a. 95.2381%, 91.2409%, 86.3558%, 85.6164%
    b. $36,225; $37,812; $39,951; $40,296

Chapter 4 Review

1. $1,196.25
2. $3,500
3. $18,186.75
4. $1,250.89
5. −7.4007%
6. 2004 RI = $22,000; 2009 RI = $24,194.95; Δ% = 9.977%
7. 2nd Emp. Higher = $36,855; higher by $1,215.48
8. $3,031.92
9. $128.00
10. BC = $2,950.69; ON = $3,298.52; Ontario pays 11.7881% more
11. $2,823,220
12. $31.32

13. a. $768.00
    b. Federal = $169.67; Provincial = $88.30
14. $47,979.31
15. $255,000
16. $356.85
17. Federal = $1,144.07; Provincial = $476.12
18. $26,477.92
19. British Columbia = $13,795.59 or 20.6954%; Alberta = $15,146.37 or 22.7218%; Saskatchewan = $16,426.64 or 24.6424%; Manitoba = $17,167.25 or 25.7535%
20. Total Regular Earnings = $4,791.01; Total Holiday Earnings = $349.77; Total Overtime Earnings = 272.96; Total Statutory Worked Earnings = $75.39; Total Payroll = $5,489.13

Chapter 5

Section 5.1

1. a. Blended; FC = $15, VC = $0.33
   b. Fixed cost
   c. Variable cost
   d. Fixed cost
   e. Blended; FC = $40; VC = $0.25
   f. Variable cost
   g. Blended, FC = $1,000; VC = 15% of sales
2. TR=$16,000; VC=$5.50; n=1,200; CR=57.69%;
3. n=1,000; TVC=$5,000; NI=$3,000; CR=50%; CM=$5
4. n=800; CM=$26.25; VC=$48.75; TVC=$39,000; TFC=$6,500
5. VC=$25; S=$47; NI=$21,600; CR=46.81%; CM=$22
6. S=$26; CR=50%; VC=$13; TVC=$39,000;TFC=$21,000
7. n=1,250; S=$121.35; TR=$151,687.50; CM=$46.11; TFC=$77,137.50
8. $1,034,400
9. −$275
10. a. $6.50 increase
    b. $125
11. S=$46 (maximizes NI)
12. CR=54.24%
13. 4,667
14. $262,000 increase
15. Total CM = $2,800,000; CR=46.67%
16. $13,750,000
17. Increase price since NI rises
18. Put dresses on sale since NI rises
19. VC = $38.24; TR = $13,096,070; NI = $4,886,630; CM = $61.73; Total CM = $8,086,630; CR = 61.75%
20. a. $4,556,630
    b. $4,019,410
    c. $4,969,078

Section 5.2

1. 229
2. 6,452 units; TR = $14,839.60
3. $122,000
4. $50,000
5. $300,000
6. 1,200 pages

7. $91,125
8. $5
9. $750
10. 38%
11. a. 918
    b. 1,124
    c. 1,101
    d. $11.04
12. $25
13. 2,301,277 barrels; TR = $382,006,032.77
14. PepsiCo = $13,442,088,008.47; The Coca-Cola Company = $21,077,238,188.98; PepsiCo is 36.22% lower
15. a. $-0.\overline{9900}\%$
    b. −1%
    c. lowering TFC always better
16. 7,667 books
17. $5,937,500
18. a. $14.795 billion
    b. −30.2%
19. n=6,250; TR=$625,000
20. a. n=5,744; TR=$574,000
    b. n=5,707; TR=$570,700
    c. n=5,883; TR=$588,300
    d. n=5,868; TR=$586,600
    e. Option B

Chapter 5 Review

1. $420,000
2. a. 11,361
   b. $698,800
3. a. $240
   b. −$150
   c. 60
   d. 82
4. 649
5. a. 34.54%
   b. $179,829.40
6. Toyota = $118,971.86 million; Honda = $94,456.94 million; Honda is 20.61% lower
7. a. $3,181,818.18
   b. $2,727,272.73
   c. $300,000
8. a. Svetlana = 200; Yoric = 200
   b. Svetlana = $9,075; Yoric = $4,000
   c. Svetlana = −$6,975; Yoric = −$4,000
   d. same b/e point; high CM dramatically impacts NI
9. a. $58.\overline{3}\%$
   b. $2,500,000
   c. $3,100,000
10. a. VC = $558.62; TFC = $10,572,000; CM = $190.38; CR = 25.42%; NI = $3,731,868; break-even n = 55,531; breakeven TR = $41,592,719
    b. NI=$3,591,467.95; Do not lower price.

Chapter 6

Section 6.1

1. D$ = $411.60; N = $568.40
2. L = $800; D$ = $200
3. d = 42.625%; N = $1,133.16; D$ = $841.84
4. d = 26.7904%; L=$500; D$ = $133.95
5. $259.97
6. $24.32
7. 31.985%
8. $19.99
9. d = 43.972%; L = $51.76
10. L = $24; N = $13.92
11. d = 62.8%; L = $700; D$ = $439.60
12. d = 26%; N = $303.99; D$ = $96
13. $1,026
14. d = 63.14%; N = $29.49
15. $1,194,553.06
16. 2 years 216 days
17. Reduced by 17.4154%
18.
    a. 3%
    b. $2.50
19.
    a. 52.2328%
    b. $3,073.82
20.
    a. $74.99
    b. $60.91
    c. L = $78.74; N = $63.96; D$ = $14.78

Section 6.2

1. S=$321.95; S_{BE} =$236.95; M$=$133.53; MoC%=70.8683%; MoS%=41.4754%
2. C=$653.59; S_{BE} =$849.67; M$=$346.41; MoC%=53.0011%; MoS%=34.6413%
3. C=$118.25; S=$301.53; E=$153.13; MoS%=60.7833%; S_{BE} =$271.38
4. M$=$96.25; P=$41.25; C=$178.74; S_{BE} =$223.74
5. C=$744.83; S=$1,284.83; P=$204.83; MoC%=72.5%; MoS%=42.0289%
6. M$=81.60; S=$201.60; E=$51.36; MoS%=40.4762%; S_{BE} =$171.36
7. P=$175; S=$1,447.37; C=$1,172.37; MoC%=23.4568%; S_{BE} =$1,272.37
8. C=$220; S=$287.50; M$=$67.50; MoS%=23.4783%; MoC%=30.6818%
9. $112.50
10. 125%
11. $27,996.79
12. $57.05
13.
    a. $733.03
    b. $219.91
    c. 53.3151%
    d. $698.03
14.
    a. $119.95
    b. $40.85
    c. 51.6435%
    d. 34.0559%
    e. 28.8887%
15. $20.82
16. $102.40

17. 42.8571%
18. $488.09
19. a. $40.47
    b. $21.79
    c. $62.26
    d. $14.51
    e. $12.20; $2.31 reduction
20. a. $12.65
    b. $17.71
    c. $0.63
    d. MoS% = 28.5714%
    e. C = $15.18; S = $21.25; P = $0.76; MoS% = 28.5647%

Section 6.3

1. D$ = $153.95; $S_{\text{on sale}}$ = $285.90
2. S = $299.95; d = 33.3389%
3. D$ = $275; d = 26.1905%
4. $S_{\text{on sale}}$ = $25,525; d = 11.2945%
5. S = $19,701.42; D$ = $6,501.47
6. S = $319.42; $S_{\text{on sale}}$ = $281.09
7. $S_{\text{on sale}}$ = $53.99
8. D$ = $45; d = 52.9474%
9. d = 14.9%
10. S = $133.30; D$ = $93.31
11. $S_{\text{on sale}}$ = $384.99; S = $699.98
12. d = 36.7893
13. a. $P_{\text{on sale}}$ = $2.59
    b. P = $74.55
14. d = 10.9512%
15. a. $S_{\text{on sale}}$ = $1,519.05
    b. $P_{\text{on sale}}$ = $303.81
    c. S = $2,025.40
    d. P = $810.16
16. $S_{\text{on sale}}$ = $367.20
17. a. $S_{\text{on sale}}$ = $282.91
    b. d = 29.2637%
    c. MoS% = 23.8026%
18. D = 32.4325%
19. a. $S_{\text{on sale}}$ = $1,680
    b. MoS% when on sale = 32.8571%
    c. $P_{\text{on sale}}$ = $192.00
20. a. $S_{\text{on sale}}$ = $1,614.06
    b. S = $2,152.08
    c. M$ = $956.48; MoC% = 80%

Section 6.4

1. M$=$50; C=$50; P=$38; S_{BE} = $62; D$ = $33; $S_{\text{on sale}}$ = $67
2. S = $50; C = $20; E = $10; S_{BE} = $30; MoC% = 150%; MoS% = 60%; d = 26%
3. M$ = $18.75; S = $75; E = $11.25; S_{BE} = $67.50; MoS% = 25%; D$ = $10.01; d = $13.34\overline{6}\%$
4. E = $35.50; M$ = $56.99; P = $21.49; MoC% = 132.5348%; D$ = $10; Sonsale = $89.99
5. P = $3.46; $P_{\text{on sale}}$ = $0.46

6. P = \$17; $P_{\text{on sale}}$ = \$3.50
7. C = \$2.30; Coupon Redemption = \$1
8. S = \$74.95; Rebate Redemption = \$15
9. a. M\$ = \$10.45; S = \$19.95; P = \$2.99; E = \$7.46; MoS% = 52.381%
   b. $S_{\text{on sale}}$ = \$15.96; D\$ = \$3.99; M$\$_{\text{on sale}}$ = \$6.46; MoS$\%_{\text{on sale}}$ = 40.4762%
10. a. P = \$1.36
    b. $P_{\text{on sale}}$ = \$0.51
    c. S_{BE} = \$1.48
11. S = \$425; $S_{\text{on sale}}$ = \$225
12. $P_{\text{on sale}}$ = \$1.17
13. $P_{\text{on sale}}$ = \$0.08; The promotion is profitable and generates a total profit of \$0.08 × 300,000 = \$24,000
14. $P_{\text{on sale}}$ = −\$1.10; The promotion is not profitable and generates a loss of \$1.10 × 50,000 = \$55,000
15. a. E = \$2,250
    b. M\$ = \$4,500
    c. S = \$19,500
    d. MoS% = 23.0769%
    e. $S_{\text{on sale}}$ = \$17,550
    f. D\$ = \$1,950
    g. $P_{\text{on sale}}$ = \$250
    h. MM = \$3,768.75
16. a. Use the coupon.
    b. Increase in profit = \$3,980
17. Choose Supplier 2 since \$75.85 more profit is earned.
18. $S_{\text{on sale}}$ = \$720
19. $P_{\text{on sale}}$ = \$6.06
20. a. E = \$16.00
    b. P = \$27.99
    c. M\$ = \$43.99
    d. MoS% = 54.9944%
    e. MoC% = 122.1944%
    f. $S_{\text{on sale}}$ = \$55.99
    g. D\$ = \$24.00
    h. $P_{\text{on sale}}$ = \$3.99
    i. MM = \$41.59

Chapter 6 Review

1. N = \$20.70
2. N = \$110.22
3. d = 24.836%; N = \$582.52 (for both)
4. S = \$5.99
5. M\$ = \$22.17; MoC% = 79.6909%; MoS% = 44.3489%
6. d = 11.7647%
7. Profits reduced by \$2.40; $P_{\text{on sale}}$ = \$1.60
8. MM = \$18.65
9. a. N = \$619.99; D\$ = \$155.00
   b. N = \$526.99; \$93 less
   c. d = 32%
10. a. S = \$1.19
    b. P = \$0.69
    c. MoC% = 750%
11. $P_{\text{on sale}}$ = −\$1.20; Do not proceed. The promotion is unprofitable.

12. a. $S_{\text{on sale}}$ = \$133.67
    b. D\$ = \$65.83
13. d = 30%
14. a. $P_{\text{on sale}}$ = \$63.29
    b. \$31 less
    c. MM = \$139.16
15. d_3 = 10.0167% (due to rounding, most likely 10%)
16. a. Previous Day = 9,805.61; Current Day = 9,246.69
    b. Start of Year = 13,209.56
17. Total profit = \$42,240; MM = \$129
18. Regular: = \$475,140; Scenario A = \$484,807.75; Scenario B = \$481,320; Recommend Scenario A
19. a. P = \$9.59
    b. P = −\$3.91. Not a profitable approach.
20. Offer #2 has lowest price of \$101.14

Chapter 7

Section 7.1

1. S_{tax} = \$682.49
2. S_{tax} = \$750.74
3. S_{tax} = \$747.49
4. S_{tax} = \$727.99
5. S = \$796.45; HST Tax Amount = \$103.54
6. S = \$818.17; GST Tax Amount = \$40.91; PST Tax Amount = \$40.91
7. January Remit \$5,471.40; February Refund \$2,202.96
8. Winter Remit \$7,171.20; Spring Refund \$1,025.25; Summer Remit \$9,803.30; Fall Refund \$1,191.65
9. a. Alberta is best = \$1,396.45
   b. Savings = \$17.48
10. S = \$529.99; GST Tax Amount = \$26.50
11. Can't purchase at S_{tax} = \$3,050.94; Down Payment = \$45.08
12. S_{tax} = \$4,477.30; GST Tax Amount = \$199.88; PST Tax Amount = \$279.83
13. Q1 Remit \$17,865; Q2 Remit \$35,370; Q3 Refund \$8,105; Q4 Refund \$24,210
14. March Remit \$951.15; April Refund; \$1,257.15; May Remit \$2,953.65
15. Higher in Ontario by 1.5274%.
16. Total PST = \$13.02; Total GST = \$52.46; Total HST = \$13
17. Remit \$378,227.85
18. PST Rate = 9.505%
19. a. S = \$352.72
    b. S = \$1,682.81
    c. S = \$19,465.09
    d. S = \$4,736.57
20. Remit \$31,331.89

Section 7.2

1. AV = \$176,000; Property Tax = \$4,719.35
2. Market Value = \$247,272.73; Property Tax = \$2,699.00
3. Assessed Value = \$400,000; Tax Policy = 80%
4. Market Value = \$1,300,000; Tax Rate = 0.675832
5. Property Tax = \$5,241.17
6. AV = \$9,000,000
7. Property Tax = \$2,250.05

8. Municipal Property Tax = $1,975.97; Library Property Tax = $220.01; School Property Tax = $1,481.76; Total Property Tax = $3,677.74
9. Tax Rate = 1.109305
10. Mill Rate = 8.9929
11. Increase the mill rate by 0.3626
12. Mill Rate = 14.5412
13. 10.0247 and 5.0124
14. $192,318.89 and $179,524.86
15. Mill Rates Under 70% Policy: 5.6872 then 5.6060; Mill Rates Under 75% Policy: 5.3081 then 5.2322

Section 7.3

1. 69,482.40 US$
2. 130,500€
3. 37,744.20 AU$
4. 170,046 AU$
5. 322,976.60 C$
6. 8,780,200 ¥
7. 0.0103 ¥ per C$
8. 53,130 C$
9. Purchase $518.75 less than before
10. a. 2,784 £
    b. 4.6739%
11. 649.44 C$
12. Elena lost $29,489.81 on her investment
13. The products cost $12,800 C$ less than before
14. 3.5%.
15. Buy in the U.S. to save $196.60
16. a. Scotiabank's sell rate fee is 2.3%; The kiosk's sell rate fee is 4.5%
    b. The airport kiosk charges 2.52% more than Scotiabank
17. Booking it themselves saves $1,824.98
18. Currency fluctuations have decreased profits by $38,432.12
19. a. 639.38 US$
    b. €15,153.19
    c. ¥166,983
    d. 140,447.70 MXN$
20. Columbia Pesos = $1,084.71 C$

Section 7.4

1. $133,568.68
2. $96,513.10
3. $46,744.67
4. Balance Remaining = $21,928.01
5. Balance Remaining = $434,964.99
6. Balance Remaining = $7,065.01
7. $13,692.78
8. $232,444.92
9. $204,664.43
10. $35,901.77
11. $12,872.93
12. $23,372.94
13. $2,654.57
14. $5,479.14

15. $4,241.10
16. $24,772.64
17. $40,608.80
18. N = $13,511.24; Late penalty amount = $519.66
19.
    a. Choose Plan #2
    b. Savings = $724.92
20. $250,214.41

Chapter 7 Review

1. $3,815.95
2. 92,260 BEF
3.
    a. $1,125.42
    b. $1,045.75
    c. $1,095.55
    d. $1,150.33
4. $14,700
5. Purchase $112.70 more than before
6.
    a. 1st Quarter = $34,707.62 Refund; 2nd Quarter = $74,618.99 Remit; 3rd Quarter = $257,843.54 Remit; 4th Quarter = $36,129.04 Refund
    b. 1st Quarter = $90,239.80 Refund; 2nd Quarter = $194,009.38 Remit; 3rd Quarter = $670,393.19 Remit; 4th Quarter = $93,935.50 Refund
7. Mill Rate = 20.0562
8. $7,140.83
9. GST Tax Amount = $85.16; PST Tax Amount = $169.90
10. 150,188.69 SKK
11. Mill Rate = 30.3740
12. $4,319
13.
    a. $0.950 per litre
    b. 30.42% more
14. $10,584
15. Increase by 5.9287%.
16.
    a. GST =4.3488%; PST = 8.6758%
    b. GST =4.4643%; PST = 6.25%
17. Banco do Brasil is better option. The traveller has 19.31 Euro more.
18. $34,200
19. $940.37 US$
20. Remit $13,886.83

Chapter 8

Section 8.1

1. $7,800
2. $25,000
3. 7.5%
4. 11 months
5. $3,802.50
6. $80,000
7. $1,619.18
8. $349.86
9.
    a. $1,315.07
    b. $2,630.14

   c. $3,945.21
   d. The amount of simple interest on the same principal over the same time frame is directly proportional to the interest rate.
10. 5 months
11. 6.25%
12. $34,285.71
13. $553.36
14. March 20, 2011.
15. The second interest rate is 1.25% higher.
16. 460 units
17. $416
18. Recommend back-to-back 3 month investments resulting in $1.26 more.
19.
   a. $1,421.78
   b. 7.3092%
20. Recommend alternative #3 since it is 0.1283% more than the worst alternative #1.

Section 8.2

1. S = $17,278.25; I = $503.25
2. P = $61,000; I = $915
3. P = $22,223; t = 11 months
4. S = $43,752.67; r = 8%
5. S = $1,779.33; I = $79.33
6. P = $3,371.78; I = $428.22
7. Replacement payment = $2,009.82; I = $59.82
8. Replacement payment = $3,767.81; I = $257.19
9. $16,232.43
10. P = $9,998; I= $269.21.
11. 7.8%
12. 32 weeks
13. Replacement payment = $5,911.32
14. Payment today = $8,512.50
15. P = $778.10; S = $818.95
16.
   a. Choose to buy the lawn mower on September 30 since the price of $399.75 is lower.
   b. Savings = $10.49
17. 14.9454%
18. $1,306.99
19. $12,600
20. $5,455.03

Section 8.3

1. $6.54
2. $25,530.74
3. $8.57
4. S = $4,530.77; I = $30.77
5. $3.63
6. $30,636.49
7. $31.63
8. 200 day GIC is better than back-to-back 100 day GICs by $69.70
9. $868.55
10. The 3 back-to-back GICs are better than the 360 day GIC by $35.75
11. $3,764.53
12. $3,944.80
13. 0.4995%

14. $3.07
15. Option 1 = $90,710.14; Option 2 = $90,666.99; Option 3 = $90,640.64; Option 4 = $90,636.36

Section 8.4

1.
    a. Borrower = Joan d'Arc; Lender = Candlelight Industries; Face Value = $13,500; Issue Date = January 7, 2010; Term = 9 months; Interest Rate = 9% annually
    b. October 10, 2010
    c. $14,418.74
2. December 5, 2011
3. 334 days
4. December 10, 2011
5. January 5, 2012; $34,156.93
6. September 18, 2011; 6.75% annually
7. October 29, 2011; $8,500
8. 320 days; February 3, 2011
9. Maturity Value = $4,908.37; Proceeds = $4,706.01
10. Face Value = $14,995; Sale Date = September 1, 2011
11. Maturity Value = $43,862.92; Discount Rate = 5.12% annually
12. Interest Rate = 6.55% annually; Proceeds = $19,265.05
13. $20,470.14
14. $14,120.02
15. $4,994.58
16. 12.75% annually
17.
    a. 186 days
    b. 6.7199% annually
18. $27,000
19. Choose Offer #2 since it realizes 0.136% more in proceeds.
20.
    a. Pick 6 months; I = $273.70
    b. Pick 8 months; I = $284.22

Section 8.5

4. Total Interest = $2,040.70

8. Total Interest = $1,990.76
9. Total Interest = $774.46
10. Total Interest = $2,590.56
11. Total Interest = $3,852.20
12. Total interest = $405.04
13. Total interest = $858.58
14. Total Interest = $3,197.08
15. Option 1 = $218.93; Option 2 = $227.25; Option 3 = $251.45; Option 4 = $262.52; Option 1 is best by $43.59 over Option 4

Section 8.6

1. $99,570.35
2. 3.89%
3. $479,872.95
4. 2.85%
5. 182 days
6. 90 days
7. $200,000
8. $89,691.85
9. P (2000) = $98,604.41; P (2010) = $99,960.56; Amount more = $1,356.15
10. P (2000) = $51,743.90; P (2009) = $54,770.59; Amount more = $3,026.69

11. $488,907.82; Cannot purchase the commercial paper with $485,000. $3,907.82 more is required.
12. $247,542.21
13. $1,970.63
14. 5.08%
15. Amount earned = $133.24; r = 4.99%
16. Amount earned = $1,710.69; r = 6.34%
17. a. 6.74%
    b. 5.35%
    c. 6.10%
    d. Phillipe purchased at 6.74%. Damien purchased it from Phillipe at 5.35%. As a result, Phillipe earned a higher return on his investment of 6.74% since a lower future rate results in a higher price.
18. 3.89% & 2.85%
19. The purchase price is lower the longer the term; The purchase price is not a straight line.
20. c. The difference in principal between the various rates continually decreases the closer the T-bill moves towards maturity.

Chapter 8 Review

1. $33.32
2. $4.05
3. Legal Due Date = September 19; S = $52,141.06
4. $946.16
5. $979,439.29
6. $8,062.47
7. The principal is $23,325 and the simple interest on the loan is $834.18
8. -
9. 2.96%
10. $2.92
11. Total Interest = $4,981.30
12. a. 6.8%
    b. $476.85
13. $17,614.33
14. a. With $3,993.30, you are $6.70 short of meeting the goal
    b. 1.229%
15. $0.67
16. Total Interest = $198.60
17. -
18. 2.05% then 3.29% then 2.79% then 3.15%
19. Savings Account Balance = $8,006.50; HELOC Balance = $24,000 20. Best: Back to back 60 day GICs with Maturity Value = $25,195.59; 2nd Best: 120 day GIC with Maturity Value = $25,193.15; Worst: 90 and 30 day T-bill with investment where Maturity Value = $25,188.99

Chapter 9

Section 9.1

1. 0.6% per month
2. 2.925% per half year
3. 7.8% quarterly
4. 29.2% daily
5. Quarterly
6. Monthly
7. 14.375% annually
8. 0.975% per quarter
9. Daily

10. 0.6458% per month
11. Monthly
12. 8.35% quarterly
13. Monthly
14. 0.0534% per day
15. 7.9% monthly
16. $50
17. The lowest nominal rate is the 1.49% every quarter, equalling 5.96% quarterly
18. 7.2% quarterly
19. 5.2% per half year; 2.6% per quarter; $0.8\overline{6}$% per month; 0.0285% per day
20. 13.2% monthly; 4.4% quarterly; 2.2% semi-annually

Section 9.2

1. $9,511.81
2. $68,351.02
3. $27,465.13
4. $4,804.20
5. $18,218.24
6. $33,638.67
7. $987.24
8. $8,397.45
9. $2,235.82
10. $15,017.33; There is enough money with $17.33 left over
11. $5,254.44
12. FV=$10,374.33; I = $1,374.33
13. $12,171.92
14. Jason has 274.5% more money
15. $3,073.30
16.
    a. $11.07 per unit
    b. −27.0066%
17. $404,769.32 more
18. $11,888.46
19. Best: 19c with FV=$14,023.26; 2nd Best: 19d with FV=$13,999.47; 3rd Best: 19b with FV=$13,933.20; Worst: 19a with FV=$13,927.43
20. Total FV = $29,270.56; Total Interest = $6,770.56

Section 9.3

1. $4,081.99
2. $86,216.21
3. $55,762.07
4. $33,014.56
5. $4,340.00
6. $5,502.51
7. $8,434.18
8. $137,397.54
9. $8,307.68
10. $606,976.63
11. $33,023.56
12. $27,736.24
13. $6,427.59
14. Choose the 2 semi-annual payments saving $6.08 in today's dollars
15. $18,788.24
16. $85

17. 40.315%
18. $58,499.97
19. $836,206.54
20. Best Offer is 20b where PV = $993,846.61; 2nd Best Offer is 20c where PV = $993,391.12; 3rd Best Offer is 20a where PV = $987,884.82

Section 9.4

1. $10,501.28
2. $10,714.59
3. $43,045.16
4. $9,966.25
5. $102,495.17
6. $9,564.76
7. $22,975.60
8. $29,488.31
9. 1st payment = $15,498.71; 2nd payment = $30,997.42
10. Payment plan is better by $9,388.40
11. $5,201.24
12. $7,465.59
13. 1st payment = $11,105.19; 2nd payment = $2,776.30
14. 1st payment = = $99,675.19; 2nd payment = $199,350.38; 3rd payment = $149,512.79
15. $8,160.07
16. $14,900.01
17. The first alternative is better by $735.14
18. 1st payment = $7,232.96; 2nd payment = $3,616.48
19. $30,703.92
20. Genstar's offer is better by $1,422,325.78

Section 9.5

1. 6.44% monthly
2. 8.26% quarterly
3. 4.99% semi-annually
4. 9% annually
5. 28.5% daily
6. 6.3177% annually
7. 4.3486% quarterly
8. 4.8464% monthly
9. 18% monthly
10. 8.35% quarterly
11. 6.9% monthly
12. 5.6933% annually
13. $24,225 investment earned 4.5138% quarterly
14. BoM earns 0.0753% semi-annually more over the five years
15. 13.9433% monthly
16. 9.7361% annually
17. 3.8127% annually
18. 6.9183% monthly
19. Ranking:

City	Growth
Calgary	3.6326%
Vancouver	2.8998%

City	Growth
Toronto	2.8031%
Halifax	1.3860%
Montreal	1.3269%
Winnipeg	0.9465%
Regina	0.8276%

20. Ranking:

Company	EAR
GIC Max	3.1971%
Laurentian	3.1820%
MGI	3.0149%
Affinity	2.9686%
CIBC	2.8240%

Section 9.6

1. 4.8352% effectively
2. 7.4424% effectively
3. 3.989% effectively
4. 9.7618% semi-annually
5. 11.3866% monthly
6. 7.7706% quarterly
7. 6.0755% semi-annually
8. 4.4832% monthly
9. 7.9216% quarterly
10. 7.1544% semi-annually
11. 29.9% effectively
12. 2.0151% effectively
13. 19.7164% effectively
14. 3.832% monthly; 3.8442% quarterly
15. Competitor growth = 0.434% per month; Your growth is higher by 0.016% per month
16. Ranking:

Rank	Company	Rate
1	ING	8.6285%
2	TD Canada	8.6231%
3	Conexus	8.6045%

17. 9% effectively
18. 49.3642% effectively
19. 19.8905% daily
20. Ranking:

Investor	Rate
3	8.0976%

Investor	Rate
1	8.085%
2	8.0841%
5	8.08%
4	8.0573%

Section 9.7

1. 5 years, 6 months
2. 12 years, 9 months
3. 40 years
4. 25 years, 6 months
5. 27 years, 6 months, 28 days
6. 1 year, 88 days
7. 12 years, 44 weeks, 4 days
8. 21 years, 79 days
9. 6 years, 11 months
10. 2 years, 3 months
11. 1 year, 6 months
12. 8 years, 44 weeks, 5 days
13. 3 years, 6 months, 66 days
14. 5 years, 8 months, 11 days
15. 1 year, 10 months, 15 days
16. Rule of 72 = 8 years, 188 days; Actual = 8 years, 6 months, 14 days; The Rule of 72 is 8 days less than the actual amount of time
17. 15 months from today
18. 3 years, 1 month, 5 days
19. Final Rankings:

#		Length
1	A	5 Years, 3 Months, 0 Days
2	B	5 Years, 0 Months, 127 Days
3	C	4 Years, 11 Months, 0 Days
4	D	4 Years, 9 Months, 18 Days

20. 1 Year, 6 Months, 27 Days After Start Of Loan

Chapter 9 Review

1. $21,524.50
2. $67,313.13
3. $5,450.83
4. 27.7516% effectively
5. 4.0604% weekly
6. 18 years, 6 months
7. $70,432.59
8. $21,600
9. $3,817.05
10. 13.5131% annually
11. 7.5% quarterly and 7.7135% effectively
12. 1st payment = $5,256.23; 2nd payment = $6,307.48

13. 21 years, 3 months, 2 days
14. RBC: 5.9% quarterly and 6.0318% effectively; CIBC: 5.54% monthly and 5.6829% effectively
15. $25,596.53
16. 1st payment = $29,886.09; 2nd payment = $59,772.18; 3rd payment = $119,544.36
17. PV = $4,957.60
18. 1 year, 3 months, 70 days from today
19. Rankings

Opt.	Nom.	Eff.
B	5.83% annually	5.83%
A	5.6469% weekly	5.8062%
D	5.715% semi-annually	5.7966%
E	5.639% monthly	5.7871%
C	5.65% quarterly	5.7708%

20. 1 Year, 3 Months, 16 Days from today

Chapter 10

Section 10.1

1. I = $645; Total interest = $10,320
2. I = $209.55; Total interest = $5,029.20
3. FV = $20,627.96; I = $3,227.96
4. FV = $26,751.28; I = $1,751.28; 3.3997% quarterly
5. FV = $61,961.27; I = $10,961.27;
6. FV = $78,974.91; I = $6,599.91; 2.2057% annually
7. FV = $2,784.43; 1.2824% annually; I = $104.43
8. FV = $157,900.58; 3.3096% semiannually; I = $23,900.58
9. FV = $95,347.54; 3.0259% annually; I = $10,717.54
10. FV = $18,144.75; 2.567% annually; I = $2,159.75
11. $41.76
12. RBC earns $13.60 more
13. FV = $7,792.07; 1.6791% monthly
14. Best option = 4.83% quarterly; Worst option = 4.845% semi-annually; Extra interest earned = $21.96
15. FV = $45,679.23; 2.6909% annually
16. Stepper = $2,661.06; Fixed = $2,598.54
17.
    a. $174.11, $298.63, $499.60, $756.88, $1,113.40
    b. 3.5017% annually
18. $733.59
19. Stay with the original 5-year GIC; switching earns $659.52 less interest
20. Ranking

Rank	Option	Interest
1	A	$3,503.14
2	B	$3,498.3
3	C	$3,375.00
4	D	$3,201.54

Section 10.2

1. $8,013.31
2. $15,078.59
3. $25,500
4. 5.35% monthly
5. $60,402.69
6. $22,289.64
7. 12.5% semi-annually
8. 18.3503% monthly
9. $11,125.50
10. 2 years and 2 months
11. $1,180.12
12. $15,800
13. $6,972.52
14. $1,716.10
15. 5.4499% monthly
16. $9,194.41
17. PV = $25,874.62; I = $5,924.62
18. 10.25% quarterly
19. $101,890
20. Ranking

#	Sell	Proceeds
1	2 Years From Today	$128,941.02
2	Today	$125,256.22
3	1 Year From Today	$124,764.20

Section 10.3

1. $2,475
2. $916.70
3. $2,196.60
4. $1,336.88
5. FV = $22,250.70; I = $3,350.70
6. FV = $87,278.81; I = $10,878.81
7. FV = $32,709.45; I = $2,809.45
8. FV = $2,848.12; I = $348.12
9. FV = $7,255.61; I = $1,055.61
10. FV = $48,320.12; I = $2,620.12
11. $3,289.29
12. FV = $75,165.44; I = $9,665.44
13. FV = $105,203.70; I = $2,103.70
14. $8,294.40
15. $4,220.78
16. FV = $25,027.93; I = $2,827.93
17. $35.51
18. $4,100
19. Ranking

Rank	Series	Maturity Value
1	S92	$11,294.18
2	S96	$11,235.67

Rank	Series	Maturity Value
3	S97	$11,225.71
4	S93	$11,207.07
5	S94	$11,181.48
6	S95	$11,159.34

20. $41,480.30

Section 10.4

1. PV = $84,475.66; I = $73,024.34
2. PV = $176,595.71; I = $73,404.29
3. PV = $3,626.26; I = $8,423.74
4. PV = $38,941.67; I = $42,433.33
5. IY = 9.4403% semi-annually; I = $90,912.87
6. IY = 8.9241% semi-annually; I = $32,462.61
7. IY = 5.1856% semi-annually; I = $50,052.03
8. IY = 3.5447% semi-annually; I = $6,056.11
9. IY = 10.7437% semi-annually; I = $1,947.67
10. IY = 6.5444% semi-annually; I = $144,783.43
11. PV = $4,124.24
12. PV = $80,156.75
13. IY = 6.0102% semi-annually
14. IY = 5.165% semi-annually
15. a. IY = 4.95% semi-annually
    b. IY = 4.5155% semi-annually
    c. IY = 5.9226% semi-annually
16. a. $22,170.05
    b. $30,699.38
    c. 4.3874% semi-annually
    d. $8,529.33
17. Yields on the date of purchase were 0.3817% semi-annually higher than on the date of sale.
18. $133,000,000
19. 158 bonds
20. d. When the yield remains constant, the purchase price follows a straight line trend. When the yield rises, the purchase price drops creating a line that is flatter initially and rising faster closer to maturity. When the yield decreases, the purchase price increases creating a line that is rising fast initially and becoming flatter as it nears maturity.

Section 10.5

1. $50,914.14
2. FV = $54,662.21
3. $9,840.05
4. 7.1993% annually
5. 19 years
6. 1983 PPD compared to 1978 is 63.0032%
7. 1994 PPD compared to 1989 is 87.7189%
8. 138,238.68
9. 249,207.20
10. 61,386.88
11. 3.3075% annually
12. $1.1024

13. $357,000
14. a. $289,929.94
    b. 2057 PPD compared to 2012 is 13.7964%
15. 1975 dollar had its purchasing power decreased by 54.5297%
16. 63 years
17. Do not launch, forecasted annual sales after 15 years are only 20,035.405 units
18. $6,729.71
19. a. 2017
    b. 2045
    c. 2072
20. c. a-c.

Age	Income Range
65	$97,514.17 to $359,400.31
70	$107,663.52 to $458,695.9
75	$118,869.23 to $585,425.24

    d. Since the future is uncertain, developing a range of best-case to worstcase scenarios helps develop a complete understanding of what may be required and aids in the development of a savings plan.

Chapter 10 Review

1. $8,914.50
2. $27,173.48
3. $1,019.20
4. $3,982.49
5. FV = $65,105.42; I = $15,105.42
6. $1,475.64
7. $13.81
8. 9,378
9. FV = $6,817.80; 2.5885% annually; I = $817.80
10. $6,503.10
11. FV = $22,195.71; I = $1,695.71
12. $4,274.90
13. a. FV = $38,248.50; I = $4,248.50
    b. $20.64
14. a. 9.7003% semi-annually
    b. 4.2141% semi-annually
    c. 13.4394% semi-annually
15. 19.1%
16. 1933 PPD compared to 1930 is 129.5585%
17. 4.3435% semi-annually
18. From 2004 to 2008, after-tax annual earnings decreased by $280.57
19. Bond yields must increase by 0.3464% in order to achieve the goal.

Chapter 11

Section 11.1

1. Annuity
2. Not an annuity - unequal payments
3. Not an annuity - non-periodic
4. Not an annuity - discontinuous

5. General annuity due
6. Ordinary simple annuity
7. Simple annuity due
8. Ordinary general annuity
9. Ordinary simple annuity
10. General annuity due
11. Ordinary general annuity; N=20
12. General annuity due; N=60
13. Simple annuity due; N=36
14. General annuity due; N=24
15. Ordinary simple annuity; N=12
16. Ordinary simple annuity; N=48
17. Ordinary general annuity; N=10
18. General annuity due; N=312
19. General annuity due; N=12
20. Simple annuity due; N=13

Section 11.2

1. \$116,471.46
2. \$250,457.58
3. \$305,305.23
4. \$272,152.25
5. \$56,486.35
6. \$22,278.17
7. \$267,678.52
8. \$198,708.14
9. \$683,712.33
10. \$8,612.62
11. \$306,680.93
12. \$984,888.25; No, the fund is \$15,111.75 short.
13. \$24,035.26
14. \$18,452.55
15. \$3,680.30
16. \$61,718.23
17.
    a. \$79,687.12301
    b. \$159,374.246; The maturity value increases proportionately to the size of the contribution.
    c. \$163,263.6827; Given the same amount of total annual contribution, the maturity value increases exponentially to the increase in the frequency of contributions.
18. \$1,827,832.95
19.
    a. The longer the money is invested, the interest earned exponentially increases.
    b. Increasing the interest rate results in exponential increases in both the maturity value and interest earned. The interest change is always greater than the change in the interest rate.
20. The more time that an investment is allowed to compound, the maturity value increases exponentially. Start RRSP contributions as early as possible.

20a	\$1,508,179.91
20b	\$590,417.31
20c	\$259,410.60

Section 11.3

1. $58,189.26
2. $89,707.35
3. $310,189.42
4. $28,870.80
5. $92,181.19
6. $89,676.72
7. Balance Owing = $14,438.98; I = $5,046.68
8. Balance Owing = $12,999.13; I = $13,452.01
9. $8,721.96
10. $44,184.00
11. $387,444.19
12. $28,166.41
13. $128,398.17
14. Balance Owing = $22,786.85; I = $3,639.92
15. $2,117.16
16.
    a. $13,340.93
    b. $11,745.07
17. $45,898.34
18. $88,880.80
19. Less money is required to fund a longer term annuity (all else held equal) since it has more time to earn compounding interest. Also, the change in time is not directly proportionate to the reduction of principal.

19a	$43,610.57
19b	$63,942.03
19c	$105,221.32

Section 11.4

1. $335.72
2. $12,580.41
3. $5,652.40
4. $6,953.76
5. $4,013.56
6. $1,044,548.68
7. $261.57
8. $802.70
9. $571.60
10.
    a. $5,133.93
    b. $5,054.93
11.
    a. $713.95
    b. Balance Owing = $22,569.38; I = $8,271.58
    c. $898.88
12. $42.29
13. $500
14. $1,084,268
15. $3,943.82
16. $6,810.60
17. $1,796.23
18. $364.88
19. $62.65

20. With each 1% increment in the interest rate, the monthly payment rises by an amount that is constantly increasing with each interest rate change. The amount of the increase is a percent increase that is substantially higher than just 1%, ranging from approximately 9% to 10.5%.

Section 11.5

1. 6 years, 9 months
2. 22 years, 11 months
3. 20 years
4. 18 years
5. 11 years
6. 15 years
7. 12 years, 9 months
8. 9 years, 3 months
9. 8 years, 3 months
10. 6 years, 1 month
11. 2 years
12. 1 year, 11 months
13. 8 years, 2 months
14. 3 years, 2 months
15. 3 years, 6 months
16. 3 years, 9 months
17. a. 9 months
    b. The regular payments of $250 requires $2,250 less principal to reach the goal
18. a. 48 fewer monthly payments
    b. $26,521.44
19. c. The total amount of money required to pay off a loan increases with each increase in the compounding frequency (while holding all else the same).
20. c. The total amount of money required to pay off a loan increases with each decrease in the payment frequency (while holding the nominal annual payment the same).

Section 11.6

1. 17.4719% monthly; 18.9412% annually
2. 3.1172% quarterly; 3.1538% annually
3. 9.044% semi-annually; 9.2484% annually
4. 7.8981% monthly; 8.1904% annually
5. 9.7591% semi-annually; 9.9972% annually
6. 4.2704% annually
7. 6.657% monthly; 6.8639% annually
8. 3.769% monthly; 3.8226% annually
9. 8.6063% monthly
10. a. 4.525% monthly; 4.62% annually
    b. $16.89
11. 3.9019% semi-annually
12. 8% annually
13. 6.5014% semi-annually
14. 11.5556% annually
15. 7.2474% monthly
16. 5% semi-annually
17. Monthly payments: 9.6535% annually; Quarterly payments: 10.6223% annually; Semi-annual payments: 9.541% annually
18. 7.6118% monthly; Decision rule: If you can obtain a loan from the bank for less than 7.6118% monthly, your payments are lower and you borrow from the bank. If unable to obtain a rate lower than 7.6118% from the bank, forego the cash rebate and use the dealership financing.

19.

Option	Effective Rate
$299	4.7080%
$319	18.2370%
$334	29.0693%
$349	40.5145%

a. Choose the $319 payment plan with an effective rate of 18.237%
b. Choose the $334 payment plan with an effective rate of 29.0693%

20.

Risk	Effective Rate
Low	5.0001%
Medium	9.0002%
High	13.9998%

Chapter 11 Review

1. $110.36
2. $652.42
3. $3.00
4. $429.84
5. 16 years
6. 7.9% semi-annually; 8.056% annually
7. 1 year, 2 months
8. $19,695.13
9. 2.9% monthly
10. $29,894.24
11. $394.14
12. $52,351.32
13. 8 years
14. PMT = $1,799.23; I = $2,196.92
15. $1,551.14
16. 9 years
17. 5.575% monthly
18. $131.36
19. (i) Each increment in the interest rate results in a proportionately larger increase in the payment regardless for all lengths of the mortgage. (ii) Each increment in the length of the mortgage results in a proportionately smaller decrease in the payment, and the decrease is smaller at higher interest rates.
20.

Starting Lump-Sum	Monthly Contribution
$5,000	$360.92
$10,000	$324.06
$15,000	$287.19
$20,000	$250.32

Each equal increase in the present value results in an equitable decrease in the contribution required (in this case about $36.87 per $5,000 increase)

Chapter 12

Section 12.1

1. $44,667.03
2. 12 years, 9 months
3. 17 years, 6 months
4. $12,101.52
5. 25 years, 6 months
6. 8 years, 6 months
7. $106,142.91
8. $897.51
9. $39,070.09
10. 13 years, 5 months, 9 days
11. 19 years
12. $21,609.06
13. July 13, 2024
14. $752.78
15. $964.29
16. 1 year
17. a. Scully's higher = $1,216,489.09
    b. Scully's deposits = $56,100; Mulder's deposits = $99,000
    c. The earlier the deposits are made, the exponential power of compound interest occurs. Each doubling of money is a large double compared to a small double.
18. $37,571.53
19. $6,895.79
20. 1.

Interest Rate	Present Value
6%	$41,098.34
8%	$31,080.66
10%	$23,689.97

2.

Interest Rate	Annuity Payment
6%	$12,165.94
8%	$16,087.17
10%	$21,105.98

3.

Interest Rate	Term
6%	6 years
8%	8 years
10%	11 years

4. The interest rate in a deferred annuity situation has exponential implications on any amount calculated. The higher the interest rate, the less present value is required, the higher the annuity payments, and the longer the term.

Section 12.2

1. a. $186,267.92
   b. $102,682.49

2. a. $339,630.53
   b. $102,702.93
3. a. $266,532.98
   b. $22,541.90
4. a. $418,925.39
   b. $96,472.02
5. a. $2,074,882.90
   b. $56,814.45
6. a. $984.40
   b. $1,057.83
7. a. $27,600.21
   b. $31,996.21
8. a. $1,205.23
   b. $2,449.39
9. a. $1,117.87
   b. $5,235.46
10. $399,390.56
11. $1,761.99
12. $318,918,112
13. $3,761,909.60
14. a. $2,001.15
    b. $1,242.06
    c. $767.82
15. $22.93
16. a. $4,897.29
    b. $8,129.90
    c. $1,301,174.06
17. a. $327,005.73
    b. $5,367.68
    c. $2,394,027.27
18. a. $731,157.19
    b. 30.32%
19. $27.99; The difference from 35 years of payments is $4.06. This is because today's money becomes worthless so far into the future (what would a penny today buy thousands of years from now?).
20. a. Increasing the growth rate from 1% to 5% (a 400% increase) results in total contributions that increase approximately 250%, however the benefit is a maturity value and the interest earned both increasing by approximately 50%.
    b. Increasing the growth rate from 1% to 5% (a 400% increase) results in a present value that only increases by approximately 33%. As a result, the individual receives approximately 37% more income which accrues almost 50% more interest.

Section 12.3

1. $515,463.92
2. $496,190.48
3. $37,999.04
4. $26,400
5. $371,348.38
6. $13,207.62
7. $38,371.42
8. $503,025.11
9. $300,884.96
10. $60,000

11. $283,224.19
12. $222,589.59
13. $3.75
14. $52,942
15. $25.89
16. $12,791.51
17. $45,757,082.97
18. $602.12
19.

Rate	Principal Required
2%	$500,000.00
3%	$333,333.33
4%	$250,000.00
5%	$200,000.00
6%	$166,666.67

The percent decrease in the principal required to fund the perpetuity is inversely proportionate to the percent change in the interest rate expressed as a change of the new interest rate.

20.

Rate	Perpetuity Payment
2%	$2,000
3%	$3,000
4%	$4,000
5%	$5,000
6%	$6,000

The percent increase in the payment made by the perpetuity is proportionate to the percent change in the interest rate expressed as a change of the old interest rate.

Section 12.4

1. $29,978.35
2. $14,874.88
3. $2,345.21
4. 11% annually
5. $6,250.03
6. $404,770.95
7. 7.75% quarterly
8. $9,506.92
9. $29,506.30
10. $5,208.61
11. $1,900,645.54
12. $592,137.21
13. 5.72% quarterly
14. $70,193.84
15. $5,198.45
16. $18,507.31
17. a. 1,593.06% annually; $1,538.12
    b. PMT = $20.74; $118.76

18. Go with the lease, savings = $1,370.45
19. $212,283.48
20.

Rank	Dealer	Present Value
1.	#3	$15,999.92
2.	#1	$16,000.05
3.	#2	$16,249.79

Section 12.5

1. Choose bank loan with payments of $729.24 monthly
2. Choose dealership lease with payments of $406.02 monthly
3. Choose dealership lease with payments of $1,147.62 monthly
4. Choose bank lease with payments of $185.03 bi-weekly
5. Choose dealership lease with payments of $616.13 monthly

Section 12.6

1. $4,921.95
2. $369.60
3. $2,333.22
4. $3,164.27
5. $8,146.44
6. $872.66
7. $806.57
8. 1.

Rate	RRSP Contribution
2%	$352.16
3%	$570.3
4%	$921.31
5%	$1,484.27

Each percentage increase in the inflation rate results in a higher percent change required for the RRSP contribution.

2.

Rate	RRSP Contribution
0.5%	$462.08
0.7%	$414.82
0.9%	$369.60
1.0%	$326.70

Each change in the growth rate results in a smaller negative percent change for the required contribution.

Chapter 12 Review

1. $9,082.22
2. $699.28
3. $1,018,577.58
4. $6,391.62
5. $29,459.49
6. $1,447.48
7. 8.2651% monthly
8. $50,505.32

9. 26 monthly payments
10. $1,081.79
11. 4 years, 101 days
12. $69.07
13. Choose bank loan with payments of $579.78 monthly
14. 75.0229%
15. a. $761.45
    b. $57,360.07
    c. $354.52
    d. $46,899.39
16. $672,390.21
17. $13,512.24
18. 37 months
19.

Company	PV of lease offer
#1	$951,255.03
#2	$973,809.83
#3	$967,954.15
#4	$976,442.25

Recommendation: Choose Company #1 with lowest PV = $951,255.03

20.

Option	PV of offer
#1	$4,289,771.59
#2	$4,417,189.74
#3	$4,253,639.42
#4	$4,350.150.00

Recommendation: Choose Option #2 with highest PV = $4,417,189.74

Chapter 13

Section 13.1

1. PRN = $135.79; INT = $20.27
2. PRN = $710.78; INT = $194.14
3. PRN = $2,588.13; INT = $448.81
4. PRN = $14,464.64; INT = $1,452.26
5. PRN = $4,633.70; INT = $914.02
6. PRN = $3,567.70; INT = $911.66
7. PRN = $7,772.80; INT = $1,303.60
8. PRN = $8,574.79; INT = $1,711.33
9. PRN = $77,544.60; INT = $13,576.08
10. PRN = $157,242.37; INT = $42,967.93
11. a. PMT = $328.79
    b. PRN = $291.70
    c. INT = $25.22
    d. INT = $556.26
    e. PRN = $3,598.25
12. a. PMT = $2,359.13

    - b. PRN = $1,792.76
    - c. INT = $257.53
    - d. PRN = $8,109.20
    - e. INT = $2,618.23
13. a. PMT = $604.25
    - b. PRN = $533.03
    - c. INT = $253.73
    - d. NT = $3,332.61
    - e. PRN = $4,241.39
14. a. PMT = $1,392.37
    - b. INT = $238.65
    - c. PRN = $4,991.29
    - d. INT = $2,704.16
15. a. PMT = $781.07
    - b. INT = $3,179.72
    - c. PRN = $7,303.28
16. INT = $1,342.93
17. a. PRN = $408.13; INT = $460.70
    - b. INT = $5,250.65
    - c. PRN = $5,796.37
18. a. 3 years, 1 month
    - b. PRN = $270.84; INT = $29.16
    - c. INT = $403.33
19. Note that the percent increase in interest is always larger than the percent decrease in the payment. Although payments get continually lower, the amount of interest paid is continually higher.
20. Note that in both the payment and the interest paid that the percent change is constantly decreasing due to the larger base upon which it is calculated. However a look at the nominal dollar changes reflects that the both the payment and interest paid increase at a relatively constant amount for each percent increase in the interest rate.

Section 13.2

1. PMT = $1,462.27
2. PMT = $1,273.03
3. PMT = $4,822.82
4. PMT = $1,765.68
5. PMT = $1,546.70; PRN = $5,889.27; INT = $297.38
6. PMT = $1,329.49; PRN = $29,720.92; INT = $2,183.15
7. PMT = $1,537.79; PRN = $5,472.70; INT = $678.55
8. PMT = $1,650.26; PRN = $41,339.04; INT = $1,548.22
9. PMT = $6,343.90; PRN = $17,636.35; INT = $1,395.32
10. PMT = $2,007.05; PRN = $7,555.71; INT = $472.16
11. a. PMT = $585.50
    - b. PRN = $6,697.62; INT = $328.50
12. a. PMT = $7,016.43
    - b. PRN = $25,619.32; INT = $2,446.01
13. a. PRN = $53,459.54; INT = $54,952.06
    - b. PMT = $1,805.72
    - c. PRN = $96,004.52; INT = $12,405.95
14. a. PMT = $5,662.21
    - b. PRN = $15,849.42; INT = $1,137.16
15. a. PRN = $14,820.95; INT = $10,606.57

b. PMT = $1,059.89
c. PRN = $23,193.79; INT = $2,234.15

16. a. PMT = $6,248.88
    b. PRN = $133,465.32; INT = $16,505.73
17. a. PMT = $636,117.25
    b. PRN = $11,095.973.76; INT = $1,626,345.59
18. a. PMT = $14,045,354.50
    b. PRN = $59,748,845.61; INT = $10,477,930.69
19. a. PMT = $512.66
    b. PRN = $5,366.34; INT = $146.32
20. b. Although the exact impact on the final payment adjustment is a factor of the rounding difference in the annuity payment, generally the longer the term (and hence more payments made), the larger the difference required in the final payment.

Section 13.3

17. Interest Saved = $3,572.25
18. b. Interest Saved = $695.40
19. Total interest for first 12 months = $3,866.97 Total interest for next 12 months = $1,394.09
20. Total Interest = $83,091.67 Total Principal = $$24,439.61

Section 13.4

1. $3,906.35
2. $1,251.96
3. $286.12
4. $2,370.32
5. $2,410.35
6. $985.74
7. $2,079.35
8. $601.52
9. Balance = $284,498.75; PMT = $1,945.52
10. a. $508,947.54
    b. PRN = $19,252.46; INT= $99,737.98
    c. $3,211.32
    d. $475,372.69
11. a. $758,979.36
    b. PRN = $75,020.64; INT = $386,368.68
    c. $1,280.56
    d. $639,406.26
12. a. $361,847.62
    b. PRN = $18,152.38; INT = $30,523.46
    c. $580.30
    d. $290,670.41
13. a. $97,557.39
    b. PRN = $83,142.61; INT = $96,137.39
14. a. $129,370.79
    b. PRN = $166,529.21; INT = $150,786.43
15. $207,331.02
16. Balance = $234,944.63; PRN = $30,055.37; INT = $63,250.83
17. a. $270,417.34
    b. $1,487.42
18. -

19. Start of 3rd term principal = \$299,756.24; Remaining balance at end of 3rd term = \$220,328.74; Total interest = \$411,499.50; Total principal = \$188,321.26
20. a. Comments: Each one percent increase in the mortgage rate upon renewal results in an exponentially larger increase in the payment required. In this case, it is approximately a 9% increase in the required payment.
    b. Comments: While an increase in income is needed just to pay for rising costs of taxes and heating, the percent increase required in the income becomes quite significant if the interest rate rises by 2% or more.

Chapter 13 Review

1. PRN = \$1,504.27; INT = \$501.75
2. PRN = \$6,810.95; INT = \$1,786.45
3. \$917.82
4. PRN = \$26,763.03; INT = \$1,187.52
5. -
6. -
7. \$95,615.95
8. \$1,735.84
9. a. \$7,604.85
    b. \$4,771.37
    c. \$4,026.56
    d. \$35,827.23
    e. \$47,183.46
10. a. \$3,118.35
    b. \$2,465.45
    c. \$266.96
    d. \$8,480.07
    e. \$1,866.95
11. a. PRN = \$4,687.84; INT = \$1,750.40
    b. \$268.13
    c. PRN = \$6,097.71; INT = \$340.41
12. a. \$263.47
    b. \$53.85
13. -
14. -
15. \$750.22
16. a. \$829,698.12
    b. PRN = \$41,301.88; INT = \$229,441.65
    c. \$4,779.82
    d. \$797,181.08
17. a. \$146,200.92
    b. \$928.70
    c. \$1,511.58
18. b. Worker = \$2,649.62; Supervisor = \$5,299.25; Manager = \$10,598.49

Chapter 14

Section 14.1

15. \$4,622.62
16. \$10,122.56 increase
17. \$9,480.97 decrease
18. \$10,000
19.

Date	Market Price	Cash Price

Date	Market Price	Cash Price
June 1, 2006	$14,722.88	$14,722.88
July 1, 2006	$14,711.11	$14,776.69
August 1, 2006	$14,699.15	$14,832.49
September 1, 2006	$14,687.41	$14,888.50
October 1, 2006	$14,676.24	$14,942.91
November 1, 2006	$14,664.91	$14,999.34
December 1, 2006	$14,654.15	$14,654.15

Comments: The Market Price constantly gets closer to the face value as the bond moves closer to its maturity date. The Cash Price equals the Market Price on an interest payment date and then constantly rises in between the interest payment dates only to decrease to the Market Price on the next interest payment.

20. a.

Market Rate	Market Price
3%	$12,991.58
4%	$11,367.77
5%	$10,000.00
6%	$8,844.26
7%	$7,864.49

b. The market prices do not change equally for each 1% change in the market rate. A bond's price is calculated through present value compound interest calculations. This results in exponential changes in the present value when an interest rate changes.

c. A 1% change does not result in the same change. For discounts, the bond is growing at 6% but paying out 5%. Since this percentage is calculated on a smaller value, it results in a smaller increase in the price over time. The market price rises slowly in small increments to its redemption price. For premiums, the bond is growing at 4% but paying out 5%. Since this percentage is calculated on larger value, it results in a larger decrease in the price over time. The market price moves rapidly downwards in large increments to its redemption price.

d.

Market Rate	Market Price
3%	$11,716.86
4%	$10,817.57
5%	$10,000.00
6%	$9,256.13
7%	$8,578.76

e. The time decreased by half, but the prices did not decrease by half due to the compounding nature of the calculations. Therefore, the more time to maturity, the more signif

f. icant the change in the bond price.

Section 14.2

1. 2.91%
2. 8.25%
3. 10.5%
4. 3.25%
5. 6.8363%

6. 8.2713%
7. 11.8364%
8. 5.8774%
9. 4.2%
10. 8%
11. 3.69%
12. 5.6742%
13. 3.8687%; Original yield to maturity is the market rate on the date of purchase = 4%. Do not sell the bond since Usama will yield only 3.8687%.
14. 5.2091%
15. 10.1617%
16. 3.7018%
17. a. 4.16%
    b. 4.07%
    c. IY = 4.5041%; Δ% = 8.2716%
18. a. Purchase price = $73,809.92; $1,190.08 discount
    b. Purchase price = $75,263.06; $263.06 premium
    c. IY = 4.6701%; Δ% = 11.1927%
    d. 4.2441% & 4.7246%; Δ% = 11.3216%
19.

Bond	Yield	Rank
I	4.85%	2
ii	4.83%	3
iii	4.89%	1

20.

Held	Yield	Rank
5 years	8.413%	1
10 years	8.0936%	4
15 years	8.2734%	3
20 years	8.3886%	2
25 years	7.9518%	5
30 years	7.55%	6

Section 14.3

15. b. After 48th payment (4 years), balance in account is $12,670.56. 5% down payment on $250,000 home is $12,500. The home can be purchased.
16. -
17. -
18. a. 664 bonds
19. -
20. a. Total interest earned = $74,788,928.85; Total payments = $25,211,070.60
    b. Total interest earned = $6,851,889.63; Total payments = $4,201.845.10

Section 14.4

1. a. $128,239.72
   b. $216,400.17
2. a. $161,290.28

  - b. $1,844,903.68
3. a. $102,952.34
   b. $248,468.94
4. a. $101,110.08
   b. $448,898.62
5. -
6. -
7. -
8. -
9. $1,222,300
10. $40,175,431.68
11. a. $9,844,581.28
    b. $66,498,231.56
12. a. $17,144,400
    b. $153,840,884.09
13. Nominal Net Income = $7,964.98
14. Nominal Net Income = $17,793.48
15. Capital Gain = $916.15
16. Capital Loss = $48.68
17. $2,078,051.88; No, do not proceed with project as ACD is $78,051.88 higher than the maximum budget.
18. Do not purchase bond since fifth year capital gains tax exceeds $150.

Chapter 14 Review

1. $107,523.91
2. $64,614.18
3. 3.8501%
4. 10.9376%
5. -
6. -
7. $2,480,127.02
8. -
9. 10.9414%
10. $411,697.06 $136,697.06 premium
11. -
12. ACD = $3,615,638.66; BVD = $35,493,436.28
13. -
14. 7.3211%
15. 15. ACD = $79,055.96; BVD = $121,371.58
16. -
17. 17. Fourth year taxes deducted are $123.41.
18. b. $37,087,119.77
    c. $3,992,153.70
    d. $61,351,698.69
19. b. Investor’s Yield = 7.5423%
    c. Yield to maturity = 4.18%
20. b. Investor’s yield = 4.9669%
    c. Yield to maturity = 4.3798%

Chapter 15

Section 15.1

1. NPV=\$149,603; $\text{NPV}_{\text{RATIO}} = 0.29$
2. NPV= −\$66,518.47; $\text{NPV}_{\text{RATIO}} = -0.07$
3. NPV= −\$125,893; $\text{NPV}_{\text{RATIO}} = -0.17$
4. NPV= \$187,245; $\text{NPV}_{\text{RATIO}} = 0.62$
5. Projects Chosen in order = Project #3, Project #1; Total NPV = \$600,500; Budget Used = \$450,000
6. Projects Chosen in order = Project #3, Project #1; Total NPV = \$735,500; Budget Used = \$640,000
7. NPV of Purchase=−\$10,794; NPV Of Lease = −\$10,857; Choose Purchase, saves \$63 in current dollars
8. \$1,703,103; Pursue the project since NPV is positive.
9. Projects Chosen in order = Project #4, Project #2; Total NPV = \$460,000; Budget Used = \$475,000
10. NPV Project A = \$575,516; NPV Project B = \$739,933; Choose Project B, NPV is higher by \$164,417.
11. \$181,282; Pursue the project since NPV is positive.
12. NPV Of Farmer A = \$185,860; NPV Of Farmer B = \$187,147; NPV Of Farmer C = \$188,701; Choose Farmer C with highest NPV.
13.
    a. \$68,289; Pursue the project since NPV is positive.
    b. \$14,632; Pursue the project since NPV is positive.
    c. −\$32,290; Do not pursue the project since NPV is negative.
14. Projects Chosen in order = B, D, A, E; Total NPV = \$4,345,000; Budget Used = \$4,270,000
15. −\$54,401; Do not pursue the project since NPV is negative. Since the cost of capital is much higher than question 11, inflows farther in the future are significantly discounted.
16. Projects Chosen in order = B, C, D; Total NPV = \$55,861; Budget Used = \$100,000
17. \$15,496; Purchase the machine since NPV is positive.
18.
    a. Capital NPV Rank 7.5% \$224,764 1 13.5% \$69,341 2 17.5% −\$11,245 3
    b. \$236,009
    c. The cost of capital plays a significant role in the NPV decision. As the costs of achieving the initial funding rise, the future cash flows are discounted by larger amounts meaning that larger cash inflows in the future are required to cover the costs.
19. Machine A NPV = \$40,950; Machine B NPV = \$46,269; Purchase Machine B since NPV is higher.
20. Projects Chosen in order = 1,2,4,6; Total NPV = \$10,237,500; Budget Used = \$9,700,000

Section 15.2

1. \$2,956
2. \$3,839
3. \$9,482
4. −\$9,546
5. 19.5945%
6. 30.1215%
7. 19.8435%
8. 11.3079%
9. Walmart=\$115; HD-\$112; CT=\$113; Choose Sno-Tek model at Home Depot
10. EC120 = −\$475,529; Eurocopter= −\$472,838; Choose Eurocopter model
11. 7.9136%; Do not pursue project since IRR 7.9136% < 12% cost of capital.
12. 26.7790%; Pursue the project since IRR 26.7790% > 21% cost of capital.
13. Product A = \$73,014; Product B = \$65,716; Choose Product A
14. New Flyer = −\$48,355 per bus; Motorcoach = −\$50,809 per bus; Choose New Flyer. This saves \$185,550 annually.
15. 22.7842%; Invest since the IRR of 22.7842% exceeds the cost of capital of 20%.
16. 22.9124%; Do not purchase licence since the IRR of 22.9124% is below the cost of capital of 25%.
17. 16.4736%
18.
    a. Project A = \$48,532; Project B = \$53,792; Choose Project B, NPV is higher by \$5,260.
    b. Project A = \$11,804; Project B = \$13,084; Choose Project B, Equivalent Annual Cash Flow is higher by \$1,280.
    c. Both the NPV method and the Equivalent Annual Cash Flow method arrive at the same decision when the timelines are of equal length between various investment decisions.

19. a.

Project	EACF
A	$31,207
B	$28,293

b. $2,914

20. a.

Project	IRR	Recommend
A	16.5438%	
B	32.0026%	Yes

b.

Project	NPV	Recommend
A	$15,201	
B	$43,103	Yes

c.

Project	NPV	Recommend
A	$73,945	Yes
B	$61,645	

d. NPV makes more sense since it is the actual value being added to the company today using the actual cost of capital involved.

Chapter 15 Review

1. NPV Of Purchase = −$105,932; NPV Of Lease = −$106,491; Choose Purchase, saves $559 in current dollars
2. $16,241; Pursue the venture since NPV is positive.
3. Projects Chosen in order = Projects B, C, A; Total NPV = $329,000; Budget Used = $295,000
4. New Car = −$28,117; Used Car = −$30,209; Choose New Car saving $2,092 in current dollars.
5. 17.4712%; Pursue project since 17.4712% IRR exceeds 15% cost of capital.
6. Vanilla is better with a highest equivalent annual cash flow that is $9,501 higher.
7. $4,017; Pursue the automation since NPV is positive
8. a. $61,263; Take the course since NPV is positive.
   b. The maximum cost of capital is 22.2581%, which is the break-even.
9. a. FR#1=$133,721; FR#2=$145,079; Choose Fishing Rights #2 since NPV is higher.
   b. FR#1=$45,984; FR#2= $49,792; Choose Fishing Rights #2 since Equivalent Annual Cash Flow is higher.
10. Projects Chosen in order = Projects #1, #5; Total NPV = $680,000; Budget Used = $658,000
11. a. Best Project is Project B with equivalent annual payment of $1,326. Project B is better than worst Project A by $2,713 annually.
    b. Project C is the best with equivalent annual payment of $21,558.
12. a. Best Project is Project B with equivalent annual payment of $5,539. Project B is better than worst Project C by $5,960 annually.
    b. Best Project is Project B with equivalent annual payment of $19,912
13. The maximum cost of capital is 16.0722%.
14. NPV = $108.240; IRR = 16.0722%; Equiv. Annual Cash Flow = $24,121
15. Projects Chosen in order = Projects #3, #4, #5; Total NPV = $1,229,000; Budget Used = $966,000

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16.3: Rounding Rules

Global Textbook Rounding Rules

1. All interim solutions never get rounded unless there is a logical reason or business process that forces the number to be rounded.
2. When writing non-terminating decimals in this textbook, only the first six (or up to six) decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
3. All final numbers are rounded to six decimals in decimal format, and four decimals in percentage format unless instructions indicate otherwise.
4. Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
5. Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line-up a sequence of numbers.

Topic-Specific Rounding Rules

Section 4.1, Gross Earnings, Salaries: When calculating overtime pay, maintain all decimals in the hourly rate and only round the final pay amount.

Section 4.3, Indexes, Index Numbers: Index numbers with a 100 base are rounded to one decimal. If the index number represents dollars, then a two decimal standard is used. All other indexes require a specific rounding instruction.

Section 5.2, Break-Even Analysis, Break-even Units: All break-even units are always rounded upwards to the next integer regardless of the actual value of the decimal.

Section 7.1, Sales Taxes: The calculation of any sales tax amount is rounded to two decimals.

Section 7.2: Property Taxes: Mill rates are commonly expressed in four decimals and tax rates are commonly expressed in six decimals.

Section 7.3: Exchange Rates: In true markets, exchange rates are commonly expressed with ten decimals or more. A four decimal standard is used for mathematical purposes in this book.

Section 8.5: Demand Loan Repayment Schedules: For simplicity when writing the numbers into repayment schedules and performing all calculations, round all interest calculations to two decimals throughout the table.

Chapter 9 onwards: The calculated value of the periodic interest rate is never rounded in any interim calculation.

Section 9.5, Determining The Interest Rate, Calculating IY: Due to an imprecise single payment used in the calculation, IY is rounded to an integer or early decimal position only if it has a marginal effect. If unable to produce a marginal effect, round as per global rounding rules.

Section 9.7, Determining The Number Of Compounds, Calculating N: Due to an imprecise single payment used in the calculation, N is rounded to the nearest integer if it would be the same result as rounding to three decimals. If rounding to three decimals is not equal to rounding to an integer, then all decimals must be carried forward in further calculations.

Section 9.7, Determining The Number Of Compounds, Calculating Days: Any calculation of days is always rounded to the nearest integer since interest can only be compounded daily at minimum.

Section 10.3, Savings Bonds, Regular Interest Savings Bond Interest Payments: Each interim interest payment must be rounded to two decimals.

Section 11.5, Number Of Annuity Payments, Calculating N: N is rounded to the nearest integer if it would be the same result as rounding to three decimals. If rounding to three decimals is not equal to rounding to an integer, then N is always rounded up to the next nearest integer regardless of the actual value of the decimal.

Section 11.6, Annuities Interest Rates, Calculating IY: Similar to Section 9.5, due to imprecise variable(s) used in the calculation IY is rounded to an integer or early decimal position only if it has a marginal effect. If unable to produce a marginal effect, round as per global rounding rules.

Section 12.1, Deferred Annuities, Moving Money Between Accumulation and Payment Stages: Money is commonly transferred between bank accounts between the two stages requiring the interim amount to be rounded to two decimals.

Section 13.3, Amortization Schedules: When writing amounts in the table, round the answers to two decimals. However, behind the scenes all calculations must be unrounded.

Section 14.1, Bond Coupon Payment Amount: The variable $\mathrm{PMT}_{\mathrm{BOND}}$ must be unrounded throughout all calculations.

Section 14.1, Bond Cash Price On A Non-Interest Date: Formula 14.5 must use the unrounded value of the Date Price in its calculation.

Section 14.3, Sinking Fund Schedules: When writing amounts in the table, round the answers to two decimals. However, behind the scenes all calculations must be unrounded.

Section 15.1, Net Present Value: Due to the imprecision of future cost and profit estimates, the net present value is rounded off to the nearest dollar unless otherwise specified.

Section 15.1, Net Present Value Ratio: When performing this calculation in hard capital rationing, it is round this variable to two decimals.

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16.4: Formula Sheet

Part 1: Mathematics Fundamentals

- (2.1) Percentage Conversion

$$\% = \text{dec} \times 100$$

- (2.2) Rate, Portion, Base

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- (3.1) Percent Change:

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$$

- (3.2) Rate Of Change Over Time

$$\text{RoC} = \left(\sqrt[n]{\frac{\text{New}}{\text{Old}}} - 1\right) \times 100$$

- (3.3) Simple Average

$$\text{SAvg} = \frac{\Sigma x}{n}$$

- (3.4) Weighted Average

$$\text{WAvg} = \frac{\Sigma wx}{\Sigma w}$$

- • (3.5) Geometric Average

$$\text{GAvg} = \left[\sqrt[n]{(1 + \Delta\%_1) \times (1 + \Delta\%_2) \times \cdots \times (1 + \Delta\%_n)} - 1\right] \times 100$$

Part 2: Business Applications

- (4.1) Salary and Hourly Gross Earnings

$$\begin{aligned} \text{GE} \ = \ & \text{Regular Earnings} + \text{Overtime Earnings} + \text{Holiday Earnings} \\ & + \text{Statutory Holiday Worked Earnings} \end{aligned}$$

- (4.2) Annual Income Tax

$$\text{Income Tax} \ = \Sigma(\text{Eligible Income In Tax Bracket} \times \text{Tax Bracket Rate})$$

- (4.3) Index Numbers

$$\text{Index Number} \ = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$$

- (4.4) Purchasing Power Of A Dollar

$$\text{PPD} \ = \frac{\$1}{\text{CPI}/100}$$

- (4.5) Real Income

$$\text{RI} \ = \frac{\text{Nominal Income}}{\text{CPI}/100}$$

- (5.1) Unit Variable Cost

$$\text{VC} \ = \frac{\text{TVC}}{n}$$

- (5.2) Net Income Using A Total Revenue And Cost Approach

$$\mathrm{NI} \ = \mathrm{n(S)} - (\mathrm{TFC} + \mathrm{n(VC)})$$

- (5.3) Unit Contribution Margin

$$\mathrm{CM} \ = \ \text{S–VC}$$

- (5.4) Net Income Using Total Contribution Margin Approach

$$\mathrm{NI} \ = \text{n(CM)–TFC}$$

- (5.5) Contribution Rate If Unit Information Known

$$\mathrm{CR} \ = \frac{\mathrm{CM}}{\mathrm{S}} \times 100$$

- (5.6) Contribution Rate If Aggregate Information Known

$$\mathrm{CR} \ = \frac{\mathrm{TR} - \mathrm{TVC}}{\mathrm{TR}} \times 100$$

- (5.7) Unit Break-even

$$n = \frac{\mathrm{TFC}}{\mathrm{CM}}$$

- (5.8) Dollar Break-even

$$\mathrm{TR} = \frac{\mathrm{TFC}}{\mathrm{CR}}$$

- (6.1) Single Discount

$$\mathrm{N} = \mathrm{L} \times (1 - \mathrm{d})$$

- (6.2a & 6.2b) Discount Amount

$$\mathrm{D\$} = \mathrm{L} \times \mathrm{d} \text{ or } \mathrm{D\$} = \text{L–N}$$

- (6.3) Multiple Discounts

$$\mathrm{N} = \mathrm{L} \times (1 - \mathrm{d}_1) \times (1 - \mathrm{d}_2) \times \cdots \times (1 - \mathrm{d}_n)$$

- (6.4) Single Equivalent Discount

$$\mathrm{d} \ = 1 - (1 - \mathrm{d}_1) \times (1 - \mathrm{d}_2) \times \cdots \times (1 - \mathrm{d}_n)$$

- (6.5) The Selling Price Of A Product

$$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$$

- (6.6) Markup Amount

$$\mathrm{M\$} = \mathrm{E} + \mathrm{P}$$

- (6.7) The Selling Price Of A Product Using Markup Amount

$$\mathrm{S} = \mathrm{C} + \mathrm{M\$}$$

- (6.8) Markup On Cost Percentage

$$\mathrm{MoC\%} = \mathrm{M\$C} \times 100$$

- (6.9) Markup On Selling Price Percentage

$$\mathrm{MoS\%} = \frac{\mathrm{M\$}}{\mathrm{S}} \times 100$$

- (6.10) The Sale Price Of A Product

$$\mathrm{S}_{\text{on sale}} = \mathrm{S} \times (1 - \mathrm{d})$$

- (6.11a & 6.11b) Markdown Amount

$$\mathrm{D\$} = \mathrm{S} \times \mathrm{d} \text{ or } \mathrm{D\$} = \mathrm{S} - \mathrm{S}_{\text{on sale}}$$

- (6.12) Markdown Percentage

$$\mathrm{d} = \frac{\mathrm{D\$}}{\mathrm{S}} \times 100$$

- (6.13) Maintained Markup

$$\mathrm{MM} = \frac{\mathrm{M\$}\left(\mathrm{n}_1\right) + \left(\mathrm{M\$} - \mathrm{D\$}\right)\left(\mathrm{n}_2\right)}{\mathrm{n}_1 + \mathrm{n}_2}$$

- (7.1) Selling Price Including Tax

$$\mathrm{S}_{\text{tax}} = \mathrm{S} + (\mathrm{S} \times \text{Rate})$$

- (7.2) GST/HST Remittance

$$\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$$

- (7.3) Property Taxes

$$\text{Property Tax} = \Sigma(\mathrm{AV} \times \mathrm{PTR})$$

- (7.4) Currency Exchange

$$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$$

Part 3: Single Payment Financial Applications

- (8.1) Simple Interest

$$\mathrm{I} = \mathrm{Prt}$$

- (8.2) Simple Interest For Single Payments

$$\mathrm{S} = \mathrm{P}(1+\mathrm{rt})$$

- (8.3) Interest Amount For Single Payments

$$\mathrm{I} = \mathrm{S} - \mathrm{P}$$

- (9.1) Periodic Interest Rate

$$\mathrm{i} = \frac{\mathrm{IY}}{\mathrm{CY}}$$

- (9.2) Number of Compound Periods For Single Payments

$$\mathrm{N} = \mathrm{CY} \times \text{Years}$$

- (9.3) Compound Interest For Single Payments

$$\mathrm{FV} = \mathrm{PV}(1+\mathrm{i})^{\mathrm{N}}$$

- (9.4) Interest Rate Conversion

$$\mathrm{i}_{\text{New}} = \left(1 + \mathrm{i}_{\text{Old}}\right)^{\frac{\mathrm{CY}_{\text{Old}}}{\mathrm{CY}_{\text{New}}}} - 1$$

- (10.1) Periodic Interest Amount

$$\mathrm{I} = \mathrm{PV} \times \mathrm{i}$$

- (10.2) Purchasing Power Of A Dollar (Compound Interest Method)

$$\mathrm{PPD} = \frac{\$1}{(1+\mathrm{i})^{\mathrm{N}}} \times 100$$

Part 4: Annuity Payments Financial Applications

- (11.1) Number Of Annuity Payments

$$\mathrm{N} \; = \; \mathrm{PY} \times \mathrm{Years}$$

- (11.2) Ordinary Annuity Future Value

$$\mathrm{FV}_{\mathrm{ORD}} = \; \mathrm{PMT}\left[\frac{\left[(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}\right]^{\mathrm{N}}-1}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}\right]$$

- (11.3) Annuity Due Future Value

$$\mathrm{FV}_{\mathrm{DUE}} = \; \mathrm{PMT}\left[\frac{\left[(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}\right]^{\mathrm{N}}-1}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}\right] \times (1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}$$

- (11.4) Ordinary Annuity Present Value

$$\mathrm{FV}_{\mathrm{ORD}} = \; \mathrm{PMT}\left[\frac{1-\left[\frac{1}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}\right]$$

- (11.5) Annuity Due Present Value

$$\mathrm{FV}_{\mathrm{DUE}} = \; \mathrm{PMT}\left[\frac{1-\left[\frac{1}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}\right] \times (1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}$$

- (12.1) Future Value Of A Constant Growth Ordinary Annuity

$$\mathrm{FV}_{\mathrm{ORD}} = \; \mathrm{PMT}(1+\Delta\%)^{\mathrm{N}-1}\left[\frac{\left[\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}\right]^{\mathrm{N}}-1}{\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right]$$

- (12.2) Future Value Of A Constant Growth Annuity Due

$$\mathrm{FV}_{\mathrm{DUE}} = \; \mathrm{PMT}(1+\Delta\%)^{\mathrm{N}-1}\left[\frac{\left[\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}\right]^{\mathrm{N}}-1}{\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right] \times (1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}$$

- (12.3) Present Value Of A Constant Growth Ordinary Annuity

$$\mathrm{PV}_{\mathrm{ORD}} = \frac{\mathrm{PMT}}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right]$$

- (12.4) Present Value Of A Constant Growth Annuity Due

$$\mathrm{PV}_{\mathrm{DUE}} = \frac{\mathrm{PMT}}{1+\Delta\%}\left[\frac{1-\left[\frac{1+\Delta\%}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}\right]^{\mathrm{N}}}{\frac{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{1+\Delta\%}-1}\right] \times (1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}$$

- (12.5) Ordinary Perpetuity Present Value

$$\mathrm{PV}_{\mathrm{ORD}} = \frac{\mathrm{PMT}}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}$$

- (12.6) Perpetuity Due Present Value

$$\mathrm{PV_{DUE}} = \mathrm{PMT}\left(\frac{1}{(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1}+1\right)$$

Part 5: Amortization & Special Financial Concepts

- (13.1) Interest Portion Of An Ordinary Single Payment

$$\mathrm{INT} \ = \ \mathrm{BAL} \times ((1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1)$$

- (13.2) Principal Portion Of A Single Payment

$$\mathrm{PRN} \ = \ \mathrm{PMT}-\mathrm{INT}$$

- (13.3) Principal Portion For A Series Of Payments

$$\mathrm{PRN} \ = \mathrm{BAL_{P1}}-\mathrm{BAL_{P2}}$$

- (13.4) Interest Portion For A Series Of Payments

$$\mathrm{INT} \ = \ \mathrm{N} \times \mathrm{PMT}-\mathrm{PRN}$$

- (13.5) Interest Portion Of A Due Single Payment

$$\mathrm{INT_{DUE}} = (\mathrm{BAL}-\mathrm{PMT}) \times ((1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1)$$

- (14.1) The Cash Price For Any Bond

$$\text{Cash Price} \ = \ \mathrm{PRI}+\mathrm{AI}$$

- (14.2) Bond Coupon Annuity Payment Amount

$$\mathrm{PMT_{BOND}} = \ \text{Face Value} \times \frac{\mathrm{CPN}}{\mathrm{CY}}$$

- (14.3) Bond Price On An Interest Payment Date

$$\text{Date Price} \ = \frac{\mathrm{FV}}{(1+\mathrm{i})^{\mathrm{N}}}+\mathrm{PMT_{BOND}}\left[\frac{1-\frac{1}{[1+\mathrm{i}]^{\mathrm{N}}}}{\mathrm{i}}\right]$$

- (14.4) Bond Premium or Discount

$$\text{Premium or Discount} \ = \ \mathrm{PRI}-\text{Face Value}$$

- (14.5) Bond Cash Price On A Non-Interest Payment Date

$$\text{Cash Price} \ = (\text{Date Price})(1+\mathrm{i})^{\mathrm{t}}$$

- (14.6) Accrued Interest On A Non-Interest Payment Date

$$\mathrm{AI} = \mathrm{PMT_{BOND}} \times \mathrm{t}$$

- (14.7) Interest Portion Of A Sinking Fund Single Payment Due

$$\mathrm{INT} \ = (\mathrm{BAL}+\mathrm{PMT}) \times ((1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}-1)$$

- (14.8) The Annual Cost Of The Bond Debt

$$\mathrm{ACD} = (\text{Face Value} \times \mathrm{CPN})+(\mathrm{PMT} \times \mathrm{PY})$$

- (14.9) The Book Value Of The Bond Debt

$$\mathrm{BVD} = \text{Bonds Outstanding}-\mathrm{BAL}$$

- (15.1) Net Present Value

$$\mathrm{NPV} \ = (\text{Present Value Of All Future Cash Flows})-(\text{Initial Investment})$$

- (15.2) Net Present Value Ratio

$$\mathrm{NPV}_{\mathrm{RATIO}}=\frac{\mathrm{NPV}}{\mathrm{CFo}}$$

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16.NS: Notation Summary

% = percentage

Δ% = constant growth rate per payment interval or the percent change from one period to another

∑ = summation symbol meaning to add everything up

ACD = annual cost of the bond debt

AI = accrued interest

AV = assessed value of a property

BAL = principal balance immediately after an annuity payment

BAL_{P1} = principal balance immediately prior to the first payment in a series of annuity payments

BAL_{P2} = principal balance immediately after the last payment in a series of annuity payments

Base = the whole quantity

BVD = book value of the bond debt

C = cost Cash Price = bond price summing market price & accrued interest

CFo = initial cash flow

CM = unit contribution margin

CPI = Consumer Price Index

CPN = nominal bond coupon rate of interest

CR = contribution rate

Current Currency = currency being converted from

CY or C/Y= compounds per year, or compounding frequency

CY_{New} = the new compounding frequency

CY_{Old} = the old compounding frequency

D$ = discount amount or markdown amount

d = discount rate or markdown rate

Date Price = bond price on the interest payment date preceding the sale date

dec = any number in decimal format

Desired Currency = currency you are converting to

Discount = bond discount amount

E = expenses

Exchange Rate = per unit conversion rate for current currency

Face Value = face value

FV = maturity value or future value in dollars

FV_{DUE} = future value of annuity due

FV_{ORD} = future value of ordinary annuity

GAvg = geometric average

GE = gross earnings

I = interest amount in dollars

i = periodic interest rate

i_{New} = the new periodic interest rate

INT = interest portion

INT_{DUE} = interest portion of a due single annuity payment

INT_{SFDUE} = the interest portion of any single annuity due sinking fund payment

i_{Old} = the old periodic interest rate

IY or I/Y = nominal interest rate per year

L = list price ln = natural logarithm

M$ = markup amount

MM = maintained markup

MoC% = markup on cost percentage

MoS% = markup on selling price percentage

n = a frequency or count or total

N = in merchandising it is the net price; in single payment compound interest it is the number of compound periods; in annuity compound interest it is the number of annuity payments

New = the value that a quantity has become

NI = net income

NPV = net present value

NPV_{RATIO} = net present value equivalent annual ratio

Old = the value that a quantity used to be

P = profit; or principal

PMT = annuity payment amount

PMT_{BOND} = bond annuity payment

Portion = the part of the whole

PPD = purchasing power of a dollar

Premium = bond premium amount

PRI = bond market price

PRN = principal portion

Property Tax = property tax amount

PTR = property tax rate

PV = principal or present value

PV_{DUE} = present value of annuity due

PV_{ORD} = present value of ordinary annuity

PY or P/Y = payments per year or payment frequency

r = interest rate (in decimal format)

Rate = portion and base relationship; sales tax rate

Remit = tax remittance

RI = real income

RoC = rate of change per time period

S = regular or unit selling price; or sum of principal/interest

SAvg = simple average

S_{BE} = the selling price at the break-even point

S_{onsale} = sale price

S_{tax} = selling price including taxes

t = time or term; time ratio

Tax Collected = total tax collected through sales

Tax Paid = total tax paid through purchases

TFC = total fixed costs

TR = total revenue

TVC = total variable cost

VC = unit variable cost

w = weighting factor

WAvg = weighted average

x = any individual piece of data

Years = the term of an annuity

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